Chess Tableaux

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Submitted: Feb 15, 2005; Accepted: May 12, 2005; Published: Jun 14, 2005
Mathematics Subject Classifications: 05A15, 05E10

Dedicated to Richard Stanley on the occasion of his 60th birthday

Abstract

A *chess tableau* is a standard Young tableau in which, for all i and j, the parity of the entry in cell (i, j) equals the parity of $i + j + 1$. Chess tableaux were first defined by Jonas Sjöstrand in his study of the sign-imbalance of certain posets, and were independently rediscovered by the authors less than a year later in the completely different context of composing chess problems with interesting enumerative properties. We prove that the number of $3 \times n$ chess tableaux equals the number of Baxter permutations of $n - 1$, as a corollary of a more general correspondence between certain three-rowed chess tableaux and certain three-rowed Dulucq-Guibert nonconsecutive tableaux. The correspondence itself is proved by means of an explicit bijection. We also outline how lattice paths, or rat races, can be used to obtain generating functions for chess tableaux. We conclude by explaining the connection to chess problems, and raising some unanswered questions, e.g., there are striking numerical coincidences between chess tableaux and the Charney-Davis statistic; is there a combinatorial explanation?

1 Introduction

A *chess tableau* is a standard Young tableau in which, for all i and j, the parity of the entry in cell (i, j) equals the parity of $i + j + 1$. If $i + j + 1$ is even (respectively, odd), then cell (i, j) is called an *even cell* (respectively, an *odd cell*), and in a chess tableau it necessarily contains an even (respectively, odd) entry. See Figure 1, in which the odd cells are shaded as a visualization aid.

1	2	3	6	7	10	15
4	5	8	9	16		
11	12	13	14			

Figure 1: Example of a chess tableau

We write $\text{Chess}(\lambda)$ for the number of chess tableaux of shape λ. If the parts of λ are (for example) a, b, and c, then we often write a, b, c instead of λ; for example, we sometimes write $\text{Chess}(a, b, c)$ for $\text{Chess}(\lambda)$. We also adopt the convention that $\text{Chess}(\lambda) = 0$ if λ is not a partition (i.e., if it contains negative integers or does not decrease monotonically).

Chess tableaux were first defined by Jonas Sjöstrand [8] in his study of the sign-imbalance of certain posets. In a remarkable coincidence, chess tableaux were independently rediscovered shortly thereafter by the present authors in the completely different context of composing chess problems with interesting enumerative properties. The connection with chess problems is explained in Section 4 below; here it suffices to remark that our investigations led us to search for a nice formula for $\text{Chess}(\lambda)$—if not for all λ, then at least for some λ. Sjöstrand considered the *signed* enumeration of chess tableaux, but we are the first to consider their *direct* enumeration.

As we shall see shortly, $\text{Chess}(\lambda)$ is easy to compute if λ has only one or two parts. In Section 2, we consider the much subtler case when λ has three parts, showing that there is a surprising and mysterious relationship between chess tableaux and so-called "nonconsecutive tableaux." In particular, we compute $\text{Chess}(\lambda)$ exactly when λ is a $3 \times n$ rectangle. In Section 3, we show how to derive a rational generating function for $\text{Chess}(\lambda)$ when λ has a bounded number of parts. Finally, in Section 5, we describe some further miscellaneous results and open problems. Specifically, a formula for $\text{Chess}(\lambda)$ for arbitrary λ remains an open question.

We now present a few basic facts about chess tableaux. If λ has only one row, then the unique standard Young tableau of shape λ is also a chess tableau; hence $\text{Chess}(\lambda) = 1$. If λ has two rows, then it turns out that computing $\text{Chess}(\lambda)$ reduces to counting standard Young tableaux with two rows. More precisely, if we let $\text{SYT}(\lambda)$ denote the number of standard Young tableaux of shape λ, then we have the following proposition.

Proposition 1. *Let $a \geq b > 0$. If a is even and b is odd, then* $\text{Chess}(a, b) = 0$. *Else,* $\text{Chess}(a, b) = \text{SYT}(\lfloor (a-1)/2 \rfloor, \lfloor b/2 \rfloor)$. *In particular,* $\text{Chess}(2k+1, 2k+1)$ *equals the kth*

Catalan number and $\mathrm{Chess}(2k, 2k) = 0$.

Proof. If a is even and b is odd, then there are more even cells than odd cells, so a chess tableau of shape a, b cannot exist. Otherwise, if one constructs a chess tableau of shape a, b by writing down the entries in consecutive order, then one quickly sees that for $i \geq 1$, the entry $2i + 1$ (if it exists at all, i.e., if $2i + 1 \leq a + b$) is forced to appear immediately to the right of the entry $2i$. Therefore, in row 1, we may "glue together" the 2nd and 3rd cells, the 4th and 5th cells, and so on, leaving the last cell in the row unglued if a is even. Similarly, in row 2, we may glue together the 1st and 2nd cells, the 3rd and 4th cells, etc., leaving the last cell in the row unglued if b is odd. When constructing a chess tableau, we enter 1, and then for all i we enter the numbers $2i$ and $2i + 1$ together into a pair of glued-together cells, and put the largest entry $a + b$ in the remaining unglued cell (if it exists). From this construction one sees readily that chess tableaux of shape a, b are equivalent to standard Young tableaux of shape $\lfloor (a - 1)/2 \rfloor, \lfloor b/2 \rfloor$. $\square$

The above argument that $\mathrm{Chess}(a, b) = 0$ if a is even and b is odd is a special case of an argument of Sjöstrand that applies more generally. The key observation is that the set of numbers from 1 to n either has an equal number of odd and even numbers or has an excess of one odd number. This fact easily yields the following proposition.

Proposition 2. *If λ has three nonempty rows and all three rows end with a cell of the same parity (in other words, if the parities of the parts of λ alternate even-odd-even or odd-even-odd), then* $\mathrm{Chess}(\lambda) = 0$.

Proof. If all three rows end in an odd cell, then each of the first and third rows has one more odd cell than it has even cells, while the second row has the same number of even and odd cells, for an overall excess of two odd cells. Similarly, if all three rows end in an even cell, then each of the first and third rows has the same number of even and odd cells, while the second row has one more even cell than it has odd cells, for an overall excess of one even cell. $\square$

The importance of the next definition will become clear in the next section.

Definition 1. *A partition or a tableau is* balanced *if it has three parts (not necessarily nonzero) and the parities of its parts are even-even-odd or odd-odd-even.*

Note that if T is a chess tableau with three parts and the lengths of row 2 and row 3 have opposite parity, then T is balanced, by Proposition 2.

Proposition 3. *Let T be a balanced chess tableau of shape d, e, f. Let $T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$ be a maximal chain where each T_k is a balanced chess tableau and each containment is strict. Then the chain is unique. Furthermore, the chess tableau T_0 has an odd number of entries in row 1, a single entry in row 2, and an empty row 3. For any $1 \leq k \leq n$, the number of entries in rows 2 and 3 of T_k is precisely two more than the number of entries in rows 2 and 3 of T_{k-1}. We have $n = (e + f - 1)/2$.*

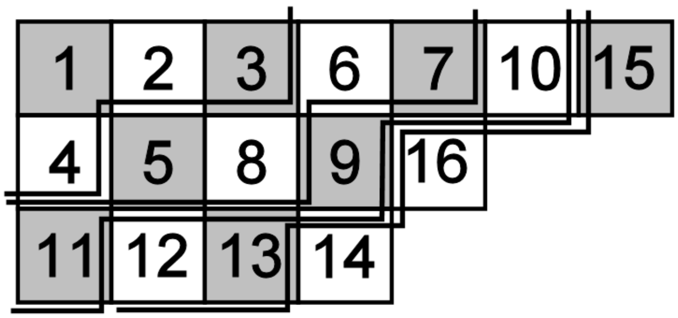


Figure 2: Decomposition of a balanced chess tableau into a chain of balanced subtableaux

The chess tableau of Figure 1 is balanced, and Figure 2 illustrates the chain described in Proposition 3 for this tableau.

Proof (of Proposition 3). Because any subtableau of T must consist of a consecutive set of entries beginning with 1, the chain is unique.

Note that there are no balanced tableaux with a single row. Let y_0 be the first entry in row 2 of T. Note that y_0 is even because T is a chess tableau. Therefore, the first y_0 entries of T comprise a balanced chess tableau, which must be T_0. If $T_0 = T$, the rest of the proposition follows and we are done. Otherwise, suppose we have constructed T_j for $j < k$ and $T_{k-1} \neq T$. Let x_k be the first entry in T not in T_{k-1}. A simple parity argument shows that because T_{k-1} is balanced, its largest entry must be in row 2 or 3. Therefore, the parity of the last cell in row 1 of T_{k-1} is the same as the parity of x_k. This implies that x_k must be in row 2 or 3. Let y_k be the smallest entry in T greater than x_k which is also in row 2 or 3. Such an entry must exist because if it did not exist, rows 2 and 3 of T would have the same parity, contrary to the fact that T is balanced.

Let T_k consist of all entries in T less than or equal to y_k. We claim that T_k is balanced. Because an even number of entries (in fact, exactly two) are added to rows 2 and 3 of T_{k-1} to obtain T_k, the parity of the sum of the lengths of rows 2 and 3 remains odd. Hence, the lengths of rows 2 and 3 of T_k have opposite parity.

Finally, note that there is no balanced tableau strictly between T_{k-1} and T_k, for such a tableau would differ in rows 2 and 3 from T_{k-1} by only one entry and therefore would not be balanced.

Because each step in the chain increases the number of entries in rows 2 and 3 (combined) by two and T_0 has exactly one entry in rows 2 and 3, it follows that $n = (e + f - 1)/2$. $\square$

2 Nonconsecutive Tableaux and the Main Result

A *nonconsecutive tableau* is a standard Young tableau in which, for all i, the entries i and $i + 1$ are in different rows. See Figure 3.

1	3	5	7	12	15
2	6	9	11	13	
4	8	10	14		

Figure 3: Example of a nonconsecutive tableau

We write $\mathrm{NCon}(\lambda)$ for the number of nonconsecutive tableaux of shape λ. We write $\mathrm{NCon}_i(\lambda)$ for the number of nonconsecutive tableaux of shape λ whose largest entry appears in row i (necessarily at the end of the row). If λ is not a partition, or if λ is empty, then we set $\mathrm{NCon}(\lambda) = \mathrm{NCon}_i(\lambda) = 0$. As far as we know, nonconsecutive tableaux were first studied by Dulucq and Guibert [4].

Our main result is the following striking connection between nonconsecutive tableaux and chess tableaux.

Theorem 1. *For any integers a, b, and c,*

$$\mathrm{NCon}_1(a, b, c) = \mathrm{Chess}(a + b - c, a - b + c, 1 - a + b + c) \tag{1}$$

Theorem 1 indicates an intimate relationship between nonconsecutive tableaux and chess tableaux, and one might suppose that there must be an obvious bijective proof. Our proof is bijective, but not obvious, and the reader is encouraged to find a simpler bijection.

Before proving Theorem 1, we deduce some easy corollaries.

Corollary 1. *If a, b, c is a nonempty partition, then* $\mathrm{NCon}(a, b, c) = \mathrm{Chess}(a + b - c, a - b + c, 1 - a + b + c) + \mathrm{Chess}(1 + a + b - c, 1 + a - b + c, -a + b + c)$.

Proof. If T is a nonconsecutive tableau of shape $a+1, b, c$ whose largest entry is in row 1, then by nonconsecutivity, the second-largest entry of T must be in either row 2 or row 3. So by deleting the largest entry, we see that

$$\mathrm{NCon}_1(a + 1, b, c) = \mathrm{NCon}_2(a, b, c) + \mathrm{NCon}_3(a, b, c).$$

On the other hand, it is obvious that

$$\mathrm{NCon}(a, b, c) = \mathrm{NCon}_1(a, b, c) + \mathrm{NCon}_2(a, b, c) + \mathrm{NCon}_3(a, b, c).$$

Therefore $\mathrm{NCon}(a, b, c) = \mathrm{NCon}_1(a, b, c) + \mathrm{NCon}_1(a + 1, b, c)$. Now apply Theorem 1 to complete the proof. $\square$

It is proved in [4] that

$$\mathrm{NCon}(n,n,n) = \frac{2}{n(n+1)^2}\sum_{k=0}^{n-1}\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}.$$

(More specifically, Dulucq and Guibert prove that $\mathrm{NCon}(n,n,n)$ equals the number of *Baxter permutations of* n*,* for which the above explicit formula was already known. The definition of a Baxter permutation is somewhat complicated and we do not need it, so we omit it, but the reader can find it in [4].)

This fact immediately yields an explicit formula for $\mathrm{Chess}(n,n,n)$.

Corollary 2. *For* $n > 1$*,*

$$\mathrm{Chess}(n,n,n) = \frac{2}{(n-1)n^2}\sum_{k=0}^{n-2}\binom{n}{k}\binom{n}{k+1}\binom{n}{k+2}. \tag{2}$$

Proof. Setting $a = b = c = n-1$ in Corollary 1 yields

$$\mathrm{NCon}(n-1,n-1,n-1) = \mathrm{Chess}(n-1,n-1,n) + \mathrm{Chess}(n,n,n-1).$$

But $n-1, n-1, n$ is not a partition so $\mathrm{Chess}(n-1,n-1,n) = 0$. Moreover, the largest entry of a chess tableau of shape n,n,n must go at the end of row 3, so $\mathrm{Chess}(n,n,n-1) = \mathrm{Chess}(n,n,n)$. Using the known explicit formula for $\mathrm{NCon}(n-1,n-1,n-1)$ yields the corollary. □

It would be nice to have a direct proof of Corollary 2.

Proof (of Theorem 1). Let d,e,f be a balanced partition and let $a = (d+e)/2$, $b = (d+f-1)/2$, and $c = (e+f-1)/2$. We construct a bijection φ from chess tableaux of shape d,e,f to nonconsecutive tableaux of shape a,b,c with largest entry in row 1.

Step 1: Description of the bijection φ. Given a chess tableau T of balanced shape d,e,f, first decompose it into a maximal set of balanced chess subtableaux $T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$ and define x_k and y_k as in the proof of Proposition 3. That is, let y_k be the largest entry of T_k and let $x_k = y_{k-1}+1$. Recall that x_k and y_k must be in row 2 or 3.

The image $U = \varphi(T)$ of T under our bijection will be constructed in stages; let U_k denote the tableau produced after stage k of the construction.

To construct U_0, place the entries 1 through $y_0 - 1$ alternating in rows 1 and 2.

For $k > 0$, exactly one of the four cases below holds; carry out stage k of the construction accordingly.

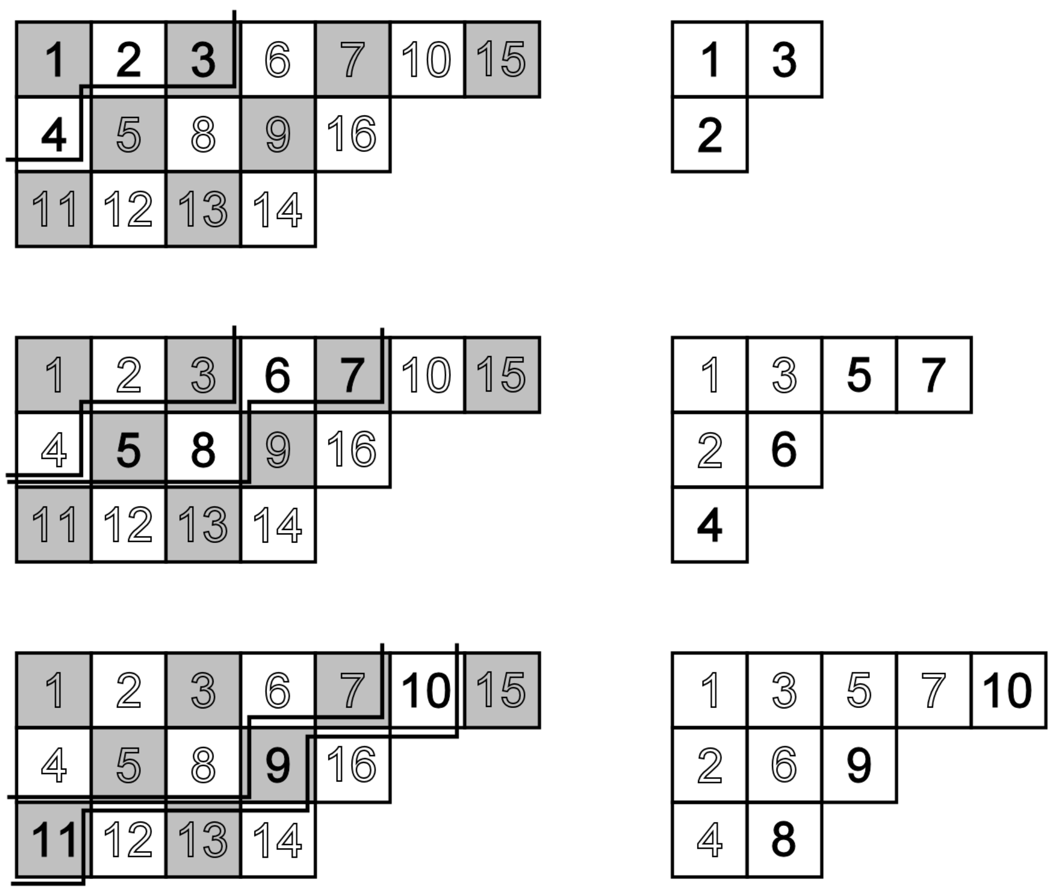
Case 1: x_k and y_k are both in row 2. Construct U_k by taking U_{k-1}, appending $x_k - 1$ to row 3, and then alternating x_k through $y_k - 1$ in rows 1 and 2, starting with x_k in row 1.

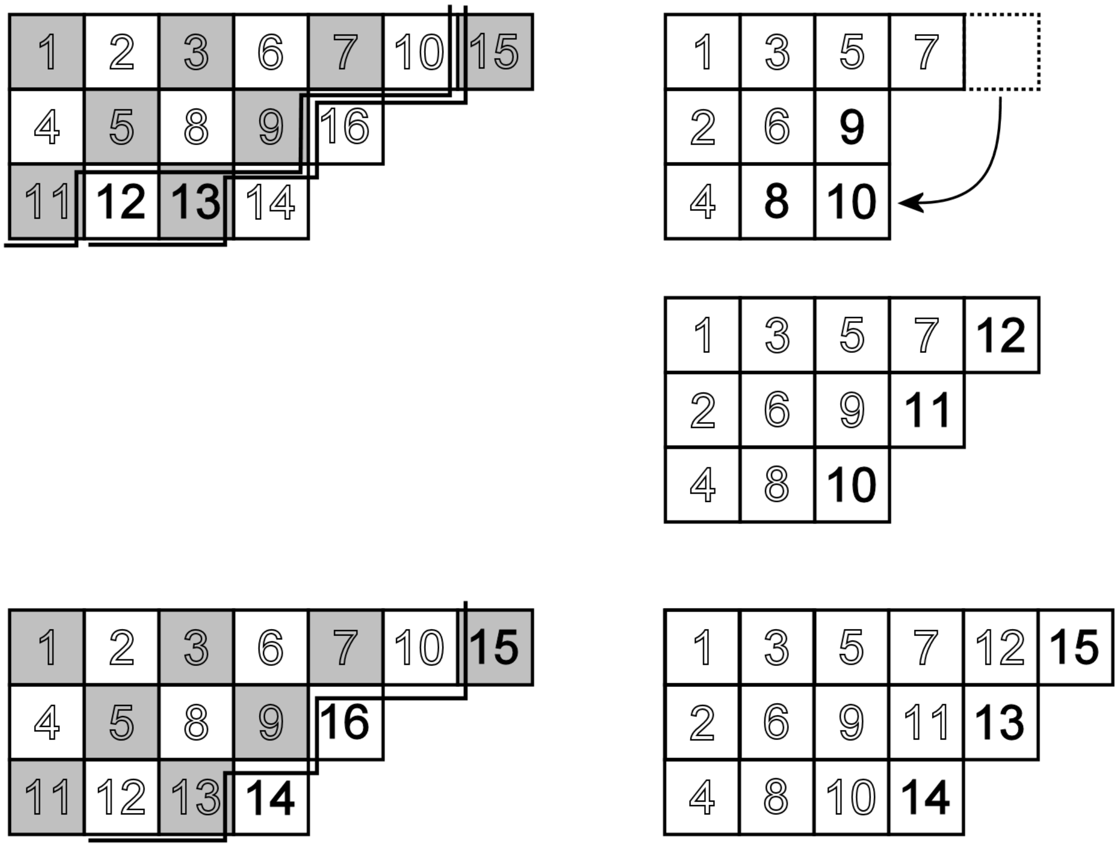
Case 2: x_k is in row 2, y_k is in row 3. Construct U_k by taking U_{k-1}, appending $x_k - 1$ to row 3, and then alternating x_k through $y_k - 1$ in rows 1 and 2, starting with x_k in row 2.

Case 3: x_k is in row 3, y_k is in row 2. Construct U_k by taking U_{k-1}, appending $x_k - 1$ to row 2, x_k to row 3, and then alternating $x_k + 1$ through $y_k - 1$ in rows 1 and 2, starting with $x_k + 1$ in row 1.

Case 4: x_k and y_k are both in row 3. This case is unique in that U_k is *not* just an extension of U_{k-1}. Begin with U_{k-1}, but first move its largest entry $x_k - 2$ (along with the cell it's in) from row 1 to row 2 or row 3, whichever choice preserves nonconsecutivity. Then append $x_k - 1$ to row 2 or row 3 (preserving nonconsecutivity), and then alternate x_k through $y_k - 1$ in rows 1 and 2, starting with x_k in row 1, to obtain U_k.

Before continuing with the proof, we give an example, showing how the chess tableau of Figure 1 is carried to the nonconsecutive tableau of Figure 3.





Step 2: Proof that the description of φ makes sense. Denote the shape of T_k by d_k, e_k, f_k and let $a_k = (d_k + e_k)/2$, $b_k = (d_k + f_k - 1)/2$, and $c_k = (e_k + f_k - 1)/2$. Note that $a_k > b_k$. We show by induction on k that U_k is a nonconsecutive tableau of shape a_k, b_k, c_k with largest entry in row 1.

Observe that in Cases 1 and 4, x_k and y_k have opposite parity (so that $y_k - x_k$ is odd), while in Cases 2 and 3, x_k and y_k have the same parity (so that $y_k - x_k$ is even). This fact will be used implicitly below, justifying our treating certain expressions of the form $(\cdot)/2$ as integers.

In Case 1, x_k, the smallest integer not in T_{k-1}, is in row 2; this means that $d_{k-1} > e_{k-1}$. Therefore, $b_{k-1} > c_{k-1}$, so there is no danger of creating an illegal shape by appending $x_k - 1$ to row 3 of U_{k-1}. Note that $d_k = d_{k-1} + y_k - x_k - 1$, $e_k = e_{k-1} + 2$, and $f_k = f_{k-1}$. To U_{k-1}, we have added $(y_k - x_k + 1)/2$ entries to row 1, $(y_k - x_k - 1)/2$ entries to row 2, and a single entry to row 3. So U_k has shape a_k, b_k, c_k.

In Case 2, the same argument as in Case 1 shows that there is no danger of creating an illegal shape by appending $x_k - 1$ to row 3 of U_{k-1}. We have $d_k = d_{k-1} + y_k - x_k - 1$, $e_k = e_{k-1} + 1$, and $f_k = f_{k-1} + 1$. To U_{k-1}, we have added $(y_k - x_k)/2$ entries to row 1, $(y_k - x_k)/2$ entries to row 2, and a single entry to row 3. So U_k has shape a_k, b_k, c_k.

In Case 3, there is no danger of creating an illegal shape by appending $x_k - 1$ to row 2 of U_{k-1}, because, as we have already observed, $a_{k-1} > b_{k-1}$. We have $d_k = d_{k-1} + y_k - x_k - 1$,

$e_k = e_{k-1} + 1$, and $f_k = f_{k-1} + 1$. To U_{k-1}, we have added $(y_k - x_k)/2$ entries to row 1, $(y_k - x_k)/2$ entries to row 2, and a single entry to row 3. So U_k has shape a_k, b_k, c_k.

In Case 4, we have $e_{k-1} > f_{k-1}+2$ (strict inequality since T_{k-1} is balanced). Therefore, $a_{k-1} > b_{k-1} + 1$. In U_{k-1}, if $x_k - 3$ is in row 3, then $x_k - 2$ must be moved to row 2, and there is no danger of creating an illegal shape. On the other hand, if $x_k - 3$ is not in row 3 of U_{k-1} (and so, by nonconsecutivity, is necessarily in row 2), then we claim that $b_{k-1} > c_{k-1}$. If to the contrary, $b_{k-1} = c_{k-1}$, then the last entry in row 2 of U_{k-1}, namely $x_k - 3$, could not exceed the last entry in row 3 of U_{k-1}. However, the only entry in U_{k-1} that is higher than $x_k - 3$ is $x_k - 2$, which must be in row 1. This contradiction shows that $b_{k-1} > c_{k-1}$, so moving $x_k - 2$ to row 3 will not create an illegal shape. Moreover, since $a_{k-1} > b_{k-1} + 1$, row 1 is still longer than row 2 after the move, so appending $x_k - 1$ to row 2 also does not create an illegal shape. We have $d_k = d_{k-1} + y_k - x_k - 1$, $e_k = e_{k-1}$, and $f_k = f_{k-1} + 2$. To obtain U_k, we have increased row 1 of U_{k-1} by $(y_k - x_k - 1)/2$ entries, row 2 by $(y_k - x_k + 1)/2$ entries, and row 3 by a single entry. So U_k has shape a_k, b_k, c_k.

Note that in Cases 1 through 3, $x_k - 1$ is not placed next to $x_k - 2$, which is in row 1 by induction. Also note that in all four cases, $y_k - 1$ ends up in row 1 for parity reasons.

Step 3: Proving that φ is a bijection. We construct φ^{-1} by induction on the length of row 3. But first, we remark that for parity reasons, if we wish to enlarge any given balanced chess tableau of shape d, e, f, we can always add $d + e + f + 1$ to row 2 or row 3, provided that $d > e$ or $e > f$, respectively. However, it is never possible to add $d + e + f + 1$ to row 1. In other words, the parity of the last cell in rows 2 and 3 is equal to the parity of $d + e + f$, whereas the last cell in row 1 has parity that of $d + e + f + 1$.

Let U be a nonconsecutive tableau of shape a, b, c with largest entry in row 1.

If $c = 0$, then we must have $a = b + 1$ and U is the unique nonconsecutive tableau with the entries 1 through $a + b + 1$ alternating in rows 1 and 2 starting with a 1 in row 1. Map U to the balanced chess tableau with entries 1 through $a + b + 1$ in row 1, $a + b + 2$ in row 2, and empty row 3.

Now assume that $c > 0$ and, by induction, for all shapes a', b', c' with $c' < c$, we have constructed a map from nonconsecutive tableaux of shape a', b', c' with largest entry in row 1 to balanced chess tableaux of shape $a' + b' - c'$, $a' - b' + c'$, $1 - a' + b' + c'$.

Let V be the tableau obtained from U by removing every entry greater than or equal to the last entry in row 3.

If V has its largest entry in row 1, then let $U' = V$. Let the shape of U' be given by a', b', c'. By induction, U' is mapped to a balanced chess tableau T'. Furthermore, the shape of T' is given by $d' = a' + b' - c'$, $e' = a' + c' - b'$, $f' = 1 - a' + b' + c'$. Let $x_n - 1$ and $y_n - 1$ be the smallest and largest entries removed from U to obtain U'.

Note that $c' < b'$ by construction of V. Therefore $d' > e'$. Therefore, we can construct a chess tableau by adding x_n to row 2 of T', and then adding the entries $x_n + 1$ through $y_n - 1$ to row 1. Finally, if x_n and y_n have the same parity, add y_n to row 3; otherwise, add y_n to row 2. Map U to the resulting tableau T.

Now, suppose that the largest entry of V is not in row 1, but in row 2. (By nonconsecutivity, the largest entry of V cannot be in row 3.)

If the penultimate entry in V is in row 1, let U' be the tableau obtained from V by deleting the largest entry (which is in row 2). Let the shape of U' be given by a', b', c'. Since U' is a nonconsecutive tableau with largest entry in row 1, by induction, U' is mapped to a balanced chess tableau T'. Furthermore, the shape of T' is given by $d' = a' + b' - c'$, $e' = a' + c' - b'$, $f' = 1 - a' + b' + c'$. Let $x_n - 1$ and $y_n - 1$ be the smallest and largest entries removed from U to obtain U'. Observe that since the largest entry of V is in row 2, an even number of entries must have been removed from U to obtain V. This means that x_n and y_n have the same parity. Also, note that $a' > b'$ because U' was obtained from V by removing the last entry in row 2. This implies that $e' > f'$. Therefore, we can construct a chess tableau by adding x_n to row 3 of T', and then adding the entries $x_n + 1$ through $y_n - 1$ to row 1. Because x_n and y_n have the same parity, we can finish by adding y_n to row 2 to obtain a chess tableau T. Map U to this T.

Finally, assume that the penultimate entry in V is in row 3. If an even number of entries were removed from U to obtain V, then let U' be the nonconsecutive tableau obtained by shifting the largest entry in V, along with its cell, into row 1. If an odd number of entries were removed from U to obtain V, then let U' be the nonconsecutive tableau obtained by adding to row 1 the largest entry in row 3 of U. Let the shape of U' be given by a', b', c'. Since U' is a nonconsecutive tableau with largest entry in row 1, by induction, U' is mapped to a balanced chess tableau T'. Furthermore, the shape of T' is given by $d' = a' + b' - c'$, $e' = a' + c' - b'$, $f' = 1 - a' + b' + c'$. Let $x_n - 1$ and $y_n - 1$ be the smallest and largest entries removed from U to obtain U'. Because U' has an even number of entries fewer than there are in U, we know that x_n and y_n have opposite parity. We claim that $a' > b' + 1$. To see this, first note that if the number of entries in U and V have the same parity, then U' was obtained by shifting the last entry in row 2 to row 1, so that $a' > b' + 1$. If the number of entries in U and V have opposite parity, then the lengths of rows 1 and 2 of V separately differ from the lengths of rows 1 and 2 of U by equal amounts, and since $a > b$, we know that the length of row 1 of V is greater than the length of its row 2. Since U' is obtained from V by adding an entry to row 1, we conclude again that $a' > b' + 1$. Since $a' > b' + 1$, we must have $e' > f' + 1$. Therefore, we can construct a chess tableau T from T' by adding x_n to row 3, adding the entries $x_n + 1$ through $y_n - 1$ to row 1, and finally adding y_n to row 3. Map U to this T.

It is straightforward to check that this map and φ are inverse to each other, and hence are both bijections.

This completes the proof of the theorem for the case in which $a+b-c, a-b+c, 1-a+b+c$ is a balanced partition. Since $a - b + c$ and $1 - a + b + c$ automatically have opposite parity, the only remaining cases are those in which $a + b - c, a - b + c, 1 - a + b + c$ is not a partition at all. But note that if $a > b \geq c \geq 0$ and $a \leq b + c + 1$, then $a + b - c \geq a - b + c \geq 1 - a + b + c \geq 0$. Therefore if $a + b - c, a - b + c, 1 - a + b + c$ is not a partition, then $\mathrm{NCon}_1(a, b, c) = 0$ (if $a > b + c + 1$ then every standard Young tableau of shape a, b, c must have consecutive entries in row 1), and both sides of (1) are zero. $\square$

We conclude this section with an alternative description of φ^{-1} that is surprisingly similar in form to the description of φ itself. We leave it to the reader to check that the following description of a map makes sense and indeed coincides with φ^{-1}.

Let U be a nonconsecutive tableau with largest entry in its first row of shape a, b, c. Let V be the nonconsecutive subtableau of U obtained by removing the entries of U that are greater than the second largest entry in row 1. Let a', b', c' be the shape of V. Note that $a = a' + 1$.

Assume by induction that we have already defined the bijection on subshapes of a, b, c. Let S be the chess tableau that V is mapped to by this bijection. We show how to add entries to S to get a chess tableau T that will be the image of U.

We consider four cases.

Case 1. We have $b - b' = c - c'$ and the largest entry of U is in row 2. In this case, form T by first adding an entry to row 2 of S, then adding another entry to row 1 of S, and finally adding $2(b - b') - 1$ entries to row 3 of S.

Case 2. We have $b - b' = c - c' + 1$ (in which case the largest entry of U must be in row 2). In this case, let R be the row of the largest entry in S. Form T by first moving the largest entry of S from row R to row 1, then adding a new entry to row 1, then adding an entry to row R, then adding $2(c - c')$ entries to row 3.

Case 3. We have $b - b' = c - c'$ and the largest entry of U is in row 3. In this case, form T by first adding an entry to row 3 of S, then adding an entry to row 1, followed by another entry to row 2, followed by adding $2(b - b') - 2$ entries to row 3.

Case 4. We have $b - b' + 1 = c - c'$ (in which case the largest entry of U must be in row 3). In this case, form T by first adding two entries to row 2 of S, and then adding $2(b - b')$ entries to row 3.

3 Generating Functions From Rat Races

For $\mathrm{SYT}(\lambda)$ there is the miraculous hook length formula, but there seems to be no analogue for $\mathrm{Chess}(\lambda)$. However, when λ has two rows, there are nice expressions. For example:

$$\mathrm{Chess}(2r+1, 2s) = \binom{r+s}{r} - \binom{r+s}{r+1} \tag{3}$$

Similar alternating sum formulas can be obtained for any number of rows by interpreting tableaux as nontouching rat races, a concept defined in [5]. It is equivalent to noncrossing lattice paths, but suits our applications in the next section better.

A *rat race* is an event that takes place on the real line. The rats have separate starting points, one unit apart to the left of the origin; in fact, rat i starts at coordinate $-i$. The running distance for rat i being λ_i units, it will finish at coordinate $\lambda_i - i$. Every time step, one of the rats moves one unit to the right. After $|\lambda|$ time steps, all rats have reached their final position.

The rat race is completely specified by recording which rat moves in each time step. In Figure 4, rat one moves in time steps 1, 2, 3, 6, 7 and rat two in time steps 4, 5, 8.

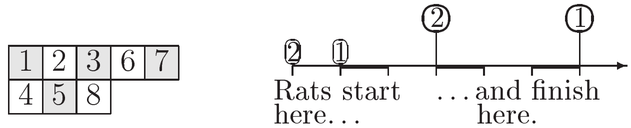


Figure 4: A chess tableau and its nontouching rat race

Standard Young tableaux correspond to *nontouching rat races*, where two rats never occupy the same coordinate simultaneously, for this condition means that the columns are increasing. *Touching rat races* are those that are not nontouching.

For a general rat race, there is no column condition and no condition on the row lengths. (Rat one may be a sprinter and rat two a marathon runner.) However, the track segments are colored black and white, and our rats run black segments only in odd time steps and white segments only in even time steps. Therefore, nontouching rat races will correspond exactly to chess tableaux. We let $\mathrm{Rats}(\lambda)$ denote the number of rat races (with the color restriction just described) in which the ith rat moves λ_i steps.

For the benefit of the rat crowd, all rats wear yellow T-shirts that carry the row index in red digits. Once a year, on Involution Day, the rat athletes play a prank on the unsuspecting crowd. At the very first moment during the race that two rats meet, the two rats in question switch T-shirts. The spectators now believe that they are watching a very different rat race. If rats one and two switch shirts, rat one seems to run a distance $\lambda_2 - 1$ while rat two seems to run $\lambda_1 + 1$. Nontouching races are not touched by the prank, but for the others a bijection is established. If $\lambda = a, b$ has just two parts $a \geq b$, then the prank occurs if and only if rat two catches rat one, and switching shirts gives a bijection between touching rat races of shape a, b and all rat races of shape $b-1, a+1$ (since the latter are necessarily touching). Thus

$$\mathrm{Chess}(a, b) = \mathrm{Rats}(a, b) - \mathrm{Rats}(b-1, a+1) \tag{4}$$

Counting rat races is much easier than counting chess tableaux, and we readily obtain formula (3) and similar ones for the other parity cases. A bivariate generating function $C(x, y)$ for $\mathrm{Chess}(a, b)$ is also easy to compute.

$$C(x, y) = \frac{(1 + x - y)(1 - 2y^2)}{(1 - y)(1 - x^2 - y^2)} \tag{5}$$

Here, $\mathrm{Chess}(a, b)$ is the coefficient of the term $x^a y^b$, unless $a < b$, in which case the term is uninteresting.

Rat races work for any number of rows; for three rows, the touching rat races are of two types: those in which rat two catches rat one first, and those in which rat three catches rat two first. So we must subtract $\mathrm{Rats}(b-1, a+1, c)$ and $\mathrm{Rats}(a, c-1, b+1)$ from $\mathrm{Rats}(a, b, c)$, but then to handle double counting we must follow through with alternating inclusion/exclusion terms. We obtain the following relation.

$$\begin{aligned}\text{Chess}(a,b,c) &= \text{Rats}(a,b,c) - \text{Rats}(b-1,a+1,c) - \text{Rats}(a,c-1,b+1)\\ &\quad + \text{Rats}(c-2,a+1,b+1) + \text{Rats}(b-1,c-1,a+2)\\ &\quad - \text{Rats}(c-2,b,a+2)\end{aligned} \tag{6}$$

Counting rat races is not too difficult if you use the obvious recursions. Splitting the generating function for $\text{Rats}(a,b,c)$ into parity cases, we can write

$$R(x,y,z) = R_{bbw} + R_{bwb} + R_{wbb} + R_{wwb} + R_{wbw} + R_{bww}. \tag{7}$$

Here R_{bbw} includes all shapes where the first row ends with a black square, the second with a black square and the third with a white square (so the row lengths are odd, even, even). Note that R_{bbb} and R_{www} are zero.

Now we have the following relations.

$$\begin{array}{lcllll} R_{bbw} &=& xR_{wbw} &+& yR_{bww} & \\ R_{bwb} &=& xR_{wwb} &+& zR_{bww} & \\ R_{wbb} &=& yR_{wwb} &+& zR_{wbw} & \\ R_{wwb} &=& xR_{bwb} &+& yR_{wbb} & \\ R_{bww} &=& yR_{bbw} &+& zR_{bwb} & \\ R_{wbw} &=& xR_{bbw} &+& zR_{wbb} & + \quad 1 \end{array}$$

Solving the system, we obtain the generating function for $\text{Rats}(a,b,c)$.

$$R(x,y,z) = \frac{(1+y)(1+x-y+z)}{1-x^2-y^2-z^2-2xyz} \tag{8}$$

If we substitute this into (6), we get a rational expression for $C(x,y,z)$, but with a forbidding numerator and no obvious interesting properties.

$$C(x,y,z) = \frac{1 - \cdots \text{a hundred and forty-seven terms} \cdots - 8y^3z^8}{(1-x^2-y^2-z^2-2xyz)(1-y^2-z^2)^2(1-x^2-z^2)(1-z)} \tag{9}$$

But we know that $\text{Chess}(a,a,a)$ are interesting numbers, so we may try to compute the *diagonal* of the ugly generating function. Indeed, for even a, (6) gives a simpler generating function, the $x^a y^a z^a$-coefficient of which is $\text{Chess}(a,a,a)$.

$$\begin{aligned} C_{\text{even}} &= \frac{P}{((1-x^2-y^2-z^2)^2-4x^2y^2z^2)(1-x^2-y^2)} \qquad (10)\\ \text{where } P &= 1-3x^2-4y^2-z^2+4x^4+9y^2x^2+5y^4+3z^2y^2-2x^6-10y^2x^4\\ &\quad -6y^4x^2-2y^6+2z^2x^4-2z^2y^2x^2-2z^2y^4+8y^4x^4 \end{aligned}$$

For rectangles with odd row lengths we can obtain a similar but less attractive expression.

Extracting the diagonal of this rational generating function can be done using residue calculus; for example, Akalu Tefera's program MultInt [10] can in principle automatically find a recurrence satisfied by the diagonal coefficients. This would yield a proof of

Corollary 2 that, while not 100% mechanical, would require much less human ingenuity than our bijective proof. Unfortunately, our rational generating function appears to be too complex for this plan to be carried out with today's computers, at least without any further tricks to simplify the computation.

4 The Connection With Chess Problems

There is a class of composed chess problems called *queue problems* in which each solution comprises the same *set* of moves, but in which the *order* of the moves varies from solution to solution. The goal is to find the total number of solutions.

Examples of queue problems may be found in [6] and [9]. In all these examples, the number of solutions equals the number of linear extensions of some particular finite poset. (The relations in the poset often correspond to moves that must occur in a certain order because one piece must vacate a square or a line to allow the next piece to move; this explains the term "queue problem," and by thinking of chess pieces as rats, one readily sees the connection to rat races.) Moreover, all the examples are *series-movers,* i.e., problems in which one side executes a series of consecutive moves while the other side passively stands by making no moves at all, in violation of one of the official rules of chess.

Shortly before [9] was written, it occurred to us that perhaps it would be possible to compose combinatorially interesting chess problems in which White and Black alternate moves. For those familiar with the principle behind the examples in [9], it is easy to see that instead of linear extensions of posets, we now seek posets whose elements are each colored either black or white, such that the number of *alternating linear extensions*—i.e., linear extensions in which black and white elements strictly alternate—is a combinatorially interesting number.

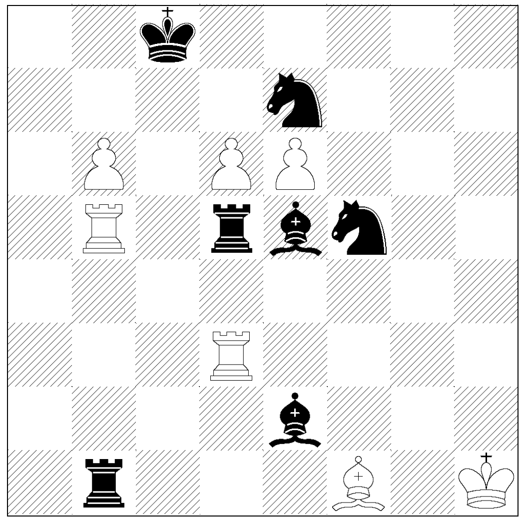
There is an unlimited number of possible posets, and black/white colorings of the posets, that one can choose to investigate. We chose to start by considering black/white colorings of Young diagrams, and this quickly led to the definition of a chess tableau. (Ironically, even though we were motivated by composing chess problems, we initially used the term "checkerboard tableau," switching to the term "chess tableau" only after learning that that was what Sjöstrand had called them.)

Corollary 2 shows that the number of $3 \times n$ chess tableaux is combinatorially interesting, so the task presents itself of composing a chess problem based on this fact. The problem shown in Figure 5 below, composed by the first author and dedicated to Richard Stanley on his 60th birthday, is a first attempt in that direction.

The stipulation "H=4.5" means that it is White to move, and that White and Black are to *cooperate* so as to *stalemate* Black after White's fifth move. White and Black alternate moves, so Black makes a total of four moves. The usual rules governing the legality of chess moves apply. The expression "7+7" is a checksum, indicating that there are 7 White units and 7 Black units on the board.

It turns out that there are precisely two solutions:

- 1.dxe7 **Rd7** 2.Rxd7 **Bc7** 3.Rxf5 **Ba6** 4.bxc7 **Rb7** 5.Bxa6



H=4.5 7+7

Figure 5: Helpstalemate with 2 solutions

- 1.dxe7 **Bc7** 2.bxc7 **Rd7** 3.Rxf5 **Rb7** 4.Rxd7 **Ba6** 5.Bxa6

(We have written Black's moves in boldface for clarity.) These can be thought of as corresponding to the two chess tableaux of shape $3, 3, 3$ as follows. Observe that the following relationships must hold between the individual moves of the solution:

- dxe7 must precede **Rd7** and **Bc7**.
- **Rd7** must precede Rxd7 and Rxf5.
- **Bc7** must precede bxc7 and Rxf5.
- Rxf5 and Rxd7 must precede **Ba6**.
- Rxf5 and bxc7 must precede **Rb7**.
- **Rb7** and **Ba6** must precede Bxa6.

(Note: The reason **Rb7** must precede Bxa6 is that the rook is pinning the bishop. By convention, even in series-movers, intermediate positions in which one of the kings is in check are forbidden.) These conditions impose a partial order on the moves; see Figure 6.

This poset has 42 linear extensions, corresponding to the 42 standard Young tableaux of shape 3,3,3. If Black and White were free to make the nine moves in question in any order and were not required to alternate moves, then these 42 tableaux would give rise to 42 solutions. However, the requirement that Black and White alternate moves imposes an additional constraint, which is readily seen to force the tableaux to be chess tableaux.

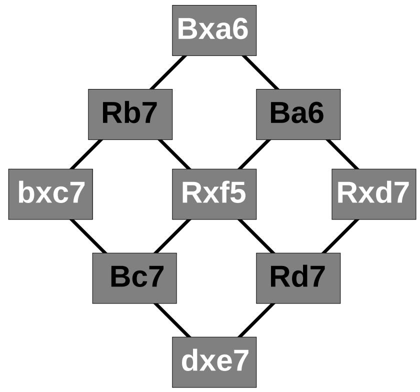


Figure 6: Ordering constraints among the moves

By Corollary 2, there are just two chess tableaux of shape 3,3,3, which give rise to the two solutions to the problem.

Clearly this example only scratches the surface of the possibilities for composing chess problems of this sort. There are presumably many other families of bicolored posets whose alternating linear extensions have a nice enumeration that can be exploited for compositional purposes.

5 The Charney-Davis Statistic and Open Problems

Given a standard Young tableau T with n cells, let $des(T)$ denote the number of *descents* of T, i.e., the number of positive integers $i < n$ such that the entry $i + 1$ appears in a lower-numbered row of T than the entry i does. Given a partition λ, we follow [7] and define the *Charney-Davis statistic* by

$$\mathrm{CD}(\lambda) = \sum_T (-1)^{des(T)},$$

where the sum is over all standard Young tableaux T of shape λ.

In [7], the value of $\mathrm{CD}(\lambda)$ is computed explicitly for $k \times n$ rectangles λ for $k = 2, 3, 4$. Strikingly, we find that $|\mathrm{CD}(n, n)| = \mathrm{Chess}(n, n)$ and $|\mathrm{CD}(n, n, n)| = \mathrm{Chess}(n, n, n)$. This cries out for a combinatorial explanation, but we do not have one. If we compare $\mathrm{CD}(\lambda)$ and $\mathrm{Chess}(\lambda)$ for other partitions λ with a small number of parts, then there are not many other numerical coincidences, so a result like Theorem 1 does not seem likely. In particular, $|\mathrm{CD}(5, 5, 5, 5)| = 580$ while $\mathrm{Chess}(5, 5, 5, 5) = 324$, and $|\mathrm{CD}(5, 5, 5, 5, 5)| = 25100$ while $\mathrm{Chess}(5, 5, 5, 5, 5) = 8716$.

As if three different relationships between $3 \times n$ tableaux and Baxter permutations were not enough, Richard Stanley helped us observe that the summand in equation (2), namely

$$\binom{n}{k}\binom{n}{k+1}\binom{n}{k+2}\Big/\binom{n}{0}\binom{n}{1}\binom{n}{2}, \tag{11}$$

is the number of $3 \times k$ semistandard Young tableaux with entries between 1 and $n-k+1$ inclusive. Dulucq and Guibert show that (11) is the number of $3 \times n$ nonconsecutive tableaux with k entries i such that i is in row 3 and $i+1$ is in row 1. In terms of balanced chess tableaux, the parameter k counts the number of balanced subtableaux T_i falling into the first two of the four possible cases in the proof of Theorem 1. However, we have not found a direct bijection with semistandard tableaux.

More generally, the number of Baxter permutations shows up in other places in mathematics, e.g., [1], [2], and [3]. Are there bijections with chess tableaux?

Finally, an intriguing observation for which we have no explanation is that the quantity $\sum_{\lambda \vdash n} \mathrm{Chess}(\lambda)^2$, where the sum ranges over all partitions of n, appears to be divisible by high powers of two. Specifically, the sequence begins

$$\begin{gathered} 1, \quad 2, \quad 2, \quad 2^2, \quad 2^3, \quad 2^4, \quad 2^4 \cdot 3, \quad 2^5 \cdot 5, \quad 2^6 \cdot 7, \quad 2^{11}, \quad 2^8 \cdot 5^2, \\ 2^9 \cdot 61, \quad 2^{10} \cdot 3 \cdot 41, \quad 2^{11} \cdot 5 \cdot 59, \quad 2^{11} \cdot 1523, \quad 2^{13} \cdot 23 \cdot 83, \\ 2^{13} \cdot 11411, \quad 2^{15} \cdot 103 \cdot 163, \quad \ldots \end{gathered}$$

Perhaps the Robinson-Schensted-Knuth correspondence can be used to help explain this phenomenon.

6 Acknowledgments

The first author thanks Joost de Heer for helping him learn to use the computer program Popeye to test his chess problem for soundness. We also thank Noam Elkies for typesetting the chess diagram, and Jim Propp for drawing our attention to [3].

References

[1] E. Ackerman, G. Barequet and R. Pinter, On the number of rectangular partitions, *SODA '04* (Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms), 2004.

[2] M. Bousquet-Mélou, Four classes of pattern-avoiding permutations under one roof: Generating trees with two labels, *Electronic J. Combin.* **9(2)** (2002–2003), #R19.

[3] H. Canary, Aztec diamonds and Baxter permutations, `math.CO/0309135`.

[4] S. Dulucq and O. Guibert, Stack words, standard tableaux and Baxter permutations, *Disc. Math.* **157** (1996), 91–106.

[5] H. Eriksson, Rat races and the hook length formula, *FPSAC '93* (Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics), 1993.

[6] A. Puusa, Queue problems, *Suomen Tehtäväniekat* (Finnish Chess Problem Society), 1992.

[7] V. Reiner, D. Stanton, and V. Welker, The Charney-Davis quantity for certain graded posets, *Sém. Lothar. Combin.* **50** (2003–2004), B50c.

[8] J. Sjöstrand, On the imbalance of partition shapes, `math.CO/0309231`.

[9] R. Stanley, Queue problems revisited, *Suomen Tehtäväniekat* (Finnish Chess Problem Society), to appear.

[10] A. Tefera, MultInt, a Maple Package for Multiple Integration by the WZ Method, *J. Symbolic Comp.* **34** (2002), 329–353.