Mean quadratic convergence of signed random measures

P. Jacob, P.E. Oliveira*

Abstract. We consider signed Radon random measures on a separable, complete and locally compact metric space and study mean quadratic convergence with respect to vague topology on the space of measures. We prove sufficient conditions in order to obtain mean quadratic convergence. These results are based on some identification properties of signed Radon measures on the product space, also proved in this paper.

Keywords: relative compactness; mean quadratic convergence
Classification: 60G57, 60F25

1. Introduction.

In this paper we propose to study mean quadratic convergence of sequences of signed Radon random measures with respect to vague convergence. This type of convergence has already been studied, in a different setting, by Bonkian [2]. Here we prove some sufficient conditions, similar to those of Bonkian, for the Radon signed measure setting. The conditions we shall prove reflect the difficulties that the vague topology on the space of signed Radon measures raises. As a matter of fact this space, in general, is not even metrizable as it follows from the classical article of Varadarajan [10] and Oliveira [8]. We will start by proving some non random results concerning the identification of signed Radon measures and the vague convergence on non negative Radon measures on the product space. This is essential to the study of the mean quadratic convergence, as this notion is defined in terms of a vague convergence in the product space. In what concerns the random case, we note that in the signed case the difficulties are always connected with the characterization of relative compactness, as it is usual in the signed measures space.

In what follows, let $(\mathbf{S}, d)$ be a separable, complete and locally compact metric space, $\mathcal{B}$ the ring of relatively compact Borel sets of $\mathbf{S}$, $\mathbf{C}_c(\mathbf{S})$ the Banach space of real valued continuous functions with compact support defined on $\mathbf{S}$. Moreover, let $\mathcal{M}_c$ be the space of real valued Radon measures on $\mathbf{S}$, endowed with the vague topology, and $\mathcal{M}_c^+$ the subspace of $\mathcal{M}_c$ of non negative Radon measures. Given a measure $\mu \in \mathcal{M}_c$, we denote $\mu = \mu^+ - \mu^-$ the Hahn–Jordan decomposition of μ. A random measure is a measurable function ξ defined on some probability space taking values almost surely on the space $\mathcal{M}_c$ with the Borel σ-algebra associated with the vague topology.

2. Some results in the non random case.

Let us define the following set of functions on $\mathbf{S} \times \mathbf{S}$

$$H = \{f \otimes g(s,t) = f(s)g(t) : f, g \in \mathbf{C}_c(\mathbf{S})\}\,.$$

* Partially supported by a scholarship from INIC – Instituto Nacional de Investigação Científica

Evidently $H \subset \mathbf{C}_c(\mathbf{S}\times\mathbf{S})$, the space of real valued continuous functions with compact support defined on $\mathbf{S}\times\mathbf{S}$. We shall prove that this set of functions identifies the elements of $\mathcal{M}_c(\mathbf{S}\times\mathbf{S})$ and determines vague convergence in $\mathcal{M}_c^+(\mathbf{S}\times\mathbf{S})$.

Theorem 2.1. *The set H identifies the elements of $\mathcal{M}_c(\mathbf{S}\times\mathbf{S})$, that is, if $\mu \in \mathcal{M}_c\mathbf{S}\times\mathbf{S}$ and $\mu h = 0$, for each $h \in H$, then $\mu = 0$.*

PROOF: Let A and B be open bounded subsets of $\mathbf{S}$. As $\mathbf{S}$ is a normal space, there exists an increasing sequence $\{h_k\}$ of functions belonging to H which converges to $A\times B$. Every open bounded subset of $\mathbf{S}\times\mathbf{S}$ is obtained by forming numerable unions of sets of the form $A\times B$, where A and B are open bounded subsets of $\mathbf{S}$. Moreover, the class of sets $A\times B$, where A and B are open bounded subsets of $\mathbf{S}$, is closed under the formation of finite intersections. Then, if the measures $\mu^+ = \mu^-$ on H, they coincide on the class of subsets of $\mathbf{S}\times\mathbf{S}$ of the form $A\times B$, A and B open bounded subsets of $\mathbf{S}$. As μ^+ and μ^- are regular measures, it follows $\mu^+ = \mu^-$, so $\mu = 0$. □

Theorem 2.2. *Let $\{\mu_n\}$ be a sequence in $\mathcal{M}_c^+(\mathbf{S}\times\mathbf{S})$. If, for each $h \in H$, the sequence $\{\mu_n h\}$ converges, the sequence $\{\mu_n\}$ is vaguely convergent in $\mathcal{M}_c^+(\mathbf{S}\times\mathbf{S})$.*

PROOF: Let $f \in \mathbf{C}_c(\mathbf{S}\times\mathbf{S})$. As $\mathbf{S}$ is a numerable union of compact sets, there exists a compact $K \subset \mathbf{S}$ such that $\operatorname{supp}(f) \subset K\times K$. Moreover, as the space $\mathbf{S}$ is normal, there exists a continuous function, g, taking the value 1 on K, the value 0 on the complement of a compact neighborhood of K and with values on $[0,1]$. Putting $t = g\mathbb{1}_K$, we obtain a continuous function with compact support such that $|f| \le t\otimes t$. By hypothesis, the sequence $\{\mu_n(t\otimes t)\}$ is convergent, so $\{\mu_n f\}$ is a bounded sequence. Then, from [7, Proposition 9.8.7], it follows that $\{\mu_n\}$ is vaguely relatively compact. Let μ_0 be a measure belonging to the closure of $\{\mu_n\}$. Then, for every $\varepsilon > 0$ and $h \in H$, the vague neighborhood of μ_0

$$\{\nu \in \mathcal{M}_c^+(\mathbf{S}\times\mathbf{S}) : \ |\nu h - \mu_0 h| < \varepsilon\}$$

contains an infinity of terms of the converging sequence $\{\mu_n h\}$. So

$$\mu_0 h = \lim_{n\to\infty} \mu_n h,$$

and this equality is verified for every function $h \in H$. According to the preceding theorem, it follows that μ_0 is the only measure belonging to the closure of $\{\mu_n\}$, so $\{\mu_n\}$ converges vaguely to μ_0. □

3. Mean quadratic convergence.

Define mean quadratic convergence as (cf. Bonkian [2])

Definition 3.1. A sequence of random measures $\{\xi_n\}$ converges in quadratic mean to the random measure ξ if the sequence of measures

$$\mathbf{E}\ [(\xi_n - \xi)\otimes(\xi_n - \xi)]$$

converges vaguely to the zero measure on $\mathbf{S}\times\mathbf{S}$.

As in the setting studied by Bonkian, this definition is justified by the following equivalences.

Theorem 3.2. *Let $\{\xi_n\}$ be a sequence of non negative random measures and ξ a non negative random measure. The following conditions are equivalent*

(1) $\mathbf{E}\ [(\xi_n - \xi) \otimes (\xi_n - \xi)] \xrightarrow{v} 0.$
(2) *For each function $f \in \mathbf{C}_c(\mathbf{S})$, $\xi_n f \longrightarrow \xi f$ in quadratic mean.*
(3) *For each $B \in \mathcal{B}$, such that $\xi(fr(B)) = 0$ p.s., where $fr(B)$ represents the frontier of the set B, $\xi_n(B) \longrightarrow \xi(B)$, in quadratic mean.*

Proof: 1⇒2: For each function $f \in \mathbf{C}_c(\mathbf{S})$, $f \otimes f \in \mathbf{C}_c(\mathbf{S} \times \mathbf{S})$. According to Fubini's theorem,

$$\mathbf{E}\ [(\xi_n - \xi) \otimes (\xi_n - \xi)](f \otimes f) = \mathbf{E}\ [(\xi_n f - \xi f)^2], \tag{1}$$

so 1⇒2 follows.

2⇒1: Taking account of (1) and the preceding theorem the implication follows.

1⇒3: Let $B \in \mathcal{B}$ be such that $\xi(fr(B)) = 0$ p.s.. Then $\xi \otimes \xi(fr(B \times B)) = 0$ p.s.. The sequences $\{\mathbf{E}\ (\xi_n \otimes \xi_n)\}$, $\{\mathbf{E}\ (\xi_n \otimes \xi)\}$ and $\{\mathbf{E}\ (\xi \otimes \xi_n)\}$ converge vaguely to $\mathbf{E}\ (\xi \otimes \xi)$, so $\{\mathbf{E}\ (\xi_n^2(B))\}$ and $\{\mathbf{E}\ (\xi_n(B)\xi(B))\}$ converge to $\mathbf{E}\ (\xi^2(B))$. It follows then that the sequence $\{\xi_n(B)\}$ converges in quadratic mean to $\xi(B)$.

3⇒1: For every $A, B \in \mathcal{B}$ such that $\xi(fr(A)) = \xi(fr(B)) = 0$ p.s., the sequences $\{\mathbf{E}\ (\xi_n(A)\xi_n(B))\}$ and $\{\mathbf{E}\ (\xi_n(A)\xi(B))\}$ converge to $\mathbf{E}\ (\xi(A)\xi(B))$. Now to finish this proof, we may proceed as in the proof of Theorem 3.1 of Billingsley [1]. □

It is evident that Condition 1 still implies Condition 2, if we are interested in signed random measures. However, the converse implication is in general false. In order to obtain this implication, we must suppose the relative compactness of the set $\{\mathbf{E}\ (\xi_n \otimes \xi_m)\}$.

Theorem 3.3. *Let $\{\xi_n\}$ be a sequence of random measures and ξ a random measure such that:*

(1) *For every $f \in \mathbf{C}_c(\mathbf{S})$, $\xi_n f$ converges in quadratic mean to ξf.*
(2) *The set $\{\mathbf{E}\ (\xi_n \otimes \xi_m)\}$ is vaguely relatively compact in $\mathcal{M}_c(\mathbf{S} \times \mathbf{S})$.*

Then ξ_n converge in quadratic mean to ξ.

Proof: Let μ be a measure belonging to the closure of the set $\{\mathbf{E}\ (\xi_n \otimes \xi_m)\}$. Proceeding as in the proof of Theorem 2.2, it follows $\mu h = \lim_{n,m\to\infty} \mathbf{E}\ (\xi_n \otimes \xi_m)h$, for every $h \in H$. So, according to Theorem 2.1, it follows that there exists only one such measure, which proves the theorem. □

Corollary 3.4. *Let $\{\xi_n\}$ be a sequence of random measures and ξ a random measure such that*

(1) *For every $f \in \mathbf{C}_c(\mathbf{S})$, $\xi_n f$ converges in quadratic mean to ξf.*
(2) *There exists a non negative random measure γ such that, for every $n \in \mathbb{N}$, $|\xi_n| \leq \gamma$.*

Then ξ_n converges in quadratic mean to ξ.

Proof: For every $n, m \in \mathbb{N}$,

$$|\mathbf{E}\ (\xi_n \otimes \xi_m)| \leq \mathbf{E}\ |\xi_n \otimes \xi_m| \leq \mathbf{E}\ (\gamma \otimes \gamma).$$

Then, according to [7, Proposition 9.8.7], the set $\{\mathbf{E}\ (\xi_n \otimes \xi_m)\}$ is vaguely relatively compact in $\mathcal{M}_c(\mathbf{S}\times\mathbf{S})$, so this corollary is proved as the preceding theorem. $\square$

If we restrict ourselves to $\mathcal{M}_c^+(\mathbf{S}\times\mathbf{S})$, the fact of supposing $\{\mathbf{E}\ (\xi_n \otimes \xi_m)\}$ vaguely convergent implies the mean quadratic convergence of the sequence $\{\xi_n\}$.

Theorem 3.5. *Let $\{\xi_n\}$ be a sequence of non negative random measures such that $\{\mathbf{E}\ (\xi_n \otimes \xi_m)\}$ is vaguely convergent to $\mu \in \mathcal{M}_c^+(\mathbf{S}\times\mathbf{S})$. Then $\{\xi_n\}$ converges in quadratic mean to a non negative random measure ξ verifying $\mu = \mathbf{E}\ (\xi \otimes \xi)$.*

Proof: For every $f \in \mathbf{C}_c(\mathbf{S})$, $\{\xi_n f\}$ is a Cauchy sequence in quadratic mean, so it is also a Cauchy sequence in probability. Let ρ be the metric corresponding to the vague topology in $\mathcal{M}_c^+$ proposed by Kallenberg [5, page 95]. As this metric depends only on a numerable amount of functions of $\mathbf{C}_c(\mathbf{S})$, it follows that $\{\xi_n\}$ is a Cauchy sequence in probability in the complete metric space $(\mathcal{M}_c^+, \rho)$. Moreover, the space of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ taking values in $(\mathcal{M}_c^+, \rho)$, with the semi-distance

$$\rho'(\xi, \eta) = \inf\{\varepsilon > 0 :\ \mathbf{P}\{\rho(\xi, \eta) \leq \varepsilon\} \geq 1 - \varepsilon\}$$

is also complete. Then $\{\xi_n\}$ converges in probability to a non negative random measure ξ ([9, page 94]). Then, the mean quadratic limit of the sequence $\{\xi_n f\}$ is ξf, for every $f \in \mathbf{C}_c(\mathbf{S})$. Then, according to Theorem 2.1, $\mu = \mathbf{E}\ (\xi \otimes \xi)$ follows. $\square$

Corollary 3.6. *Let $\{\xi_n\}$ be a sequence of non negative random measures such that for every $f \in \mathbf{C}_c(\mathbf{S})$, $\{\xi_n f\}$ converges in quadratic mean. Then the sequence $\{\xi_n\}$ converges in quadratic mean.*

Proof: As the sequence $\{\xi_n f\}$ converges in quadratic mean for every function $f \in \mathbf{C}_c(\mathbf{S})$, it follows that $\{\mathbf{E}\ (\xi_n \otimes \xi_m)h\}$ converges for every $h \in H$. So, according to Theorem 2.2 the quadratic mean convergence of the sequence $\{\xi_n\}$ follows. $\square$

This enables us to prove a generalization of Corollary 3.4.

Corollary 3.7. *Let $\{\xi_n\}$ be a sequence of random measures such that:*

(1) *For every $f \in \mathbf{C}_c(\mathbf{S})$, $\{\xi_n f\}$ converges in quadratic mean.*
(2) *There exists a non negative random measure γ such that for every $n \in \mathbb{N}$, $|\xi_n| \leq \gamma$.*

Then $\{\xi_n\}$ converges in quadratic mean.

Proof: If $\{\xi_n f\}$ converges in quadratic mean, then the non negative random measures $\eta_n = \xi_n + \gamma$ verify the conditions of the preceding corollary, so η_n converges in quadratic mean to a non negative random measure η. So $\{\xi_n f\}$ converges in quadratic mean to $\eta f - \gamma f = \xi f$. According to Corollary 3.4 it follows that $\{\xi_n\}$ converges in quadratic mean to ξ. $\square$

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Université des Sciences et Tèchniques de Lille I, 59655 – Villeneuve d'Ascq, France

Dep. Matemática – Univ. de Coimbra, Apartado 3008 – 3000 Coimbra, Portugal

(*Received* July 18, 1990)