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Convergence theorems for set-valued conditional expectations

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Abstract. In this paper we prove two convergence theorems for set-valued conditional expectations. The first is a set-valued generalization of Levy's martingale convergence theorem, while the second involves a nonmonotone sequence of sub σ-fields.

Keywords: measurable multifunction, set-valued conditional expectation, Levy's theorem, support function, Kuratowski-Mosco convergence of sets

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1. Introduction.

Set-valued random variables (random sets) have been studied recently by many authors. Selectively we mention the important works of Alo-deKorvin-Roberts [1], Hiai [9], Hiai-Umegaki [10] and Luu [12]. Furthermore the works of Artstein-Hart [2], deKorvin-Kleyle [11] and Papageorgiou [15], illustrated that set-valued random variables can be useful in the study of problems in optimization theory, information systems and mathematical economics.

In this paper we prove a set-valued analogue of the well-known Levy's martingale convergence theorem and then we go one step further and allow the sequence of sub σ-fields to vary in a nonmonotone fashion. Theorem 3.1 in this paper extends Theorem 2.1 of the author [18], where the Banach space was assumed to be reflexive. Theorem 3.2 is a new general convergence result for set-valued random variables (random sets).

2. Preliminaries.

Let (Ω, Σ, μ) be a probability space and X a separable Banach space. We shall be using the following notation:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex)}\}$$

and

$$P_{wkc}(X) = \{A \subseteq X : \text{ nonempty, weakly compact and convex}\}.$$

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For any $A \in 2^X \setminus \{\emptyset\}$, we set $|A| = \sup\{\|x\| : x \in A\}$ (the "norm" of A), $\sigma(x^*, A) = \sup\{(x^*, x) : x \in A\}$, $x^* \in X^*$ (the support function of A) and, for every $z \in X$, $d(z, A) = \inf\{\|z - x\| : x \in A\}$ (the distance function from A).

A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be measurable, if for all U open in X $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$. If in addition $F(\cdot)$ is $P_f(X)$-valued, then the above definition is equivalent to any of the following statements:

(i) for every $z \in X$, $\omega \to d(z, F(\omega))$ is measurable,

(ii) there exist measurable functions $f_n : \Omega \to X$, $n \geq 1$, s.t. $F(\omega) = \mathrm{cl}\{f_n(\omega)\}_{n\geq 1}$ for all $\omega \in \Omega$.

The above statements imply the following:

(iii) $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability).

If Σ is μ-complete, then all the statements (i)–(iii) are equivalent.

Further details on the measurability of multifunctions can be found in the survey paper of Wagner [23].

Given a measurable multifunction $F : \Omega \to P_f(X)$, S_F^1 will denote the set of integrable selectors of $F(\cdot)$; i.e., $S_F^1 = \{f \in L^1(\Omega, X) : f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. Clearly this set is closed, maybe empty and using Aumann's selection theorem (see Wagner [23, Theorem 5.10]) we can easily check that S_F^1 is nonempty if and only if $\omega \to \inf\{\|x\| : x \in F(\omega)\} \in L^1(\Omega)$.

Indeed, let $m(\omega) = \inf\{\|x\| : x \in F(\omega)\}$. Because of the property (ii) above, we have $m(\omega) = \inf_{n\geq 1} \|f_n(\omega)\|$, where $f_n : \Omega \to \Xi$, $n \geq 1$, are measurable functions s.t. $F(\omega) = \mathrm{cl}\{f_n(\omega)\}_{n\geq 1}$. So $\omega \to m(\omega)$ is measurable. If $S_F^1 \neq \emptyset$, let $g \in S_F^1$. Then $m(\omega) \leq \|g(\omega)\|$ μ-a.e. $\Rightarrow m \in L^1(\Omega)$. Conversely, suppose that $m(\cdot) \in L^1(\Omega)$. Let $\varepsilon > 0$ and set $H_\varepsilon(\omega) = \{x \in F(\omega) : \|x\| \leq m(\omega) + \varepsilon\}$. Clearly for all $\omega \in \Omega$, $H_\varepsilon(\omega) \neq \emptyset$ and $GrH_\varepsilon = GrF \cap \{(\omega, x) \in \Omega \times X : \|x\| - m(\omega) \leq \varepsilon\}$. Clearly then $(\omega, x) \to \|x\| - m(\omega)$ is measurable. So $GrH_\varepsilon \in \Sigma \times B(X)$. Apply Aumann's selection theorem to get $g : \Omega \to X$ measurable s.t. $g(\omega) \in H_\varepsilon(\omega)$ for all $\omega \in \Omega$. Then $g(\omega) \in F(\omega)$ and $\|g(\omega)\| \leq m(\omega) + \varepsilon \Rightarrow g \in S_F^1 \Rightarrow S_F^1 \neq \emptyset$.

This is the case if $\omega \to |F(\omega)| = \sup\{\|x\| x \in F(\omega)\} \in L^1(\Omega)$. Such a multifunction is called integrably bounded. Note that if $F(\cdot)$ is $P_{fc}(X)$-valued, then S_F^1 is convex, too. Using S_F^1 we can define a set-valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)\, d\mu(\omega) = \{\int_\Omega f(\omega)\, d\mu(\omega) : f \in S_F^1\}$.

Let Σ_0 be a sub σ-field of Σ. Let $F : \Omega \to P_f(X)$ be a measurable multifunction s.t. $S_F^1 \neq \emptyset$. Following Hiai-Umegaki [10], we define the set-valued conditional expectation of $F(\cdot)$ with respect to Σ_0 to be the Σ_0-measurable multifunction $E^{\Sigma_0}F : \Omega \to P_f(X)$ for which we have $S^1_{E^{\Sigma_0}F}(\Sigma_0) = \mathrm{cl}\{E^{\Sigma_0}f : f \in S_F^1\}$ (the closure taken in the $L^1(\Omega, X)$-norm). Note that by definition $S^1_{E^{\Sigma_0}F}(\Sigma_0)$ consists of all Σ_0-measurable selectors of $E^{\Sigma_0}F$. To simplify the already heavy notation, we shall simply write $S^1_{E^{\Sigma_0}F}$ instead of $S^1_{E^{\Sigma_0}F}(\Sigma_0)$. If $F(\cdot)$ is integrably bounded (resp. convex valued), then so is $E^{\Sigma_0}F(\cdot)$. Note that in Hiai-Umegaki [10], the definition was given for integrably bounded $F(\cdot)$. However, it is clear that it can be extended to the more general class of multifunctions $F(\cdot)$ used here. Recall that

$A \in \Sigma$ is said to be a Σ_0-atom if and only if for all $A' \in \Sigma$, $A' \subseteq A$ there exists $B \in \Sigma_0$ s.t. $\mu(A'\Delta(A \cap B)) = 0$ or equivalently $\chi_{A'}(\omega) = \chi_{A\cap B}(\omega)$ μ-a.e. (see Hanen-Neveu [7]).

Finally let $\{A_n\}_{n\geq 1} \subseteq 2^X \setminus \{\emptyset\}$. Following Mosco [14], we define:

$$\begin{aligned} s - \underline{\lim} A_n &= \{x \in X : x = s - \lim x_n,\ x_n \in A_n,\ n \geq 1\} \\ &= \{x \in X : \lim d(x, A_n) = 0\} \end{aligned}$$

and

$$w - \overline{\lim} A_n = \{x \in X : x = w - \lim x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < n_3 \dots < n_k < \dots\}.$$

Here $s-$ denotes the strong topology on X, while $w-$ denotes the weak topology on X. It is easy to see that we always have $s - \underline{\lim} A_n \subseteq w - \overline{\lim} A_n$. We say that the A_n's converge to A in the Kuratowski-Mosco sense to A, denoted by $A_n \xrightarrow{K-M} A$ if $s - \underline{\lim} A_n = A = w - \overline{\lim} A_n$.

3. Convergence theorems.

Assume that $\{\Sigma_n\}_{n\geq 1}$ is an increasing subsequence of sub σ-fields of Σ s.t. $V_{n\geq 1}\Sigma_n = \Sigma_0$. Recall that if $f \in L^1(\Omega, \mathbb{R})$, then $E^{\Sigma_n} f(\omega) \to E^{\Sigma_0} f(\omega)$ μ-a.e. (Levy's martingale convergence theorem). This was extended to Banach space-valued random variables; i.e., $f \in L^1(\Omega, X)$ (see for example Metivier [13, Theorem 11.2]). The following theorem is a set-valued version of this martingale convergence theorem. It improves Theorem 2.1 of [18], since we get a stronger kind of convergence for the set-valued martingale and the reflexivity hypothesis on X is relaxed.

Theorem 3.1. *If X^* is separable and $F : \Omega \to P_{fc}(X)$ is integrably bounded, then $E^{\Sigma_n} F(\omega) \xrightarrow{K-M} E^{\Sigma_0} F(\omega)$ μ-a.e. The result is also true if X^* is separable, $F : \Omega \to P_f(X)$ is integrably bounded and (Ω, Σ, μ) has no Σ_0-atoms.*

Proof: From the lemma in Section 2 of [19], we know that for all $x^* \in X^*$ and all $\omega \in \Omega \setminus N$, $\mu(N) = 0$, we have $E^{\Sigma_0}\sigma(x^*, F(\omega)) = \sigma(x^*, E^{\Sigma_0} F(\omega))$. So we have $\overline{\lim} E^{\Sigma_n}\sigma(x^*, F(\omega)) = \overline{\lim}\sigma(x^*, E^{\Sigma_n} F(\omega))$. But from the classical Levy's martingale convergence theorem, we know that $\overline{\lim} E^{\Sigma_n}\sigma(x^*, F(\omega)) = \lim E^{\Sigma_n}\sigma(x^*, F(\omega)) = E^{\Sigma_0}\sigma(x^*, F(\omega))$ for all $\omega \in \Omega \setminus N(x^*)$, $\mu(N(x^*)) = 0$. Let $\{x^*_m\}_{m\geq 1}$ be dense in X^* for the strong topology (recall that X^* is assumed to be separable). We have $E^{\Sigma_n}\sigma(x^*, F(\omega)) \to E^{\Sigma_0}\sigma(x^*, F(\omega))$ as $n \to \infty$ for all $m \geq 1$ and all $\omega \in \Omega \setminus N$, where $N = \bigcup_{m\geq 1} N(x^*_m)$, $\mu(N) = 0$. Let $x^* \in X^*$ and let $\{x^*_k\}_{k\geq 1} \subseteq \{x^*_n\}_{n\geq 1}$ be s.t. $x^*_k \xrightarrow{s} x^*$ (here s denotes the strong on X^*). From Proposition 14 of Thibault [22], we know that for all $\omega \in \Omega \setminus N_1$, $\mu(N_1) = 0$, $E^{\Sigma_0}\sigma(\cdot, F(\omega))$ is continuous and so $E^{\Sigma_0}\sigma(x^*_k, F(\omega)) \to E^{\Sigma_0}\sigma(x^*, F(\omega))$ for all $\omega \in \Omega\setminus N_1$, $\mu(N_1) = 0$. Let $N_2 = N \cup N_1$, $\mu(N_2) = 0$ and let $\omega \in \Omega \setminus N_2$. Invoking Lemma 1.6 of Attouch [3], we can find a map $n \to k(n)$, depending in general on $\omega \in \Omega \setminus N_2$

s.t. $E^{\Sigma_n}\sigma(x^*_{k(n)}, F(\omega)) \to E^{\Sigma_0}\sigma(x^*, F(\omega)) = \sigma(x^*, E^{\Sigma_0}F(\omega))$ as $n \to \infty$. So for any given $\omega \in \Omega \setminus N_2$, $\mu(N_2) = 0$, we have

$$\begin{aligned}
&|E^{\Sigma_n}\sigma(x^*, F(\omega)) - E^{\Sigma_0}\sigma(x^*, F(\omega))| \\
&\leq |E^{\Sigma_n}\sigma(x^*, F(\omega)) - E^{\Sigma_n}\sigma(x^*_{k(n)}, F(\omega))| \\
&+ |E^{\Sigma_n}\sigma(x^*_{k(n)}, F(\omega)) - E^{\Sigma_0}\sigma(x^*, F(\omega))|.
\end{aligned}$$

For the first summand in the right hand side of the above inequality, we have

$$\begin{aligned}
|E^{\Sigma_n}\sigma(x^*, F(\omega)) - E^{\Sigma_0}\sigma(x^*_{k(n)}, F(\omega))| &\leq E^{\Sigma_n}|\sigma(x^*, F(\omega)) - \sigma(x^*_{k(n)}, F(\omega))| \\
&\leq E^{\Sigma_n}|F(\omega)| \cdot \|x^* - x^*_{k(n)}\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Also from the choice of the map $n \to k(n)$, we have

$$|E^{\Sigma_n}\sigma(x^*_{k(n)}, F(\omega)) - E^{\Sigma_0}\sigma(x^*, F(\omega))| \to 0 \text{ as } n \to \infty.$$

Thus finally we deduce that for all $x^* \in X^*$ and all $\omega \in \Omega \setminus N_2$ $\mu(N_2) = 0$, we have:

$$\begin{aligned}
&E^{\Sigma_n}\sigma(x^*, F(\omega)) \to E^{\Sigma_0}\sigma(x^*, F(\omega)) \text{ as } n \to \infty, \\
&\Rightarrow \sigma(x^*, E^{\Sigma_n}F(\omega)) \to \sigma(x^*, E^{\Sigma_0}F(\omega)) \ \mu\text{-a.e. as } n \to \infty.
\end{aligned}$$

Applying Proposition 4.1 of [16], we get

$$w - \overline{\lim} E^{\Sigma_n}F(\omega) \subseteq \overline{\text{conv}} E^{\Sigma_0}F(\omega) \ \mu\text{-a.e.}$$

If $F(\cdot)$ is $P_{fc}(X)$-valued, then $\overline{\text{conv}} E^{\Sigma_0}F(\omega) = E^{\Sigma_0}F(\omega)$. If $F(\cdot)$ is $P_f(X)$-valued and (Ω, Σ, μ) has no Σ_0-atoms, from Dynkin-Evstigneev [5], we have that $E^{\Sigma_0}F(\omega)$ is μ-a.e. convex. So in both cases we have:

$$w - \overline{\lim} E^{\Sigma_n}F(\omega) \subseteq E^{\Sigma_0}F(\omega) \ \mu\text{-a.e.} \tag{1}$$

Next, let $f \in S^1_F$. Then from Theorem 11.2 of Metivier [13], we know that $E^{\Sigma_n}f(\omega) \xrightarrow{s} E^{\Sigma_0}f(\omega)$ μ-a.e. in X as $n \to \infty$. Clearly $E^{\Sigma_n}f \in S^1_{E^{\Sigma_n}F}$ and so we have $E^{\Sigma_0}f(\omega) \in s - \underline{\lim} E^{\Sigma_n}F(\omega)$ μ-a.e. Hence we have that

$$E^{\Sigma_0}S^1_F \subseteq S^1_{s-\underline{\lim}E^{\Sigma_n}F}.$$

Recalling that $s - \underline{\lim} E^{\Sigma_n}F(\cdot)$ is closed-valued, we have that the set $S^1_{s-\underline{\lim}E^{\Sigma_n}F}$ is closed in $L^1(X)$. Hence we have:

$$\overline{E^{\Sigma_0}S^1_F} \subseteq S^1_{s-\underline{\lim}E^{\Sigma_n}F}.$$

But by definition (see Section 2), we have $\overline{E^{\Sigma_0}S^1_F} = S^1_{E^{\Sigma_0}F}$. Therefore we finally have

$$\begin{aligned} &S^1_{E^{\Sigma_0}F} \subseteq S^1_{s-\underline{\lim} E^{\Sigma_n}F} \\ \text{(2)} \qquad &\Rightarrow E^{\Sigma_0}F(\omega) \subseteq s - \underline{\lim} E^{\Sigma_n}F(\omega)\ \mu\text{a.e.} \end{aligned}$$

From (1) and (2) above we conclude that $E^{\Sigma_n}F(\omega) \xrightarrow{K-M} E^{\Sigma_0}F(\omega)$ μ-a.e. $\square$

Corollary. *If* $\dim X < \infty$ *and* $F : \Omega \to P_{fc}(X)$ *is integrably bounded, then* $E^{\Sigma_n}F(\omega) \xrightarrow{h} E^{\Sigma_0}F(\omega)$ *μ-a.e., where h denotes the Hausdorff metric on $P_{fc}(X)$. The same holds if* $\dim X < \infty$, $F : \Omega \to P_f(X)$ *is integrably bounded and* (Ω, Σ, μ) *has no Σ_0-atoms.*

Remark. The "convex" part of this corollary was proved by the author in [18, Theorem 2.1]. This result is a consequence of Corollary 3A of Salinetti-Wets [21].

In the next convergence theorem, we allow the sub σ-fields to converge in a non-monotone fashion. Recall that $\Sigma_n \to \Sigma_0$ in $L^1(\Omega, X)$ if and only if for every $f \in L^1(\Omega, X)$, we have $E^{\Sigma_n}f \xrightarrow{s} E^{\Sigma_0}f$ in $L^1(\Omega, X)$. From the vector valued version of Levy's martingale convergence theorem (see Metivier [13, Theorem 11.2]), we know that if $\Sigma_n \uparrow \Sigma_0$, then $\Sigma_n \to \Sigma_0$ in $L^1(\Omega, X)$. More generally, if $X = \mathbb{R}$ and $\Sigma = V_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Sigma_n = \bigcap_{m=1}^{\infty} V_{n=m}^{\infty} \Sigma_n$, then $\Sigma_n \to \Sigma$ in $L^1(\Omega)$ (see Fetter [6, Theorem 3]).

Recall that if X^* is separable, then X^* has the RNP and so $L^1(\Omega, X)^* = L^{\infty}(\Omega, X^*)$ (see Diestel-Uhl [4, Theorem 1, p. 98]). We shall denote the duality brackets for this pair by $\langle \cdot, \cdot \rangle$; i.e., $\langle f, h \rangle = \int_\Omega (f(\omega), h(\omega))\, d\mu(\omega)$ for every $f \in L^1(\Omega, X)$, $h \in L^{\infty}(\Omega, X^*)$.

We shall need the following two lemmata. In both we assume X^* is separable.

Lemma 3.1. *If* Σ_0 *is a sub σ-field of Σ,* $f \in L^1(\Sigma_0, X)$ *and* $h \in L^{\infty}(\Sigma, X^*)$, *then* $\langle f, E^{\Sigma_0}h \rangle = \langle f, h \rangle$.

Proof: Let $h = \chi_A x^*$, $A \in \Sigma$, $x^* \in X^*$. Then we have:

$$\begin{aligned} \langle f, E^{\Sigma_0}h \rangle &= \int_\Omega (f(\omega), E^{\Sigma_0}\chi_A(\omega)x^*)\, d\mu(\omega) \\ &= \int_\Omega E^{\Sigma_0}\chi_A(\omega)(f(\omega), x^*)\, d\mu(\omega) \\ &= \int_\Omega \chi_A(\omega)(f(\omega), x^*)\, d\mu(\omega) \\ &= \int_\Omega (f(\omega), \chi_A(\omega)x^*)\, d\mu(\omega) = \langle f, h \rangle. \end{aligned}$$

Clearly then the result is valid for countably-valued $h \in L^{\infty}(\Sigma, X^*)$. But those functions are dense in $L^{\infty}(\Sigma, X^*)$ (see Diestel-Uhl [4, p. 42]). So by a simple density argument, we conclude that the lemma holds for all $h \in L^{\infty}(\Sigma, X^*)$. $\square$

In a similar way, exploiting the density of simple functions in $L^1(\Sigma, X)$, we can prove the following lemma, whose proof is omitted.

Lemma 3.2. *If Σ_0 is a sub σ-field of Σ, $f \in L^1(\Sigma, X)$ and $h \in L^\infty(\Sigma_0, X^*)$, then $\langle f, h\rangle = \langle E^{\Sigma_0} f, h\rangle$.*

The next theorem partially generalizes Theorem 3.1. Now we have a sequence $\{F_n\}_{n\geq 1}$ of random sets, instead of just a fixed one as in Theorem 3.1, and the sequence $\{\Sigma_n\}_{n\geq 1}$ of the sub σ-fields of Σ need not be monotone increasing. Because of the nature of the convergence of the Σ_n's, our convergence result is in terms of the sets of integrable selectors of the random multifunctions. When specialized to single-valued random variables, then we get that $E^{\Sigma_n} f_n \to E^{\Sigma_0} f$ in $L^1(\Omega, X)$, which improves Theorem 4 of Fette [6], where $X = \mathbb{R}$. Note that the convexity of the values of the random sets $\{F_n(\omega)\}_{n\geq 1}$ is important, because it allows us to use the "multivalued dominated convergence theorem" established in [16, Theorem 4.4]. It remains an open question whether the almost everywhere convergence holds (even if random variables are single-valued and $X = \mathbb{R}$; see also Fetter [6]).

Theorem 3.2. *If X^* is separable, $F_n : \Omega \to P_{wkc}(X)$ $n \geq 1$ are measurable multifunctions s.t. $F_n(\omega) \subseteq G(\omega)$ μ-a.e. with $G : \Omega \to P_{wkc}(X)$ integrably bounded, $F_n(\omega) \xrightarrow{K-M} F(\omega)$ μ-a.e. and $\Sigma_n \to \Sigma_0$ in $L^1(X)$, then $S^1_{E^{\Sigma_n} F_n} \xrightarrow{K-M} S^1_{E^{\Sigma_0} F}$ as $n \to \infty$.*

Proof: From Proposition 4.3 of Hess [8], we have that $F : \Omega \to P_{wkc}(X)$ is measurable and $F(\omega) \subseteq G(\omega)$ μ-a.e. Then from Proposition 3.1 of [17], we have $S^1_{F_n}, S^1_F$ are weakly compact convex subsets of $L^1(X)$ and so $S^1_{E^{\Sigma_n} F_n} = E^{\Sigma_n} S^1_{F_n}$, $S^1_{E^{\Sigma_0} F} = E^{\Sigma_0} S^1_F$ $n \geq 1$.

Now let $h \in w - \overline{\lim} S^1_{E^{\Sigma_n} F_n}$. Then by definition we can find $h_k \in S^1_{E^{\Sigma_{n(k)}} F_{n(k)}}$ s.t. $h_k \xrightarrow{w} h$ in $L^1(X)$. Then we can find $f_k \in S^1_{F_{n(k)}}$ s.t. $E^{\Sigma_{n(k)}} f_k = h_k$. Since $\{f_k\}_{k\geq 1} \subseteq S^1_G$ and the latter is w-compact in $L^1(X)$ (see Proposition 3.1 of [17]), by passing to a subsequence if necessary, we may assume that $f_k \xrightarrow{w} f$ in $L^1(X)$. Also since $S^1_{F_n} \xrightarrow{K-M} S^1_F$ by Theorem 4.4 of [16], we have $f \in S^1_F$. Now for $v \in L^\infty(\Omega, X^*) = L^1(\Omega, X)^*$ we have using Lemmata 3.1 and 3.2:

$$\langle h_k, v\rangle = \langle E^{\Sigma_{n(k)}} f_k, v\rangle = \langle E^{\Sigma_{n(k)}} f_k, E^{\Sigma_{n(k)}} v\rangle = \langle f_k, E^{\Sigma_{n(k)}} v\rangle.$$

Invoking Lemma 4.2 of Papageorgiou-Kandilakis [20], we get $\langle f_k, E^{\Sigma_{n(k)}} v\rangle \to \langle f, E^{\Sigma_0} v\rangle$ as $k \to \infty$. Once again through Lemmata 3.1 and 3.2 above, we have $\langle f, E^{\Sigma_0} v\rangle = \langle E^{\Sigma_0} f, E^{\Sigma_0} v\rangle = \langle E^{\Sigma_0} f, v\rangle$. Therefore

$$\langle h_k, v\rangle \to \langle E^{\Sigma_0} f, v\rangle \quad \text{as} \quad k \to \infty.$$

Also $\langle h_k, v\rangle \to \langle h, v\rangle \Rightarrow \langle h, v\rangle = \langle E^{\Sigma_0} f, v\rangle$ for all $v \in L^\infty(X^*) \Rightarrow h = E^{\Sigma_0} f$ with $f \in S^1_F \Rightarrow h \in S^1_{E^{\Sigma_0} F}$. So we have:

$$w - \overline{\lim} S^1_{E^{\Sigma_n} F_n} \subseteq S^1_{E^{\Sigma_0} F}. \tag{1}$$

Next let $h \in S^1_{E^{\Sigma_0}F}$. Then $h = E^{\Sigma_0}f$, $f \in S^1_F$. Recalling that $S^1_{F_n} \xrightarrow{K-M} S^1_F$ (Theorem 4.4 of [16]), we get $f_n \in S^1_{F_n}$ s.t. $f_n \xrightarrow{s} f$ in $L^1(X)$. We have:

$$\begin{aligned}\|E^{\Sigma_n}f_n - E^{\Sigma_0}f\|_1 &\le \|E^{\Sigma_n}f_n - E^{\Sigma_n}f\|_1 + \|E^{\Sigma_n}f - E^{\Sigma_0}f\|_1\\ &\le \|f_n - f\|_1 + \|E^{\Sigma_n}f - E^{\Sigma_0}f\|_1 \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

since $\Sigma_n \to \Sigma_0$ in $L^1(X)$. Hence $E^{\Sigma_n}f_n \xrightarrow{s} E^{\Sigma_0}f = h$ in $L^1(X)$ and $E^{\Sigma_n}f_n \in S^1_{E^{\Sigma_n}F_n}$ $n \ge 1$. Therefore $h \in s - \underline{\lim} S^1_{E^{\Sigma_n}F_n}$. Thus we have:

$$S^1_{E^{\Sigma_0}F} \subseteq s - \underline{\lim} S^1_{E^{\Sigma_n}F_n}. \tag{2}$$

From (1) and (2) we conclude that

$$S^1_{E^{\Sigma_n}F_n} \xrightarrow{K-M} S^1_{E^{\Sigma_0}F} \quad \text{as} \quad n \to \infty.$$

□

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