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Prime Ideal Theorems and systems of finite character

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Dedicated to Bernhard Banaschewski on the occasion of his 70th birthday.

Abstract. We study several choice principles for systems of finite character and prove their equivalence to the Prime Ideal Theorem in ZF set theory without Axiom of Choice, among them the Intersection Lemma (stating that if $\mathscr{S}$ is a system of finite character then so is the system of all collections of finite subsets of $\bigcup \mathscr{S}$ meeting a common member of $\mathscr{S}$), the Finite Cutset Lemma (a finitary version of the Teichmüller-Tukey Lemma), and various compactness theorems. Several implications between these statements remain valid in ZF even if the underlying set is fixed. Some fundamental algebraic and order-theoretical facts like the Artin-Schreier Theorem on the orderability of real fields, the Erdös-De Bruijn Theorem on the colorability of infinite graphs, and Dilworth's Theorem on chain-decompositions for posets of finite width, are easy consequences of the Intersection Lemma or of the Finite Cutset Lemma.

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1. A wealth of Prime Ideal Theorems

Throughout this note, the logical framework is Zermelo-Fraenkel set theory (ZF) without Axiom of Choice (AC). Without particular emphasis, we shall make frequent use of the fact that the Axiom of Choice for finite families of nonempty sets is provable in ZF.

In the early fifties, Scott and Tarski have initiated the study of principles equivalent to the Ultrafilter Principle (UP), which postulates for any set-theoretical filter $\mathscr{F}$ an ultrafilter containing $\mathscr{F}$. Various forms of the Prime Ideal Theorem (PIT) for rings, distributive lattices, Boolean algebras and other structures are known to be equivalent to the Ultrafilter Principle (see, for example, [3], [38], [42], [43]) but strictly weaker than the Axiom of Choice in ZF or NBG set theory, as was demonstrated by Halpern and Lévy [21], [22]. While the weak forms of the Prime Ideal Theorem merely state the existence of at least one prime ideal in all nontrivial algebras of the given variety, the strong forms postulate the possibility to extend ideals disjoint from a given multiplicative subsemigroup (respectively, filter) to prime ideals with the same disjointness property. The breakthrough in the development of rather general variants of PIT, applicable to quite diverse situations in various mathematical areas and, in particular, to the case of non-commutative (semi)rings, was Banaschewski's observation [4] that UP is equivalent to the

Prime Element Theorem. *Every nontrivial distributive complete lattice with a compact top element contains a prime element.*

A rather comprehensive prime ideal theorem concerns arbitrary groupoids, i.e. algebras with one binary operation. By an (*associating*) *ideal* of a groupoid G, we mean a subset A satisfying the following two rules:

(I) $a \in A$ and $b \in G \Rightarrow ab \in A$ and $ba \in A$,

(A) $(a(bc))d \in A \Leftrightarrow ((ab)c)d \in A$,

where d may be any member of G but also an adjoined neutral element, so that (A) includes the association rule

(A_3) $a(bc) \in A \Leftrightarrow (ab)c \in A$,

which is formally simpler than (A) but too weak for the applications we have in mind (see [14] for details). On account of (A_3), it is unambiguous to write $abc \in A$ for $a(bc) \in A$. A *prime ideal* is a proper ideal P such that for any two ideals A, B of G,

$$AB \subseteq P \text{ implies } A \subseteq P \text{ or } B \subseteq P \text{ (where } AB = \{ab : a \in A, b \in B\}).$$

By a *distributive ideal system* on G, we mean an algebraic (= inductive) closure system $\mathscr{I}$, consisting of certain ideals of G and enjoying the distributive laws

$$A \cdot (B \vee C) = A \cdot B \ \vee \ A \cdot C \quad \text{and} \quad (B \vee C) \cdot A = B \cdot A \ \vee \ C \cdot A,$$

where $A \cdot B$ denotes the closure of AB and $B \vee C$ the closure of $B \cup C$ with respect to $\mathscr{I}$ (for related, but more restricted considerations on so-called x-systems, see Aubert [2]). It is easy to see that any distributive ideal system is a quantale; hence, one may invoke, as an intermediate step, the Separation Lemma for quantales (see [6]) in order to derive the following theorem from the Ultrafilter Principle:

Prime Ideal Theorem for groupoids. *Let $\mathscr{I}$ be a distributive ideal system and S a nonempty multiplicatively closed subset of some groupoid G. Then any member of $\mathscr{I}$ disjoint from S may be extended to a prime ideal belonging to $\mathscr{I}$ and still disjoint from S.*

The previous theorem encompasses many prime ideal theorems for bi-algebras, i.e. algebras of type $(2,2)$ like rings or lattices. Let G be such a bi-algebra with operations $+$ ("addition") and $\cdot$ ("multiplication"). As above, we use the shorthand notation ab for $a \cdot b$. A subset A of G is called a *distributing ideal* if it is an (associating) ideal A of the groupoid $(G, \cdot)$ satisfying the rule

(D) $dac \in A$ and $dbc \in A \Rightarrow d(a+b)c \in A$,

where c and d may also stand for an adjoined neutral element. Hence (D) implies that distributing ideals are additively closed. Of course, if the addition distributes over the multiplication (as in semirings) then every additively closed ideal is automatically distributing. In case of a commutative multiplication, (D) may be simplified to the implication

(D') $ac \in A$ and $bc \in A \Rightarrow (a+b)c \in A$.

In particular, the semiprime lattice ideals in the sense of Rav [38] are just the distributing ideals of the corresponding bi-algebra $(L, \vee, \wedge)$.

It is now a challenging exercise to show that the distributing ideals of any bi-algebra form a distributive ideal system. From this fact, one concludes that the following statement is a consequence of the Prime Ideal Theorem for groupoids:

Prime Ideal Theorem for bi-algebras. *If a distributing ideal A of a bi-algebra G does not meet a given nonempty multiplicatively closed subset S of G then there is a prime ideal extending A and still disjoint from S.*

Summarizing the previous implications, we arrive at

Proposition 1. *The Ultrafilter Principle is equivalent to the Prime Ideal Theorem for any class of bi-algebras containing all Boolean algebras or all Boolean rings.*

The equivalence of PIT with a still more general Prime Ideal Theorem for arbitrary (universal) algebras with at least one binary operation, including the Prime Element Theorem and the Separation Lemma as specific cases, has been established in [14].

It is a commonly observed phenomenon that maximal principles equivalent to AC turn into equivalents to PIT when maximality is replaced with suitable notions of primeness. For example, the maximal principle for distributive lattices, postulating the existence of coatoms in distributive complete lattices with compact top elements, is equivalent to AC (see Klimovsky [29]) but becomes the Prime Element Theorem when "prime" is substituted for "maximal" (here synonymous with "coatom"). However, there are some classical maximal principles like the Teichmüller-Tukey Lemma (TTL) where it is not evident how to weaken the notion of maximality in order to obtain an equivalent to PIT. Surprisingly, we shall see in Section 4 that a certain *finitary* version of TTL, the so-called *Finite Cutset Lemma*, will do the job, whereas the Axiom of Choice for families of finite sets (ACF) is strictly weaker than PIT in ZF (see [26]; the equivalence ACF $\Leftrightarrow$ PIT claimed in [8] fails in ZF). Moreover, we shall show that, like the aforementioned finitary weakening of TTL, various statements concerning *systems of finite character* are equivalent to PIT and have nice applications in algebra, topology, graph theory and order theory. One of the most simple and efficient one among these principles is what we shall call the *Intersection Lemma*. We shall prove its equivalence to the Finite Cutset Lemma, to Alexander's Subbase Lemma, and to various other important theorems from topology and set theory. Our emphasis will be on "local" implications that can be derived for a *fixed* underlying set X in the framework of ZF set theory. In other words, we shall prove statements of the form $\forall X (p(X) \Leftrightarrow q(X))$ rather than the weaker equivalence $\forall X\, p(X) \Leftrightarrow \forall X\, q(X)$. For example, $p(X)$ might stand for "*every Boolean algebra with a generating set indexed by X contains a prime ideal*" and $q(X)$ for "2^X *is a compact space*" (see Section 2 for details).

2. Local equivalents of the Prime Ideal Theorem

Various combinatorial selection lemmas (due to Rado, Engeler, Robinson, Rav and others) have been shown to be equivalent to the Prime Ideal Theorem. For a comprehensive study of the interrelations between these choice principles, we refer to Rav [37]. Perhaps the most famous one among these principles is Engeler's Selection Lemma for partial valuations [11]. While the global equivalence of this lemma to the principles stated in Proposition 2 below is known, the proof of their local equivalence for a fixed set X requires some additional care.

As usual, ω denotes the set of all natural numbers, and each natural number n is regarded as the set of all smaller numbers, i.e. $n = \{k \in \omega : k < n\}$; in particular, we have $2 = \{0, 1\}$. Furthermore, we put $nX = X \times n$ and denote by $\mathscr{P}_\omega(S)$ the collection of all finite subsets of a given set S. It will be convenient to write $E \Subset S$ for $E \in \mathscr{P}_\omega(S)$.

A subset $\mathscr{S}$ of a power set $\mathscr{P}(X)$ is referred to as a *system on* X, and $\mathscr{S}$ is said to be a *system of finite character* if $S \in \mathscr{S}$ is equivalent to $\mathscr{P}_\omega(S) \subseteq \mathscr{S}$. Compactness of a set C with respect to a collection $\mathscr{H}$ of sets may be expressed by saying that the system of all subsets of $\mathscr{H}$ whose union does not contain C is of finite character (notice that throughout this note, compactness is *not* assumed to include the Hausdorff separation property). A further remark on compactness will be opportune: the finitary version of Tychonoff's Theorem, claiming the compactness of a product of *finitely many* compact spaces, may be established in ZF without AC (or PIT), whereas the theorem in its full generality (for an arbitrary number of factors) is known to be equivalent to the Axiom of Choice (see Kelley [28]), and for Hausdorff spaces, it is equivalent to the Prime Ideal Theorem (Rubin and Scott [40], Los and Ryll-Nardzewski [32], [33]).

At the first glance, it might be tempting to guess that the Prime Ideal Theorem for Boolean algebras generated by a given set X should be an immediate consequence of the Prime Ideal Theorem for distributive lattices generated by the same set X. But a Boolean algebra generated by X (via joins, meets and complementation) need not be generated by X as a distributive lattice, i.e. there may be smaller distributive lattices containing X. However, one can prove:

Proposition 2. *The following statements on a set X are equivalent:*

(a) *The Prime Ideal Theorem for bounded distributive lattices generated by $2X$.*
(b) *The Prime Ideal Theorem for Boolean algebras freely generated by X.*
(c) *The Prime Ideal Theorem for Boolean algebras generated by X.*
(d) *Tychonoff's Theorem for X-fold powers of two-element spaces: 2^X is compact.*
(e) *Engeler's Lemma for partial valuations on X: If $\mathscr{S} \subseteq \bigcup\{2^Y : Y \subseteq X\}$ is a system of finite character then so is the system of all domains of functions belonging to $\mathscr{S}$.*

Moreover, if one of these statements holds for X then also for all sets Y equipollent to any subset of nX for some positive integer n. Furthermore, in (d) *and* (e), *2 may be replaced with any finite set having at least two elements.*

Proof: (a) $\Rightarrow$ (b): Let B be a Boolean algebra freely generated by X, denote the complement of $a \in B$ by $\neg a$, and put

$$X^+ = X \cup \{\neg x : x \in X\}.$$

Then X^+ is equipollent to $2X$ and generates B as a bounded distributive lattice.
(b) $\Rightarrow$ (c): Let B be a Boolean algebra freely generated by X (the existence of such a free algebra is easily established in ZF without any choice principles; see the first remark after the proof of Proposition 2). If A is an arbitrary Boolean algebra generated by a set Y then any surjection from X onto Y extends to a homomorphism φ from B onto A. Given an ideal I and a filter F of A with $I \cap F = \emptyset$, we obtain an ideal $\varphi^{-1}[I]$ of B and a filter $\varphi^{-1}[F]$ of B not intersecting $\varphi^{-1}[I]$. Now any prime ideal P of B containing $\varphi^{-1}[I]$ and disjoint from $\varphi^{-1}[F]$ gives rise to a prime ideal $Q = \varphi[P]$ of A with $I \subseteq Q$ and $Q \cap F = \emptyset$ (use surjectivity of φ and maximality of P). Hence, PIT holds for any Boolean algebra with a generating set Y indexed by X.
(c) $\Rightarrow$ (d): Consider the Boolean set algebra $\mathscr{B} \subseteq \mathscr{P}(2^X)$ generated by

$$\mathscr{H} = \{\pi_x^{-1}(1) : x \in X\},$$

where π_x is the x-th projection from 2^X onto 2, and consequently

$$\pi_x^{-1}(1) = \{\varphi \in 2^X : \varphi(x) = 1\}.$$

Obviously, $\mathscr{H}$ is equipollent to X, so PIT holds for the algebra $\mathscr{B}$, which consists of all finite unions formed by finite intersections of sets in $\mathscr{H}$ and their complements; hence $\mathscr{B}$ is a clopen base for the product topology on 2^X, and it suffices to prove compactness of 2^X in $\mathscr{B}$ (the passage from topologies to *bases* and vice versa does not require any choice principle, in contrast with Alexander's Subbase Lemma; see Section 3).

Assume now that $\mathscr{A}$ is a subset of $\mathscr{B}$ with $\bigcup \mathscr{E} \neq 2^X$ for all $\mathscr{E} \Subset \mathscr{A}$. Since the ideal generated by $\mathscr{A}$ in $\mathscr{B}$ is proper, there exists a prime ideal $\mathscr{P}$ of $\mathscr{B}$ with $\mathscr{A} \subseteq \mathscr{P}$. For each $x \in X$, exactly one of the complementary sets $\pi_x^{-1}(0)$ and $\pi_x^{-1}(1)$ belongs to $\mathscr{P}$. Hence there is a unique $\varphi \in 2^X$ such that $\pi_x^{-1}(\varphi(x)) \notin \mathscr{P}$ for all $x \in X$. We show that φ does not belong to any $A \in \mathscr{A}$ (and so $\bigcup \mathscr{A} \neq 2^X$, proving the compactness claim). If, on the contrary, $\varphi \in A$ for some $A \in \mathscr{A}$ then A contains a basic neighborhood U of φ which is an intersection of finitely many sets of the form $\pi_x^{-1}(\varphi(x))$. But none of these sets is a member of the prime ideal $\mathscr{P}$, so their intersection U cannot be in $\mathscr{P}$ either. This contradicts the hypothesis $U \subseteq A \in \mathscr{A} \subseteq \mathscr{P}$.

Next, we observe that (d) implies the stronger statement

(d$_n$) $\quad n^Y$ is compact for all $n \in \omega$ and all subsets Y of X.

Indeed, if 2^X is a compact space then, by an earlier remark, so are $(2^X)^k$ and the homeomorphic copies 2^{kX} and $(2^k)^X$, as well as their closed subspaces n^X ($k \in \omega$, $n \in \omega$, $n \leq 2^k$). Since for any subset Y of X, the space n^Y is homeomorphic to a closed subspace of n^X, it follows that n^Y is compact, too.

Now we show that (d_n) implies the assertion (e_n) obtained from (e) by replacing 2 with n.

Suppose $\mathscr{S} \subseteq \{n^Y : Y \subseteq X\}$ is a system of finite character and S is a subset of X such that each finite subset of S is the domain of some function $\psi \in \mathscr{S}$. We will show that S is the domain of a member of $\mathscr{S}$, too. For $x \in S$, the x-th projection from n^S onto n is denoted by π_x. For $E \Subset S$, the set

$$\mathscr{S}_E = \{\varphi \in n^S : \varphi\,|_E \in \mathscr{S}\} = \bigcup\{\{\varphi \in n^S : \varphi\,|_E = \psi\} : \psi \in \mathscr{S} \cap n^E\}$$

is clopen in the product space n^S, being a finite union of basic clopen sets

$$\{\varphi \in n^S : \varphi\,|_E = \psi\} = \bigcap\{\pi_x^{-1}(\psi(x)) : x \in E\} \quad (\psi \in \mathscr{S} \cap n^E).$$

For $\mathscr{E} \Subset \mathscr{P}_\omega(S)$, there exists a $\psi \in \mathscr{S}$ with domain $\bigcup \mathscr{E}$, hence $\psi \in \bigcap\{\mathscr{S}_E : E \in \mathscr{E}\} \neq \emptyset$. Now, by compactness of the power space n^S there is a function $\varphi \in \bigcap\{\mathscr{S}_E : E \Subset S\}$. Then $\varphi|_E$ belongs to $\mathscr{S}$ for each $E \Subset S$, and since $\mathscr{S}$ is of finite character, it follows that $\varphi \in \mathscr{S}$.

(e) $\Rightarrow$ (c): Suppose B is a Boolean algebra generated by X; let I be an ideal of B and F a filter of B disjoint from I. For each finite $E \subseteq X$, the subalgebra $\langle E \rangle$ generated by E is still finite, and consequently, there is a homomorphism φ from $\langle E \rangle$ onto 2, mapping the ideal $I \cap \langle E \rangle$ onto 0 and the filter $F \cap \langle E \rangle$ onto 1. Consider the system $\mathscr{S}$ of all maps from subsets Y of X into 2 admitting a (unique!) extension to a homomorphism on the subalgebra $\langle Y \rangle$, mapping $I \cap \langle Y \rangle$ onto 0 and $F \cap \langle Y \rangle$ onto 1. It is easy to see that $\mathscr{S}$ is a system of finite character, and by the previous consideration, every finite subset of X is the domain of some member of $\mathscr{S}$. Hence, by Engeler's Lemma, some map φ in $\mathscr{S}$ has domain X and extends, therefore, to a homomorphism on the whole algebra B. The kernel of this homomorphism is a prime ideal containing I and disjoint from F.

Similarly, one shows that Engeler's Lemma for X implies the Prime Ideal Theorem for bounded distributive lattices generated by X, and consequently, that (e_4) implies (a). Thus we have closed the implication circle:

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow \text{(e)} \Rightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow (\mathrm{d}_4) \Rightarrow (\mathrm{e}_4) \Rightarrow \text{(a)}.$$

$\square$

A few remarks are in order.

(1) The Boolean set algebra $\mathscr{B}$ in the proof of (c) $\Rightarrow$ (d) is freely generated by the set $\mathscr{H}$ (which corresponds to the set of fixed ultrafilters on X via characteristic functions).

(2) The usual argument for the compactness of 2^X invokes ultrafilters on this space — in other words, prime ideals of 2^{2^X} instead of a Boolean algebra generated by X.

(3) It can be shown without using any choice principle that every bounded distributive lattice L is a sublattice of some Boolean algebra B. Hence, if I is an ideal of L and F is a filter of L with $I \cap F = \emptyset$ then the ideal $\downarrow I$ and the filter $\uparrow F$ generated by I and F, respectively, in B are still disjoint. Hence there is a prime ideal P of B including $\downarrow I$ and not intersecting $\uparrow F$, and then $L \cap P$ is a prime ideal of L with the corresponding properties. This argument provides a direct deduction of PIT for bounded distributive lattices from PIT for Boolean algebras generated by the same set.

(4) As was shown by Rav [37], Engeler's Lemma for X implies the Prime Ideal Theorem for rings whose underlying set is equipollent to X. However, it is not clear whether the same conclusions are possible for rings with a *generating subset* equipollent to X.

(5) In the above proof, we have always referred to the Prime Ideal Theorem in its strong "extension" form. Globally, it is clear and well-known that the strong versions follow from the weak ones (requiring merely the existence of prime ideals in nontrivial bounded distributive lattices or Boolean algebras), by factorizing through suitable ideals and filters. Moreover, the quotient homomorphism sends generators to generators. But it is not clear whether the weak form of PIT for a generating set X entails the strong PIT for X (and for all sets indexed by X). The crucial obstacle is that the image of a prime ideal under an epimorphism between Boolean algebras need not be proper (while all other properties of a prime ideal are transferred). It is certainly not enough to postulate the existence of prime ideals in free Boolean algebras; indeed, some of these prime ideals are easily determined by explicit construction: for the "standard" Boolean set algebra $\mathscr{B}$ freely generated by the collection $\mathscr{H} = \{\dot{x} : x \in X\}$ of all fixed ultrafilters on X, one may pick any subset Y of X to obtain a prime ideal $\mathscr{P} = \{\mathscr{Z} \in \mathscr{B} : Y \notin \mathscr{Z}\}$. However, the previous reasonings provide the following

Corollary. *Both the weak and the strong Prime Ideal Theorem for Boolean algebras with generators indexed by X are equivalent to the statements in Proposition* 2.

3. Alexander's Subbase Lemma and the Intersection Lemma

It is well-known (see Parovičenko [36], Rubin and Scott [40]) that the Prime Ideal Theorem is globally equivalent to

Alexander's Subbase Lemma. *If a set C is compact in a subbase $\mathscr{H} = \{S_x : x \in X\}$ of a topology $\mathscr{T}$ then C is compact in $\mathscr{T}$, too.*

We are now going to establish the equivalence between Alexander's Subbase Lemma and the following useful principle concerning systems of finite character:

Intersection Lemma. *If a system $\mathscr{S} \subseteq \mathscr{P}(X)$ is of finite character then so is the system of all collections of finite subsets of X intersecting a common member of $\mathscr{S}$.*

The *section operator*, known from convergence theory and lattice theory (cf. Gähler [18]), associates with any set $\mathscr{A}$ the collection $\mathscr{A}^{\#}$ of all subsets of the union $\bigcup \mathscr{A}$ which meet each member of $\mathscr{A}$. In terms of this operator, the Intersection Lemma reads as follows:

If $\mathscr{S}$ is a system of finite character on X and $\mathscr{A}$ is a subset of $\mathscr{P}_\omega(X)$ satisfying $\mathscr{E}^{\#} \cap \mathscr{S} \neq \emptyset$ for all $\mathscr{E} \Subset \mathscr{A}$, then $\mathscr{A}^{\#} \cap \mathscr{S} \neq \emptyset$.

Proposition 3. *For any fixed set X, the Intersection Lemma is equivalent to Alexander's Subbase Lemma.*

Proof: It is routine (though a bit tedious) to check that if the Intersection Lemma holds for X then also for any set indexed by X. In order to deduce Alexander's Lemma for a given subbase $\mathscr{H} = \{S_x : x \in X\}$, observe first that the system

$$\mathscr{S} = \{\mathscr{Y} \subseteq \mathscr{H} : C \nsubseteq \bigcup \mathscr{Y}\}$$

is of finite character provided C is compact in $\mathscr{H}$. Given any subset $\mathscr{U}$ of $\mathscr{T}$ with $C \nsubseteq \bigcup \mathscr{F}$ for all $\mathscr{F} \Subset \mathscr{U}$, we have to show that $C \nsubseteq \bigcup \mathscr{U}$. For this, consider the system

$$\mathscr{A} = \{\mathscr{F} \Subset \mathscr{H} : \bigcap \mathscr{F} \subseteq U \ \text{ for some } \ U \in \mathscr{U}\}.$$

It is not hard to verify the required hypothesis $\mathscr{E}^{\#} \cap \mathscr{S} \neq \emptyset$ for all finite $\mathscr{E} \subseteq \mathscr{A}$. (Indeed, for each $\mathscr{F} \in \mathscr{E}$, find some $U_{\mathscr{F}} \in \mathscr{U}$ such that $\bigcap \mathscr{F} \subseteq U_{\mathscr{F}}$. Then $\mathscr{G} = \{U_{\mathscr{F}} : \mathscr{F} \in \mathscr{E}\}$ is a finite subset of $\mathscr{U}$, hence $C \nsubseteq \bigcup \mathscr{G}$, *a fortiori* $C \nsubseteq \bigcup\{\bigcap \mathscr{F} : \mathscr{F} \in \mathscr{E}\}$. Choose $x \in \bigcap\{C \setminus \bigcap \mathscr{F} : \mathscr{F} \in \mathscr{E}\}$ and $F(x,\mathscr{F}) \in \mathscr{F}$ with $x \notin F(x,\mathscr{F})$. Then $\{F(x,\mathscr{F}) : \mathscr{F} \in \mathscr{E}\}$ is a member of $\mathscr{E}^{\#} \cap \mathscr{S}$.)

Now, the Intersection Lemma yields a member $\mathscr{Y}$ of $\mathscr{S}$ with $\mathscr{Y} \cap \mathscr{F} \neq \emptyset$ for all $\mathscr{F} \in \mathscr{A}$, and a straightforward computation, using the subbase property of $\mathscr{H}$, gives $\bigcup \mathscr{U} \subseteq \bigcup \mathscr{Y}$, hence $C \nsubseteq \bigcup \mathscr{U}$. (See [12] for a more general order-theoretical subbase lemma equivalent to PIT.)

For the converse implication, let $\mathscr{S} \subseteq \mathscr{P}(X)$ be any system of finite character. The sets

$$\mathscr{S}_x = \{S \in \mathscr{S} : x \notin S\} \quad (x \in X)$$

form a subbase $\mathscr{H}$ of a topology $\mathscr{T}$ on $\mathscr{S}$, and $\mathscr{S}$ is compact in $\mathscr{H}$ since $\mathscr{S} \neq \bigcup\{\mathscr{S}_x : x \in Y\}$ is equivalent to $Y \in \mathscr{S}$. For each $E \in \mathscr{P}_\omega(X)$, the set

$$\mathscr{S}_E = \{S \in \mathscr{S} : E \cap S = \emptyset\} = \bigcap\{\mathscr{S}_x : x \in E\}$$

is a member of $\mathscr{T}$, and for $\mathscr{A} \subseteq \mathscr{P}_\omega(X)$, the inequality $\mathscr{S} \neq \bigcup\{\mathscr{S}_E : E \in \mathscr{A}\}$ is equivalent to $\mathscr{A}^\# \cap \mathscr{S} \neq \emptyset$. Hence, by compactness of $\mathscr{S}$ in $\mathscr{T}$, the system $\{\mathscr{A} \subseteq \mathscr{P}_\omega(X) : \mathscr{A}^\# \cap \mathscr{S} \neq \emptyset\}$ has finite character. $\square$

4. The Finite Cutset Lemma: A Finitary Version of the Teichmüller-Tukey Lemma

Recall that the Axiom of Choice is equivalent to various maximal principles (see [26], [34] or [39]), among them the following principle pointed out independently by Teichmüller [44] and Tukey [45]:

Teichmüller-Tukey Lemma (TTL)**.** *Each member of a system of finite character on a set X is contained in a maximal one.*

The following definition is motivated by the usual notion of cutsets in graphs and ordered sets but avoids the term "*maximal*". Given a set X and a system $\mathscr{S}$ of subsets of X, we mean by a *cutset* for $\mathscr{S}$ a set C such that each member of $\mathscr{S}$ may be extended to one that intersects C. In case of a system of finite character, we may characterize the cutsets of $\mathscr{S}$ by the property that for each $S \in \mathscr{S}$, there is some element $x \in C$ such that $S \cup \{x\}$ is still in $\mathscr{S}$. The following remark will be crucial:

A member of a system $\mathscr{S}$ of sets is maximal in $\mathscr{S}$ (*with respect to inclusion*) *if and only if it intersects every cutset for $\mathscr{S}$.*

Indeed, it is clear that maximal members of $\mathscr{S}$ meet every cutset. Conversely, if S is a member of $\mathscr{S}$ but not maximal in $\mathscr{S}$, say $S \subset T \in \mathscr{S}$, then the complement C of S is disjoint from S but a cutset for $\mathscr{S}$: given $R \in \mathscr{S}$, we have either $R \subseteq T$ and $C \cap T \neq \emptyset$, or $R \nsubseteq S$ and $C \cap R \neq \emptyset$.

Accordingly, TTL for a fixed set X is equivalent to the

Cutset Lemma. *If $\mathscr{S}$ is a system of finite character on X then each member of $\mathscr{S}$ is contained in a member of $\mathscr{S}$ that intersects every cutset for $\mathscr{S}$.*

Similarly, it is clear that TTL implies the

Weak Cutset Lemma. *If $\mathscr{S}$ is a system of finite character on X then the cutsets for $\mathscr{S}$ are precisely those sets which meet every maximal member of $\mathscr{S}$.*

In particular, if such a system $\mathscr{S}$ would have no maximal members at all then the empty set would be a cutset for $\mathscr{S}$, which is impossible unless $\mathscr{S}$ is empty. In other words, the Weak Cutset Lemma entails the

Weak Teichmüller-Tukey Lemma. *Every nonempty system of finite character on X has a maximal member.*

The latter in turn entails the strong version of TTL: if S belongs to a system $\mathscr{S}$ of finite character on X then the system $\mathscr{T} = \{T \setminus S : S \subseteq T \in \mathscr{S}\} \subseteq \mathscr{P}(X)$ is again of finite character, and any maximal member M of $\mathscr{T}$ gives rise to a maximal member $M \cup S$ of $\mathscr{S}$.

In all, we see that *for a fixed underlying set X, both versions of the Teichmüller-Tukey Lemma are equivalent to both versions of the Cutset Lemma.*

Now, let us call a member of a system $\mathscr{S}$ of sets *almost maximal* if it intersects every *finite* cutset for $\mathscr{S}$ (in Johnstone [27], the term "*almost maximal*" has a different meaning).

Let us consider a few extremal examples.

(1) The system $\mathscr{S}$ of all chains of the open real unit square $Q =]0,1[^2$ (ordered componentwise) is of finite character but has no finite cutsets at all. Hence every chain of Q (even the empty one) is almost maximal in $\mathscr{S}$.

(2) If in an ordered set every cutset for the system $\mathscr{S}$ of all chains contains a finite cutset then every almost maximal chain is already maximal. However, a cutset in a poset of finite width need not contain any finite cutset; a counterexample is the "doubled chain" $\omega \times 2$, ordered by $(x_1, x_2) \sqsubset (y_1, y_2) \Leftrightarrow x_1 < y_1$, with the cutset $\omega \times 1$.

(3) An antichain which is a cutset must be maximal, but a maximal antichain need not be a cutset, and a minimal cutset need not be an antichain; in both cases, a counterexample is provided by the "zigzag" poset N, obtained from the product lattice 2×3 by deleting top and bottom.

The finitary version of TTL we are interested in may now be formulated as follows:

Finite Cutset Lemma. *Each member of a system of finite character on X is contained in an almost maximal one.*

Proposition 4. *For a fixed set X, the Intersection Lemma is equivalent to the Finite Cutset Lemma.*

Proof: First, let us derive the Finite Cutset Lemma from the Intersection Lemma. Given a system $\mathscr{S}$ of finite character and a fixed member R of $\mathscr{S}$, consider the system

$$\mathscr{A}_R = \{\{x\} : x \in R\} \cup \{E \Subset X : \text{for each } S \in \mathscr{S} \text{ there is an } x \in E \text{ with } S \cup \{x\} \in \mathscr{S}\}.$$

If $\mathscr{E}$ is a finite subset of $\mathscr{A}_R$ then, by finiteness of the union $\bigcup \mathscr{E}$, we may choose a maximal member S of the system $\mathscr{T} = \{T \in \mathscr{S} : R \subseteq T \subseteq R \cup \bigcup \mathscr{E}\}$. For each $E \in \mathscr{E}$, there is an $x \in E$ with $S \cup \{x\} \in \mathscr{S}$, and by maximality of S in $\mathscr{T}$, it follows that $x \in E \cap S \neq \emptyset$. Hence, the Intersection Lemma gives a set $S \in \mathscr{S}$ with $E \cap S \neq \emptyset$ for all $E \in \mathscr{A}_R$. In particular, S meets every finite cutset for $\mathscr{S}$.

Conversely, let $\mathscr{S}$ be a system of finite character on X obeying the Finite Cutset Lemma, and let $\mathscr{A}$ be any collection of finite subsets of X such that $\mathscr{E}^{\#} \cap \mathscr{S} \neq \emptyset$ for all $\mathscr{E} \Subset \mathscr{A}$. Consider the system $\mathscr{S}_{\mathscr{A}}$ of all subsets S of X such that for each $\mathscr{E} \Subset \mathscr{A}$, there is a $C \in \mathscr{E}^{\#}$ with $S \cup C \in \mathscr{S}$. Taking $\mathscr{E} = \emptyset$, we observe that $\mathscr{S}_{\mathscr{A}}$ is a subset of $\mathscr{S}$. By hypothesis, $\emptyset \in \mathscr{S}_{\mathscr{A}}$. Clearly, $T \subseteq S \in \mathscr{S}_{\mathscr{A}}$ implies

$T \in \mathscr{S}_{\mathscr{A}}$ (because $S \cup C \in \mathscr{S}$ entails $T \cup C \in \mathscr{S}$). Moreover, $\mathscr{S}_{\mathscr{A}}$ is a system of finite character because $\mathscr{S}$ is one: indeed, if $S \in \mathscr{P}(X) \setminus \mathscr{S}_{\mathscr{A}}$ then we may choose a finite set $\mathscr{E} \subseteq \mathscr{A}$ such that $S \cup C \notin \mathscr{S}$ for all $C \in \mathscr{E}^{\#}$, and then $F_C \cup C \notin \mathscr{S}$ for suitable $F_C \Subset S$ (here no choice principle is required because $\mathscr{E}^{\#}$ is finite). Putting $F = \bigcup\{F_C : C \in \mathscr{E}^{\#}\}$, we obtain a finite subset F of S with $F \cup C \notin \mathscr{S}$ for all $C \in \mathscr{E}^{\#}$, and consequently, $F \notin \mathscr{S}_{\mathscr{A}}$.

We claim that each member of $\mathscr{A}$ is a cutset for $\mathscr{S}_{\mathscr{A}}$. Assuming the contrary, we find some $E \in \mathscr{A}$ and some $S \in \mathscr{S}_{\mathscr{A}}$ such that $S \cup \{x\} \notin \mathscr{S}_{\mathscr{A}}$ for all $x \in E$. By definition of $\mathscr{S}_{\mathscr{A}}$, there are finite subsets $\mathscr{E}_x$ of $\mathscr{A}$ with $S \cup \{x\} \cup C \notin \mathscr{S}$ for all $C \in \mathscr{E}_x^{\#}$. Then the union $\mathscr{E} = \{E\} \cup \bigcup\{\mathscr{E}_x : x \in E\}$ is a finite subset of $\mathscr{A}$. Any $C \in \mathscr{E}^{\#}$ contains some $x \in E$, so that $\mathscr{E}_x \subseteq \mathscr{E}$ entails $C_x = C \cap \bigcup \mathscr{E}_x \in \mathscr{E}_x^{\#}$ and therefore $S \cup \{x\} \cup C_x \notin \mathscr{S}$. But since $S \cup \{x\} \cup C_x$ is contained in $S \cup C$, it would follow that $S \cup C \notin \mathscr{S}$, contradicting our hypothesis $S \in \mathscr{S}_{\mathscr{A}}$.

Now, the Finite Cutset Lemma provides an almost maximal member S of $\mathscr{S}_{\mathscr{A}}$. Thus S belongs to $\mathscr{S}$ and meets every finite cutset for $\mathscr{S}_{\mathscr{A}}$. In particular, $S \in \mathscr{A}^{\#} \cap \mathscr{S}$, as desired. □

5. The Primrose Lemma for polynomial rings

By a result due to Hodges [25], the Axiom of Choice is equivalent to the existence of maximal (proper) ideals in certain localizations of polynomial rings over a field F. In [13], this equivalence has been established for a fixed set X of indeterminates and an arbitrary but fixed field F (see also Banaschewski [5]). An appropriate tool for the investigation of connections between set-theoretical and ring-theoretical choice principles is the following. Let $R = F[X]$ denote the polynomial ring over the field F with X as set of indeterminates. Then every system $\mathscr{S}$ of finite character on X gives rise to a so-called *primrose*

$$P_{\mathscr{S}} = \bigcup\{RS : S \in \mathscr{S}\}$$

where RS denotes the ideal generated by $S \subseteq X$. As shown in [13], the ideals of that form are precisely the *conservative prime ideals*, where a subset of R is said to be conservative if it contains with any polynomial a all a-monomials, that is, all monomials occurring in the canonical sum representation of a. Moreover, the primroses are just the unions of arbitrary collections of conservative prime ideals.

We observe at once that the complement

$$U_{\mathscr{S}} = R \setminus P_{\mathscr{S}} = \{u \in R : \text{for all } S \in \mathscr{S}, \\ \text{there is a } u\text{-monomial with no factor in } S\}$$

is multiplicatively closed in R, and consequently, the localization

$$F_{\mathscr{S}}(X) = \{\frac{r}{u} : r \in R,\ u \in U_{\mathscr{S}}\}$$

is a subring of the quotient field $F(X)$. The aforementioned local strengthening of Hodges' result that "*Krull implies Zorn*" reads as follows:

> *Let $\mathscr{S} \subseteq \mathscr{P}(X)$ be a system of finite character. Then there is a one-to-one correspondence $S \mapsto RS$ between the members of $\mathscr{S}$ and the conservative prime ideals contained in the primrose $P_{\mathscr{S}}$. Under this bijection, the maximal members of $\mathscr{S}$ correspond to the maximal ideals contained in $P_{\mathscr{S}}$, hence to the maximal ideals of the localization $F_{\mathscr{S}}(X)$.*

As an immediate consequence, the Teichmüller-Tukey Lemma for X is equivalent to the existence of maximal ideals in the localization $F_{\mathscr{S}}(X)$.

Although the existence of conservative prime ideals in $F[X]$ is entirely constructive, it is not clear *a priori* whether the extension of arbitrary ideals to conservative prime ideals contained in a given primrose is equivalent to the Prime Ideal Theorem. As remarked in [13], the answer is in the affirmative, although neither sufficiency nor necessity is obvious. We shall prove this equivalence via the intermediate role of the Intersection Lemma:

Proposition 5. *Given a fixed set X and any field F, the Intersection Lemma is equivalent to the following Primrose Lemma: Every ideal of $F[X]$ contained in a primrose P of $F[X]$ extends to a conservative prime ideal contained in P.*

Proof: First, let us derive the Intersection Lemma from the Primrose Lemma. Suppose $\mathscr{S}$ is a system of finite character on X and $\mathscr{A}$ is a subset of $\mathscr{P}_\omega(X)$ such that $\mathscr{S}$ intersects $\mathscr{E}^\#$ for each $\mathscr{E} \Subset \mathscr{A}$. Put $R = F[X]$ and consider the ideal RA generated by the set A of all monomials obtained by forming the product of all indeterminates in some $E \in \mathscr{A}$. Each element of RA has a representation $p = r_1 m_1 + \cdots + r_n m_n$ with $r_j \in R$ and $m_j \in A$. By the hypotheses on $\mathscr{A}$ and $\mathscr{S}$, there exists an $S \in \mathscr{S}$ such that all m_j belong to RS, and consequently $p \in RS$. Hence RA is contained in the primrose $P_{\mathscr{S}}$, and the Primrose Lemma yields a conservative prime ideal RT with $RA \subseteq RT \subseteq P_{\mathscr{S}}$. It follows that each monomial in A contains at least one indeterminate from T as a factor. In other words, T intersects each member of $\mathscr{A}$.

Now to the converse implication. By Lemma 3 of [13], it suffices to consider a *conservative* ideal I contained in a primrose $P_{\mathscr{S}}$, and by Lemma 1 of [13], we have $I = RA$ for some set A of monomials (which are products of indeterminates, i.e. elements of X). Let $\mathscr{A}$ denote the collection of all sets $V(m)$ of indeterminates occurring in m, with m ranging over A. For finite $\mathscr{E} = \{V(m_1), \ldots, V(m_n)\} \subseteq \mathscr{A}$, the sum $m_1 + \cdots + m_n$ belongs to $RA \subseteq P_{\mathscr{S}}$, hence to at least one conservative prime ideal RS with $S \in \mathscr{S}$. As each m_j lies in RS, it is then clear that $V(m_j)$ intersects S, i.e. $\mathscr{E}^\# \cap \mathscr{S} \neq \emptyset$. Now, by the Intersection Lemma, we find a $T \in \mathscr{S}$ with $V(m) \cap T \neq \emptyset$ for all $m \in A$, and therefore RA is contained in $RT \subseteq P_{\mathscr{S}}$. □

As an immediate consequence of Propositions 3 and 5, we get:

Corollary. *The Primrose Lemma is equivalent to Alexander's Subbase Lemma and therefore to the Prime Ideal Theorem.*

6. Systems of finite character and compactness

In order to analyze the precise position of the Intersection Lemma compared with certain statements on compact spaces, one may relate it to compactness properties with respect to certain "intrinsic" topologies on power sets. Let us recall some of the basic definitions. For any (partially) ordered set P, the principal ideals $\downarrow b = \{a \in P : a \leq b\}$ $(b \in P)$ form a subbase for the closed sets in the *upper* or *weak* topology υ_P. The latter is always coarser than the *Scott topology* σ_P, which consists of all subsets U such that any directed subset D of P with join x meets U iff x is an element of U. The weak topology on the dual of P is referred to as the *lower topology* and denoted by ω_P. A subbase for the (*upper*) *Alexandroff topology* α_P is constituted by the principal dual ideals $\uparrow b = \{a \in P : b \leq a\}$, while their complements generate the lower topology. The join of the upper and the lower topology is the *interval topology* ι_P, while the join of the Scott topology and the lower topology is the Lawson topology λ_P (cf. [15], [19]).

$$\begin{tikzcd}[column sep=small]
\alpha_P \arrow[dr, dash] & & \lambda_P \arrow[dl, dash] \arrow[dr, dash] & & \\
& \sigma_P \arrow[dr, dash] & & \iota_P \arrow[dl, dash] \arrow[dr, dash] & \\
& & \upsilon_P & & \omega_P
\end{tikzcd}$$

It is not hard to see that compactness of a set S with respect to the join of two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on S is equivalent to compactness of the diagonal $\{(x,x) : x \in S\}$ in the product space $(S, \mathscr{T}_1) \times (S, \mathscr{T}_2)$. Provided $\mathscr{T}_1$ and $\mathscr{T}_2$ are compact Hausdorff topologies, the diagonal is closed and therefore compact in the product space. But unfortunately, that remark does not apply to upper and lower topologies, because they are never Hausdorff on nontrivial ordered sets. In fact, every ordered set with a least element is compact in its upper topology, and so any bounded lattice is compact in the upper and in the lower topology, but by Frink's Theorem [16], only *complete* lattices are compact in the interval topology (the join of the upper and the lower topology). Later on, we shall prove a local strengthening of the fact that compactness of complete lattices in the interval topology is equivalent to Alexander's Subbase Lemma and, consequently, to the Prime Ideal Theorem.

First, let us focus on the specific situation of a power set lattice $\mathscr{P}(X)$, ordered by inclusion. Here the system $\{\dot{x} : x \in X\}$ of all fixed ultrafilters on X is an *open* subbase for the upper topology and a *closed* subbase for the lower topology. Hence, passing to complements, we see that the system $\{\mathring{x} : x \in X\}$ with $\mathring{x} = \mathscr{P}(X \setminus \{x\})$ is a *closed* subbase for the upper topology and an *open* subbase for the lower topology.

A system $\mathscr{S}$ of sets is called *descending* if $S \in \mathscr{S}$ implies $\mathscr{P}(S) \subseteq \mathscr{S}$. In other words, the descending systems on a set X are just the closed sets in the Alexandroff topology on $\mathscr{P}(X)$. Of course, every system of finite character is descending.

Proposition 6. *Consider the following statements on a set X and a system $\mathscr{S}$ on X:*

(a) *$\mathscr{S}$ is descending and compact in the Lawson topology on $\mathscr{P}(X)$.*
(b) *$\mathscr{S}$ is descending and compact in the interval topology on $\mathscr{P}(X)$.*
(c) *$\mathscr{S}$ is descending and $\{\mathscr{A} \subseteq \mathscr{P}_\omega(X) : \mathscr{A}^\# \cap \mathscr{S} \neq \emptyset\}$ is a system of finite character.*
(d) *$\mathscr{S}$ is descending and compact in the lower topology on $\mathscr{P}(X)$.*
(e) *$\mathscr{S}$ is descending and compact in the subbase $\{\mathring{x} : x \in X\}$ for the lower topology.*
(f) *$\mathscr{S}$ is closed in the upper topology on $\mathscr{P}(X)$.*
(g) *$\mathscr{S}$ is closed in the Scott topology on $\mathscr{P}(X)$.*
(h) *$\mathscr{S}$ is a system of finite character.*

The implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) *are valid in ZF. The Intersection Lemma holds for X iff the last six statements are equivalent for all systems $\mathscr{S}$ on X, and the Intersection Lemma holds for $2X$ iff all eight statements are equivalent.*

Proof: The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (e) are clear by the above inclusion diagram for the involved topologies. For the equivalence (a) $\Leftrightarrow$ (b), use the equivalence (f) $\Leftrightarrow$ (g), which will be proved below and entails the coincidence of the Lawson topology with the interval topology on power set lattices.

(c) $\Leftrightarrow$ (d): $\mathscr{A}^\# \cap \mathscr{S} \neq \emptyset$ means $\bigcap\{\bigcup\{\dot{x} : x \in E\} : E \in \mathscr{A}\} \cap \mathscr{S} \neq \emptyset$ and the sets $\bigcup\{\dot{x} : x \in E\}$ with $E \Subset X$ form a basis for the closed sets in the lower topology on $\mathscr{P}(X)$. Hence $\mathscr{S}$ is compact in that topology iff for all $\mathscr{A} \subseteq \mathscr{P}_\omega(X)$ with $\mathscr{A}^\# \cap \mathscr{S} = \emptyset$, there is a finite subset $\mathscr{E}$ of $\mathscr{A}$ with $\mathscr{E}^\# \cap \mathscr{S} = \emptyset$.

(e) $\Leftrightarrow$ (h): For descending systems $\mathscr{S}$, the inclusion $\mathscr{S} \subseteq \bigcup\{\mathring{x} : x \in Y\}$ is equivalent to $Y \notin \mathscr{S}$.

(f) $\Rightarrow$ (g): The Scott topology is finer than the upper topology.

(g) $\Rightarrow$ (h): Clear by definition of the Scott topology.

(h) $\Rightarrow$ (f): If S is a subset of X but not a member of $\mathscr{S}$ then there is a finite subset E of S with $E \notin \mathscr{S}$, and as $\mathscr{S}$ is descending, it does not intersect the system $\{Y \subseteq X : E \subseteq Y\} = \bigcap\{\dot{x} : x \in E\}$, which is an open neighborhood of S in the upper topology.

The Intersection Lemma for X states that (h) implies (c) for all $\mathscr{S} \subseteq \mathscr{P}(X)$. Furthermore, from Proposition 5 we know that the Intersection Lemma for $2X$ entails compactness of the power 2^X in the product topology, which agrees with the interval topology and with the Lawson topology on 2^X. But 2^X is isomorphic to the power set lattice $\mathscr{P}(X)$, and consequently, every closed subset of the latter (in particular, every system of finite character on X) is compact in the Lawson topology. Hence, under the hypothesis of the Intersection Lemma for $2X$, (h) entails (a), and all eight statements are equivalent. $\square$

7. Further topological equivalents of the Intersection Lemma

Van Benthem [46] obtained the equivalence of PIT and of Tychonoff's Theorem to a certain set-theoretical principle similar to our Intersection Lemma. Below we prove a stronger local version of this equivalence. As usual, $\mathscr{D}\,|_S$ denotes the set $\{D \cap S : D \in \mathscr{D}\}$.

Proposition 7. *Among the following statements on a set X, each of the first five implies the next one:*

(a) *The Intersection Lemma for $2X$.*
(b) *Alexander's Subbase Lemma for $2X$.*
(c) *Frink's Theorem for X: Every complete lattice with a join-dense subset indexed by X and a meet-dense subset indexed by X is compact in the interval topology.*
(d) *Tychonoff's Theorem for X-fold powers of two-element spaces.*
(e) *Van Benthem's Lemma for X: For any set $\mathscr{C}$, the system of all subsets $\mathscr{D}$ of $\mathscr{P}_\omega(X)$ with $\mathscr{D}\,|_S \subseteq \mathscr{C}$ for at least one set S is of finite character.*
(f) *The Intersection Lemma for X.*

Furthermore, the first four statements are equivalent to Van Benthem's Lemma for $2X$.

Proof: The equivalence of (a) and (b) is clear by Proposition 3.

(b) $\Rightarrow$ (c): Let L be a complete lattice with a join-dense subset $J = \{j_x : x \in X\}$ and a meet-dense subset $M = \{m_x : x \in X\}$. Then $\{L \setminus {\uparrow} j_x : x \in X\} \cup \{L \setminus {\downarrow} m_x : x \in X\}$ is a subbase for the interval topology on L, and by completeness, L is compact in that subbase.

(c) $\Rightarrow$ (d): The characteristic functions δ_x with $\delta_x(y) = 1 \Leftrightarrow x = y$ $(x \in X)$ are join-dense in 2^X, and the characteristic functions $1 - \delta_x$ are meet-dense in 2^X.

(d) $\Rightarrow$ (e): 2^X is homeomorphic to the power set $\mathscr{P}(X)$, endowed with the topology generated by the clopen sets $\dot{x} = \{S \subseteq X : x \in S\}$ and $\mathring{x} = \{S \subseteq X : x \notin S\}$. Hence the sets

$$K(E, F) = \{S \subseteq X : F \cap S = E\} = \bigcap\{\mathring{x} : x \in E\} \cap \bigcap\{\dot{x} : x \in F \setminus E\} \quad (E \subseteq F \Subset X)$$

form a clopen base, and the sets

$$K(F) = \{S \subseteq X : F \cap S \in \mathscr{C}\} = \bigcup\{K(E, F) : E \in \mathscr{C} \cap \mathscr{P}(F)\}$$

are clopen, too. By compactness of $\mathscr{P}(X)$,

$$\bigcap\{K(F) : F \in \mathscr{F}\} \neq \emptyset \text{ for all } \mathscr{F} \Subset \mathscr{D} \text{ implies } \bigcap\{K(D) : D \in \mathscr{D}\} \neq \emptyset \quad (\mathscr{D} \subseteq \mathscr{P}_\omega(X)).$$

But this implication is just a reformulation of statement (e).

(e) $\Rightarrow$ (f): Let $\mathscr{S}$ be any system of finite character on X, and let $\mathscr{A}$ be a nonempty subset of $\mathscr{P}_\omega(X)$ such that for all finite $\mathscr{E} \subseteq \mathscr{A}$, there is an $S \in \mathscr{S}$ with $E \cap S \neq \emptyset$ for each $E \in \mathscr{E}$. Setting

$$\mathscr{D} = \{D \Subset X : A \subseteq D \ \text{ for some } \ A \in \mathscr{A}\} \ \text{ and } \ \mathscr{C} = \mathscr{S} \setminus \{\emptyset\},$$

we find for each finite $\mathscr{F} \subseteq \mathscr{D}$ a set $S \in \mathscr{S}$ with $F \cap S \neq \emptyset$, hence $F \cap S \in \mathscr{C}$ for all $F \in \mathscr{F}$. Now (e) yields an S with $D \cap S \in \mathscr{C}$ for all $D \in \mathscr{D}$. Given any finite set $E \subseteq S$ and $A \in \mathscr{A}$, we get $A \cup E \in \mathscr{D}$ and $E \subseteq (A \cup E) \cap S \in \mathscr{C} \subseteq \mathscr{S}$, hence $E \in \mathscr{S}$, and finally $S \in \mathscr{S}$.

(d) $\Rightarrow$ (a): As we have seen in Proposition 2, (d) entails Tychonoff's Theorem for 2^{2X}, and then the implications (d) $\Rightarrow$ (e) $\Rightarrow$ (f) for $2X$ instead of X give the desired conclusion. $\square$

A combination of the implications in Propositions 2 and 7 yields a deduction of Engeler's Lemma for X from the Intersection Lemma for $2X$. A direct proof of this implication is obtained by applying the Intersection Lemma to the system $\mathscr{A} = \{\{x\} \times 2 : x \in X\}$ and to the given system $\mathscr{S}$ of finite character, consisting of partial valuations on X.

From Proposition 7 we know that compactness of $2^X \cong \mathscr{P}(X)$ in the product topology implies the Intersection Lemma for X. Although we do not know whether the converse conclusion works in ZF (for fixed X), we can now say the following:

Corollary. *Tychonoff's Theorem for X-fold powers of* 2 *and the other statements in Proposition* 2 *are equivalent to the postulate that if a descending system on X is compact in the lower topology on $\mathscr{P}(X)$ then it is also compact in the interval topology.*

Indeed, suppose that 2^X is compact in the product topology, or equivalently, that $\mathscr{P}(X)$ is compact in the interval topology, and let $\mathscr{S}$ be a descending system on X that is compact in the lower topology. Then Proposition 7 yields the Intersection Lemma for X as well as for $2X$, and Proposition 6 gives compactness of $\mathscr{S}$ with respect to the interval topology.

As a consequence of Propositions 3, 4 and 7, the Finite Cutset Lemma for $2X$ entails the Prime Ideal Theorem for Boolean algebras B generated by X. This implication may be obtained more directly, by establishing a close connection between the almost maximal members of certain systems of finite character on X and maximal ideals in B. For any subset S of B, the set

$$I(S) = \{a \in B : a \leq \bigvee E \ \text{ for some } \ E \Subset S\}$$

is the ideal generated by S, that is, the least ideal of B containing S.

As before, we put $X^+ = X \cup \{\neg x : x \in X\}$. Now, given any subset F of B, the system

$$\mathscr{S}_F = \{S \subseteq X^+ : I(S) \cap F = \emptyset\}$$

turns out to be of finite character, on account of the equation

$$I(S) = \bigcup \{I(E) : E \Subset S\}.$$

Any ideal generated by a subset of X^+ will be called X-*basic*.

Lemma. *Let B be a Boolean algebra generated by a set X, and let F be any filter of B. Then the following three conditions on a set $S \in \mathscr{S}_F$ are equivalent:*

(a) *S is maximal in $\mathscr{S}_F$.*
(b) *S is almost maximal in $\mathscr{S}_F$.*
(c) *For each $x \in X$, either x or $\neg x$ belongs to S.*

Each of these conditions implies

(d) *S generates a maximal (= prime) ideal of B.*

If B is freely generated by X then all four statements are equivalent, and the assignment $S \mapsto I(S)$ yields a one-to-one correspondence between the (almost) maximal members of $\mathscr{S}_F$ and the maximal X-basic ideals disjoint from F.

Proof: (a) $\Rightarrow$ (b) $\Rightarrow$ (c): For each $x \in X$ and $S \in \mathscr{S}_F$, we have $S \cup \{x\} \in \mathscr{S}_F$ or $S \cup \{\neg x\} \in \mathscr{S}_F$, because otherwise there would exist elements $a \in (I(S) \vee {\downarrow} x) \cap F$ and $b \in (I(S) \vee {\downarrow} \neg x) \cap F$, so that $a \wedge b$ would belong to the intersection $(I(S) \vee {\downarrow} x) \cap (I(S) \vee {\downarrow} \neg x) \cap F = I(S) \cap F$, which is impossible. Hence, by definition, any almost maximal member S of $\mathscr{S}_F$ must contain x or $\neg x$.

The implication (c) $\Rightarrow$ (a) is clear since $x \vee \neg x \in I(S) \cap F$ for all $x \in S$.

(c) $\Rightarrow$ (d): In order to show that $P = I(S)$ is a maximal ideal, it suffices to observe that the set $C = \{a \in B : a \in P \text{ or } \neg a \in P\}$ is a subalgebra of B containing X, hence the whole algebra B: suppose $a \in C$ and $b \in C$; if $a \vee b \notin P$ then w.l.o.g. $a \notin P$ and so $\neg a \in P$, hence $\neg(a \vee b) = \neg a \wedge \neg b \in P$; if $\neg(a \wedge b) \notin P$ then $\neg a \vee \neg b \notin P$, so that, like before, $a \wedge b \in P$.

Now suppose that B is freely generated by X and that $I(S)$ is a maximal (proper!) ideal of B. Then for no $x \in X$, it can happen that both x and $\neg x$ are elements of S. But $x \in I(S) \cap X^+$ means $x \in X^+$ and $x \leq \bigvee E \vee \bigvee \{\neg y : y \in F\}$ for some finite disjoint subsets E and F of $S \cap X$, which is impossible unless x was already in $E \cup F$ (see Grätzer [20, Chapter 2, Theorem 4 and Exercise 43]). This proves the equation $I(S) \cap X^+ = S$.

If E is a finite cutset for $\mathscr{S}_F$ then $S \cup \{x\} \in \mathscr{S}_F$ for some $x \in E$. By maximality, $I(S)$ must coincide with $I(S \cup \{x\})$, and consequently $x \in I(S) \cap X^+ = S$. Hence E meets S, and S is almost maximal in $\mathscr{S}_F$. Together with the previous remarks, this establishes the claimed one-to-one correspondence between the almost maximal members of $\mathscr{S}_F$ and the maximal X-basic ideals of B. $\square$

Now, under the hypothesis of the Finite Cutset Lemma for $2X$ and consequently for X^+, we find for any ideal I of B and for any filter F disjoint from I an almost maximal member S of $\mathscr{S}_F$ with $I \cap X^+ \subseteq S$ (since the ideal generated by $I \cap X^+$ is disjoint from F, i.e. $I \cap X^+$ belongs to $\mathscr{S}_F$). It follows that $I(S)$ is a maximal ideal, and if I was X-basic, then $I \subseteq I(S) \subseteq B \setminus F$. In particular, this holds for the zero ideal $I(\emptyset) = \{0\}$ and proves the weak Prime Ideal Theorem for Boolean algebras generated by X. Passing from X to sets indexed by X (which does not affect the validity of the Finite Cutset Lemma) we see that the Finite Cutset Lemma for X entails the weak Prime Ideal Theorem for Boolean algebras with generating sets indexed by X and, consequently, the Prime Ideal Theorem for Boolean algebras generated by X (see Section 2).

The previous considerations also include the observation that the Finite Cutset Lemma for $2X$ entails the following "basic" Prime Ideal Theorem:

If B is a Boolean algebra generated by X then for each X-basic ideal I and each filter F disjoint from I, there is an X-basic prime ideal containing I and disjoint from F.

Let us summarize the various implications between statements on prime ideals and systems of finite character in a diagram:

$$
\begin{tikzcd}
\mathrm{D}2X \arrow[r, leftrightarrow] & \mathrm{B}2X \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{E}2X \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{T}2X \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{V}2X \arrow[r, leftrightarrow] & \mathrm{S}2X \arrow[r, leftrightarrow] & \mathrm{F}2X \arrow[r, leftrightarrow] & \mathrm{I}2X \arrow[r, leftrightarrow] & \mathrm{P}2X \\
 & \mathrm{B}X \arrow[r, leftrightarrow] \arrow[dl] & \mathrm{E}X \arrow[r, leftrightarrow] & \mathrm{T}X \arrow[dr] & & & & & \\
\mathrm{D}X & & & & \mathrm{V}X \arrow[dr] & & & & \\
 & & & & & \mathrm{S}X \arrow[r, leftrightarrow] & \mathrm{F}X \arrow[r, leftrightarrow] & \mathrm{I}X \arrow[r, leftrightarrow] & \mathrm{P}X
\end{tikzcd}
$$

BX Prime Ideal Theorem for (free) Boolean algebras generated by X
DX Prime Ideal Theorem for distributive lattices generated by X
EX Engeler's Lemma for partial valuations on X
FX Finite Cutset Lemma for X
IX Intersection Lemma for X
PX Primrose Lemma for X
SX Subbase Lemma for X
TX Tychonoff's Theorem for X-fold powers of 2
VX Van Benthem's Lemma for X

Corollary. *If X is equipollent to $2Y$ for some set Y then the above principles are all equivalent.*

With regard to the above implication diagram, it is worth mentioning that in ZF or NBG set theory, the equipollence of X to $2X$ for infinite X is not provable but strictly weaker than the Axiom of Choice, which is known to be equivalent to the equipollence of X to X^2 for all infinite sets X (see Halpern and Howard [23], Rubin and Rubin [39], Sageev [41]). However, since $2\omega = \omega$ trivially holds, we have:

Corollary. *The following statements are equivalent:*

(A) *The Axiom of Choice for countable families of finite sets.*
(B) *The Prime Ideal Theorem for Boolean algebras with countably many generators.*
(C) *Compactness of the Cantor set.*
(D) *The Prime Ideal Theorem for distributive lattices with countably many generators.*
(E) *Engeler's Lemma for partial valuations on a countable set.*
(F) *The Finite Cutset Lemma for countable sets.*
(I) *The Intersection Lemma for countable sets.*
(K) *König's Lemma for locally finite graphs.*
(S) *The Subbase Lemma for second countable spaces.*
(V) *Van Benthem's Lemma for collections of finite subsets of a countable set.*

Of course, many of these equivalences belong to the folklore of set theory, but some of them are perhaps new, and the proofs of various known implications are considerably simplified by the previous arguments.

8. Applications of the Intersection Lemma and the Finite Cutset Lemma

One of the immediate consequences of the Intersection Lemma is the Axiom of Choice for families of finite sets (ACF): if $\mathscr{A}$ is a system of pairwise disjoint nonempty finite subsets of X then the system $\mathscr{S}$ of all subsets of X intersecting each member of $\mathscr{A}$ in at most one element is of finite character, and for finite $\mathscr{E} \subseteq \mathscr{A}$, there is an $S \in \mathscr{E}^{\#} \cap \mathscr{S}$ (choice for finite families). Hence, there is an $S \in \mathscr{S}$ intersecting each member of $\mathscr{A}$ in a singleton. Alternatively, we may invoke the Finite Cutset Lemma: each member of $\mathscr{A}$ is a finite cutset for $\mathscr{S}$, and an almost maximal member of $\mathscr{S}$ is then a set of representatives for $\mathscr{A}$.

As demonstrated in Jech [26], ACF is strictly weaker than the Ordering Principle, requiring the existence of a linear order on every set, which in turn is strictly weaker than the Order Extension Principle, stating that every (partial) order may be extended to a linear order. On the other hand, *the Finite Cutset Lemma for X^2 entails the Order Extension Principle for X*: indeed, the system $\mathscr{S}$ of all subsets S of X^2 whose transitive-reflexive closure is antisymmetric (hence an order) is of finite character, and each of the sets $\{(x,y),(y,x)\}$ $((x,y) \in X)$ is a finite cutset for $\mathscr{S}$. Hence every almost maximal member of $\mathscr{S}$ is a linear order on X.

Of great combinatorial and order-theoretical interest are Dilworth's Decomposition Theorem and its graph-theoretical variants due to Menger and Ford-Fulkerson.

Proposition 8. *The Intersection Lemma for X implies Dilworth's Theorem: If every antichain of an ordered set $(X, \leq)$ has at most n elements then X is a disjoint union of n chains (and conversely).*

Proof: The finite case is settled by induction on the size of X (see [7]). For the infinite case, let $\mathscr{S}$ denote the system of all chains in X and $\mathscr{A}$ the system of all antichains of size n. Given a finite subset $\mathscr{E}$ of $\mathscr{A}$, the argument for finite subposets yields a decomposition of $\bigcup \mathscr{E}$ into n chains. If C is one of them then C intersects each $E \in \mathscr{E}$ (otherwise, some n-element antichain $E \in \mathscr{E}$ would be contained in $\bigcup \mathscr{E} \setminus C$, a union of $n-1$ chains, which is absurd). Hence, by the Intersection Lemma, there is a chain C that intersect each $A \in \mathscr{A}$. Thus every antichain of $X \setminus C$ has at most $n-1$ elements, and induction completes the proof. Compare this with the rather complicated proof in [7], based on Zorn's Lemma. □

The step from the finite to the infinite in the proof of Dilworth's Theorem can also be achieved by using the famous n-Coloring Theorem due to De Bruijn and Erdös [9], who used Rado's Selection Lemma in connection with ACF for its proof. But the n-Coloring Theorem is also a consequence of the Intersection Lemma. However, this time the latter is needed for nX in order to obtain the desired conclusion for graphs with vertex set X. Recall that an n-coloring of a graph (X, R) is a map $\varphi : X \to n$ so that adjacent vertices have different colors, i.e. xRy implies $\varphi(x) \neq \varphi(y)$. Of course, any such coloring induces a partition of X into n independent subsets. On the other hand, we shall consider an intermediate principle suggested by Los and Ryll-Nardzewski [33] and Mycielski [35] on so-called n-block partitions: these are collections of pairwise disjoint n-element subsets of X (whose union need not be the whole set X).

Proposition 9. *Let X be a fixed set and n a natural number.*

(1) *The Intersection Lemma for X implies the Consistency Lemma for X: For any irreflexive relation R on X, the system of all n-block partitions $\mathscr{A}$ admitting a choice function $\varphi \in \Pi\mathscr{A}$ with $R \,|\, \varphi[\mathscr{A}] = \emptyset$ is of finite character.*

(2) *The Consistency Lemma for nX implies the n-Coloring Theorem for X: If (X, R) is a graph whose finite subgraphs are n-colorable then so is the whole graph.*

Proof: (1) An *R-block* is a subset B *with* xRy for any two distinct $x, y \in B$. Let R^c denote the complementary relation $X \times X \setminus R$, and consider the system $\mathscr{S}$ of all R^c-blocks intersecting each n-element R-block in at most one element. For any collection $\mathscr{A}$ of n-element R-blocks, the condition $\mathscr{A}^{\#} \cap \mathscr{S} \neq \emptyset$ means that there is an R^c-block having with each member of $\mathscr{A}$ exactly one element in common, and this is tantamount to postulating a choice function $\varphi \in \Pi\mathscr{A}$ with $R \,|\, \varphi[\mathscr{A}] = \emptyset$. Since the system $\mathscr{S}$ is of finite character, the Intersection Lemma directly applies to this situation.

(2) The set $\{n\{x\} = \{x\} \times n : x \in X\}$ is an n-block partition of nX. Every relation R on X induces a relation R^+ on nX by

$$(x,k)R^+(y,l) \Leftrightarrow xRy \quad \text{and} \quad k = l.$$

For $Y \subseteq_\omega X$, every function $\varphi \in n^Y$ gives rise to a function $\varphi^+ \in \Pi_{x \in Y} n\{x\}$ with $\varphi^+(x) = (x, \varphi(x))$. For each $F \Subset X$, the hypothesis of the n-Coloring Theorem yields a $\varphi \in n^F$ with $\varphi(x) \neq \varphi(y)$ for $(x,y) \in R \,|\, F$, or equivalently, $R^+ \,|\, \varphi^+[F] = R^+ \,|\, \varphi = \emptyset$. Now, the Consistency Lemma, applied to $\mathscr{A} = \{n\{x\} : x \in X\}$ and to R^+ instead of R, gives the conclusion of the n-Coloring Theorem (cf. [35]). □

Corollary. *The Intersection Lemma for* $2X$ *implies the* n*-Coloring Theorem for* X *and all natural numbers* n*.*

Mycielski [35] has shown that *the* n*-Coloring Theorem for* X *implies the Axiom of Choice for families of disjoint* n*-element subsets of* X (AC_n). By a deep result due to Läuchli [30], the Prime Ideal Theorem is equivalent to the (global) 3-Coloring Theorem, while the 2-Coloring Theorem is equivalent to AC_2, hence effectively weaker than PIT. Moreover, Lévy [31] has shown that the Consistency Lemma for 2-block partitions does not imply AC_3, and that AC_n for all n is weaker than PIT.

A nice algebraic application of the Intersection Lemma is the Artin-Schreier Theorem on real fields, stating that a field is totally orderable iff its zero element is not a sum of nonvanishing squares (see [1]); for generalizations to (noncommutative) rings, see Fuchs [17]. By a *strict subsemiring* of a ring R, we mean an additively and multiplicatively closed subset of R not containing the zero element 0 of R.

Proposition 10. *Consider the following statements on a ring* R *without zero divisors:*

(a) R *is totally orderable (so that the strictly positive elements form a subsemiring).*
(b) *There is a strict subsemiring* S *of* R *such that* $x \in S$ *or* $-x \in S$ *for each* $x \in R \setminus \{0\}$*.*
(c) *For any finite set* F *of nonzero elements of* R*, there is a strict subsemiring* S *of* R *such that* $x \in S$ *or* $-x \in S$ *for each* $x \in F$*.*
(d) *A nontrivial sum of products in which each element occurs an even number of times as a factor cannot be zero.*

The implications

$$\text{(a)} \Leftrightarrow \text{(b)} \Rightarrow \text{(c)} \Leftrightarrow \text{(d)}$$

hold in ZF, and the Intersection Lemma makes all four statements equivalent.

Proof: (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c): Clear.

(c) $\Rightarrow$ (d): Let s be a sum of the required type, and let F denote the set of all elements occurring as factors in the (nonzero) summands of s. Choose a strict semiring S according to (c). Then S must contain the element s, since replacing some of the elements from F by their negatives does not affect the value of s. Hence s cannot be zero.

(d) $\Rightarrow$ (c): See the proof of Theorem 7 in Chapter VI of [17].

Finally, under the hypothesis of the Intersection Lemma, one easily obtains the implication

(c) $\Rightarrow$ (b): Put $\mathscr{A} = \{\{x, -x\} : x \in R \setminus \{0\}\}$, and let $\mathscr{S}$ denote the collection of all subsets of R generating a strict subsemiring. Then $\mathscr{S}$ is a system of finite character, and (c) means that $\mathscr{S} \cap \mathscr{E}^{\#} \neq \emptyset$ for all finite $\mathscr{E} \subseteq \mathscr{A}$. Hence $\mathscr{S} \cap \mathscr{A}^{\#} \neq \emptyset$, which gives (b). $\square$

In the same way, one derives the somewhat more general extension theorem due to Fuchs [17] from the Intersection Lemma.

The Intersection Lemma or the equivalent Finite Cutset Lemma is also an appropriate tool for the only non-constructive step in the proof of the Compactness Theorem of first order logic, namely the extension of a consistent set of sentences to a *negation complete* one, that is, to a consistent set that contains, for each sentence φ, either φ or its negation $\neg\varphi$ (see, for example, [10], [26]). We need here only the fact that the consistent sets form a system of finite character, and that for any consistent set S and any sentence φ, either $S \cup \{\varphi\}$ or $S \cup \{\neg\varphi\}$ is still consistent. Hence, an application of the Finite Cutset Lemma to the cutsets $\{\varphi, \neg\varphi\}$ immediately gives

Proposition 11. *The Finite Cutset Lemma ensures that every consistent set of sentences is contained in a negation complete consistent set.*

Hence, in ZF without AC, the Compactness Theorem is a consequence of the Finite Cutset Lemma. On the other hand, it implies the Prime Ideal Theorem (see, e.g., [26]) which in turn implies the Intersection Lemma. Thus we have closed the circle and conclude:

Corollary. *The Finite Cutset Lemma is globally equivalent to the Compactness Theorem and to the Prime Ideal Theorem.*

However, it remains open whether the Finite Cutset Lemma for a fixed set X implies the Prime Ideal Theorem for Boolean algebras generated by X.

References

[1] Artin E., Schreier O., *Algebraische Konstruktion reeller Körper*, Abh. Math. Sem. Hamb. Univ. **5** (1926), 85–99.
[2] Aubert K.E., *Theory of x-ideals*, Acta Math. **107** (1962), 1–52.
[3] Banaschewski B., *The power of the ultrafilter theorem*, J. London Math. Soc. (2) **27** (1983), 193–202.
[4] Banaschewski B., *Prime elements from prime ideals*, Order **2** (1985), 211–213.
[5] Banaschewski B., *A new proof that "Krull implies Zorn"*, Mathematical Logic Quarterly **40** (1994), 478–480.
[6] Banaschewski B., Erné M., *On Krull's separation lemma*, Order **10** (1993), 253–260.

[7] Crawley P., Dilworth R.P., *Algebraic Theory of Lattices*, Prentice-Hall, N.J., 1973.

[8] Davey B.A., Priestley H.A., *Introduction to Lattices and Order*, Cambridge University Press, 1990.

[9] de Bruijn N.G., Erdös P., *A colour problem for infinite graphs and a problem in the theory of relations*, Indag. Math. **13** (1951), 371–373.

[10] Ebbinghaus H.-D., Flum J., Thomas W., *Mathematical Logic*, Springer-Verlag, New York, 1991.

[11] Engeler E., *Eine Konstruktion von Modellerweiterungen*, Z. Math. Logik Grundlagen Math. **5** (1959), 126–131.

[12] Erné M., *Semidistributivity, prime ideals and the subbase lemma*, Rend. Circ. Math. Palermo **II -XLI** (1992), 241–250.

[13] Erné M., *A primrose path from Krull to Zorn*, Comment. Math. Univ. Carolinae **36** (1995), 123–126.

[14] Erné M., *Prime ideal theorems for universal algebras*, Preprint Univ. Hannover, 1995.

[15] Erné M., Gatzke H., *Convergence and continuity in partially ordered sets and semilattices*, in: R.-E. Hoffmann and K.H. Hofmann (eds.), Continuous lattices and their applications, Lecture Notes in Pure and Appl. Math. **101**, Marcel Dekker Inc., New York-Basel, 1985, pp. 9–40.

[16] Frink O., *Topology in lattices*, Trans. Amer. Math. Soc. **51** (1942), 569–583.

[17] Fuchs L., *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.

[18] Gähler W., *Grundstrukturen der Analysis I*, Akademie-Verlag and Birkhäuser Verlag, Berlin-Basel, 1977.

[19] Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M., Scott D.S., *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[20] Grätzer G., *General Lattice Theory*, Birkhäuser, Basel, 1978.

[21] Halpern J., *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. **55** (1964), 57–66.

[22] Halpern J., Lévy A., *The Boolean prime ideal theorem does not imply the axiom of choice*, in: D. Scott (ed.), Axiomatic set theory, Proc. Symp. Pure Math., Univ. of California, Los Angeles **13** (1), (1967), 83–124.

[23] Halpern D., Howard P.E., *Cardinals m such that $2m = m$*, Bull. Amer. Math. Soc. **76** (1970), 487–490.

[24] Henkin L., *The completeness of the first order functional calculus*, J. Symbolic Logic **14** (1949), 159–166.

[25] Hodges W., *Krull implies Zorn*, J. London Math. Soc. **19** (1979), 285–287.

[26] Jech T., *The Axiom of Choice*, North-Holland, Amsterdam-New York, 1973.

[27] Johnstone P., *Almost maximal ideals*, Fund. Math. **123** (1984), 197–209.

[28] Kelley J.L., *The Tychonoff product theorem implies the axiom of choice*, Fund. Math. **37** (1950), 75–76.

[29] Klimovsky G., *Zorn's theorem and the existence of maximal filters and ideals in distributive lattices*, Rev. Un. Mat. Argentina **18** (1958), 160–164.

[30] Läuchli H., *Coloring infinite graphs and the Boolean prime ideal theorem*, Israel J. Math. **9** (1971), 420–429.

[31] Lévy A., *Remarks on a paper by J. Mycielski*, Acta Math. Acad. Sci. Hungar. **14** (1963), 125–130.

[32] Los J., Ryll-Nardzewski C., *0n the application of Tychonoff's theorem in mathematical proofs*, Fund. Math. **38** (1951), 233–237.

[33] Los J., Ryll-Nardzewski C., *Effectiveness of the representation theory for Boolean algebras*, Fund Math. **41** (1954), 49–56.

[34] Moore G.H., *Zermelo's Axiom of Choice – its Origins, Development and Influence*, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[35] Mycielski J., *Some remarks and problems on the colouring of infinite graphs and the theorem of Kuratowski*, Acta Math. Acad. Sci. Hung. **12** (1961), 125–129.

[36] Parovičenko I.I., *Topological equivalents of the Tihonov theorem*, Dokl. Akad. Nauk SSSR **184** (1969), 38–39; Soviet Math. Dokl. **10** (1969), 33–34.

[37] Rav Y., *Variants of Rado's selection lemma and their applications*, Math. Nachr. **79** (1977), 145–165.

[38] Rav Y., *Semiprime ideals in general lattices*, J. Pure and Appl. Algebra **56** (1989), 105–118.

[39] Rubin H., Rubin J.E., *Equivalents of the Axiom of Choice, II*, North-Holland, Amsterdam-New York-Oxford, 1985.

[40] Rubin H., Scott D.S., *Some topological theorems equivalent to the prime ideal theorem*, Bull. Amer. Math. Soc. **60** (1954), 389 (Abstract).

[41] Sageev G., *An independence result concerning the axiom of choice*, Ann. Math. Logic **8** (1975), 1–184.

[42] Scott D.S., *Prime ideals for rings, lattices and Boolean algebras*, Bull. Amer. Math. Soc. **60** (1954), 390 (Abstract).

[43] Tarski A., *Prime ideal theorems for Boolean algebras and the axiom of choice. Prime ideal theorems for set algebras and ordering principles. Prime ideal theorems for set algebras and the axiom of choice*, Bull. Amer. Math. Soc. **60** (1954), 390–391 (Abstracts).

[44] Teichmüller O., *Braucht der Algebraiker das Auswahlaxiom?*, Deutsche Math. **4** (1939), 567–577.

[45] Tukey J.W., *Convergence and uniformity in topology*, Annals of Math. Studies **2**, Princeton, 1940.

[46] van Benthem J.F.A.K., *A set-theoretical equivalent of the prime ideal theorem for Boolean algebras*, Fund. Math. **89** (1975), 151–153.

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