# On $\delta$-continuous selections of small multifunctions and covering properties

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*Abstract.* The spaces for which each $\delta$-continuous function can be extended to a $2\delta$-small point-open l.s.c. multifunction (resp. point-closed u.s.c. multifunction) are studied. Some sufficient conditions and counterexamples are given.

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## 1. Introduction and preliminaries.

E. Michael characterized paracompact spaces by the property that each point-convex closed l.s.c. multifunction/ from the space to some Banach space has a continuous selection. So here we investigate the situation which is a kind of opposite: for a function we are looking for some nice multivalued extension. We will indicate that properties under investigation have some curious features and some related open problems will be mentioned as well. This note extends results from [6].

Let $X, Y$ be topological spaces. A multifunction/ $\varphi : X \to Y$ is a correspondence such that $\varphi(x)$ is a non-empty subset of $Y$ for every $x \in X$. A selection of $\varphi$ is a single-valued map $f : X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.

Now let $(Y, d)$ be a metric space. The multifunction $\varphi : X \to (Y, d)$ is $\delta$-small/ for some $\delta > 0$ if $\operatorname{diam}(\varphi(x)) \leq \delta$ for all $x \in X$ [6]; $f : X \to (Y, d)$ is $\delta$-continuous/ if for every $x \in X$ there exists an open neighborhood $U$ of $x$ such that $f(U) \subset S_\delta(f(x)) = \{y \in Y \mid d(y, f(x)) < \delta\}$ [4], [5], [7]. We set $\overline{S}_\delta(f(x)) = \{y \in Y \mid d(y, f(x)) \leq \delta\}$. Obviously $f$ is continuous iff it is $\delta$-continuous/ for all $\delta > 0$; and $\varphi$ is single-valued iff it is $\delta$-small/ for every $\delta > 0$. Moreover a multifunction/ $\varphi : X \to Y$ is called usc (upper semi-continuous) if for every open set $V \subset Y$ with $\varphi(x) \subset V$ there exists an open neighborhood $U$ of $x$ such that $\varphi(U) \subset V$. It is called lsc (lower semi-continuous) if for every $x \in X$ and for every open set $V \subset Y$ with $\varphi(x) \cap V \neq \emptyset$ there exists an open neighborhood $U$ of $x$ such that $\varphi(x') \cap V \neq \emptyset$ for all $x' \in U$. A point-closed (point-open) multifunction/ is a multifunction/ $\varphi : X \to Y$ for which $\varphi(x)$ is closed (open) for each $x \in X$. For undefined notions see [2] or [3].

## 2. Spaces having the $2\delta$ open lsc extension property.

A space $X$ is said to have the $2\delta$ open lsc extension property if for every $\delta$-continuous/ map $f : X \to (Y, d)$ there exists a point-open lsc $2\delta$-small/ multifunction/ for which $f$ is a selection.

**Theorem 1.** *Every paracompact space has the $2\delta$ open lsc extension property.*

Proof: Let $X$ be paracompact and let $f : X \to (Y,d)$ be $\delta$-continuous/. For each $x_\lambda \in X$ there exists an open neighborhood $U'_\lambda$ of $x_\lambda$ such that $f(U'_\lambda) \subset S_\delta(f(x_\lambda))$. Since $X$ is regular then there exists an open neighborhood $U_\lambda$ of $x_\lambda$ such that $U_\lambda \subset \overline{U}_\lambda \subset U'_\lambda$. The open cover $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of $X$ has an open locally finite refinement $\{V_\lambda\}_{\lambda\in\Lambda}$ such that $V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$. For every $x \in X$ let $J(x)$ be the subset of $\Lambda$ given by $\lambda \in J(x)$ iff $x \in \overline{V}_\lambda$. Since the family $\{\overline{V}_\lambda\}_\Lambda$ is locally finite then every $J(x)$ is finite. The function $f$ is a selection of the multifunction/ $\psi : X \to (Y,d)$ defined by $\psi(X) = \bigcap[S_\delta(f(x_\lambda)) \mid \lambda \in J(x)]$. For each $x \in X$ $\psi(X)$ is open and $\operatorname{diam}(\psi(x)) \leq 2\delta$ by construction, hence $\psi$ is point-open and $2\delta$-small/ by construction. Now let $x \in X$ and let $V$ be an open set such that $V \cap \psi(x) \neq \emptyset$. Since the family $\{\overline{V}_\lambda \mid \lambda \in \Lambda - J(x)\}$ is locally finite we have that $\bigcup\{\overline{V}_\lambda \mid \lambda \in \Lambda - J(x)\}$ is closed, hence there exists an open neighborhood $N(x)$ of $x$ such that $N(x) \cap \left(\bigcup\left[\overline{V}_\lambda \mid \lambda \in \Lambda - J(x)\right]\right) = \emptyset$. Therefore $\psi(x) \subseteq \psi(x')$ for each $x' \in N(x)$, and hence $\psi(x') \cap V \neq \emptyset$. So $\psi$ is also lsc. □

**Example 2** (See [2]). Let $X = (\omega_1 + 1) \times (\omega + 1) - \{(\omega_1, \omega)\}$. For $\alpha \in \omega_1$ let $H_\alpha = \{\alpha\} \times (\omega + 1)$ and for $n \in \omega$ let $V_n = (\omega_1 + 1) \times \{n\}$. The topology on $X$ is defined as follows: all points in $\omega_1 \times \omega$ are isolated, a neighborhood base of $(\alpha, \omega)$ (of $(\omega_1, n)$, resp.) is formed by all cofinite subsets of $H_\alpha$ (of $V_n$, resp.). $X$ is metacompact and subparacompact. $X$ does not have $2\delta$ open lsc extension property. Define $f : X \to \mathbb{R}$ by:

$$
\begin{aligned}
f(\alpha,\omega) &= 0 && \text{for each } \alpha \in \omega_1 \\
f(\alpha,n) &= 1 - \frac{1}{n+1} && \text{for each } (\alpha,n) \in \omega_1 \times \omega \\
f(\omega_1,n) &= 2 - \frac{2}{n+1} && \text{for each } n \in \omega.
\end{aligned}
$$

Clearly $f$ is 1-continuous. Suppose that there exists a point-open and lsc multifunction/ $F$ such that $f$ is a selection of it. Then there exist $T \in [\omega]^{<\omega}, \varepsilon > 0$ and $M \in [\omega_1]^{\omega_1}$ such that for each $\alpha \in M : (-\varepsilon, 0) \subset F((\alpha,\omega))$, and for each $n \in \omega - T : F(n,\alpha) \cap (-\varepsilon, -\frac{7}{8}\varepsilon) \neq \emptyset$. Now take $n \in \omega - T$ such that $n \gg \frac{8}{\varepsilon}$. There is $W \in [\omega_1]^{<\omega}$ such that for each $\alpha \in \omega_1 - W : F(\alpha,n) \cap (2 - \frac{1}{2n}, \infty) \neq \emptyset$. Take $\alpha_0 \in M - W$. Then $\operatorname{diam} F(n,\alpha_0) > 2$.

The next example shows however that the $2\delta$ open lsc extension property cannot characterize paracompact spaces.

**Example 3.** $\omega_1$ has the $2\delta$ open lsc extension property. First we show the following fact: "let $f : \omega_1 \to (Y,d)$ be $\delta$-continuous/. Then there is $\alpha \in \omega_1$ such that for each $\beta \geq \alpha$ there exists $D$, $0 < D < 2\delta$ so that $d(f(\gamma), f(\beta)) \leq D$ for every $\gamma \geq \alpha$". Suppose not. Hence

(∗) for each $\alpha$ there exists $z_\alpha = f(\beta_\alpha), \beta_\alpha > \alpha$, and there exists $y_n^\alpha = f(\gamma_n^\alpha), \gamma_n^\alpha \geq \alpha$, such that $d(z_\alpha, y_n^\alpha) > 2\delta - \frac{1}{2^n}$.

Put $\alpha_0 = 0$. Induction: take $\alpha_n > \sup_\alpha \gamma^{\alpha_{n-1}}$ and use $(*)$ again. Put $\beta = \sup \alpha_n$. As $f$ is $\delta$-continuous/ there is $\lambda, \lambda < \beta$ such that $f((\lambda, \beta]) \subset S_\delta(f(\beta))$. Take $\alpha_i \in (\lambda, \beta]$. Then $d(f(\alpha_i), f(\beta)) = \eta < \delta$. Put $\varrho = \delta - \eta$. Take $n$ so large that $\frac{1}{2^n} \ll \varrho$. Then $d(f(\alpha_i), f(\gamma_n^{\alpha_i})) \leq d(f(\alpha_i), f(\beta)) + d(f(\beta), f(\gamma_n^{\alpha_i})) = \delta - \varrho + d(f(\beta), f(\gamma_n^{\alpha_i})) < 2\delta - \varrho < 2\delta - \frac{1}{2^n}$, a contradiction. It proves our claim. Take now some $\delta$-continuous/ $f : \omega_1 \to (Y, d)$. Find $\alpha \in \omega_1$ as in the claim. For each $\beta \geq \alpha$ let $D$ from the claim be denoted by $D(\beta)$. Put $r_\beta = \frac{1}{8}(2\delta - D(\beta))$ and define $Z = \bigcup\{S_{r_\beta}(f(\beta)) : \beta \in (\alpha, \omega_1)\}$. Then diam $Z \leq 2\delta$. As $[0, \alpha]$ is compact and clopen we can use Theorem 1 on it, for any $\beta > \alpha$, we take $F(\beta) = Z$.

## 3. Spaces having the $2\delta$ closed usc extension property.

A space $X$ is said to have the $2\delta$ closed usc extension property if for every $\delta$-continuous/ map $f : X \to (Y, d)$ there exists a point-closed usc $2\delta$-small/ multifunction/ for which $f$ is a selection. A space $X$ is called orthocompact if every open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that $\bigcap \mathcal{W}$ is open for any $\mathcal{W} \subset \mathcal{V}$.

**Theorem 4.** *Every orthocompact space has the $2\delta$ closed usc extension property.*

Proof: Let $X$ be orthocompact and let $f : X \to (Y, d)$ be $\delta$-continuous/. For each $x_\lambda \in X$ let $U_\lambda$ be an open neighborhood of $x_\lambda$ such that $f(U_\lambda) \subset S_\delta(f(x_\lambda))$. $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $X$. By hypothesis, there is an open refinement $\mathcal{V}\{V_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{U}$ such that $V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$ and $\bigcap \mathcal{W}$ is open for any $\mathcal{W} \subset \mathcal{V}$. For each $x \in X$ let $J(x)$ be the subset of $\Lambda$ given by $\lambda \in J(x)$ iff $x \in V_\lambda$. Define a multifunction/ $\varphi : X \to (Y, d)$ by $\varphi(x) = \bigcap\big[\overline{S}_\delta(f(x_\lambda)) \mid \lambda \in J(x)\big]$. For each $x \in X$ we have $x \in \bigcap\big[V_\lambda \mid \lambda \in J(X)\big]$ then $f(x) \in f\big(\bigcap\big[V_\lambda \mid \lambda \in J(x)\big]\big) \subset \bigcap\big(\big[f(V_\lambda) \mid \lambda \in J(x)\big]\big) \subset \bigcap\big[\overline{S}_\delta(f(x_\lambda)) \mid \lambda \in J(x)\big] = \varphi(x)$, hence $f$ is a selection of the multifunction/. Given $x \in X$ let $V$ be an open set in $Y$ such that $\varphi(x) \subset V$. For each $x' \in N(x) = \bigcap\big[V_\lambda \mid \lambda \in J(x)\big]$, we have $J(x) \subseteq J(x')$. Then $\varphi(x') = \bigcap\big[\overline{S}_\delta(f(x_\lambda)) \mid \lambda \in J(x')\big] \subseteq \big[\overline{S}_\delta(f(x_\lambda)) \mid \lambda \in J(x)\big] = \varphi(x)$, i.e. $\varphi(x') \subseteq \varphi(x)$ for each $x' \in N(x)$, therefore $N(x)$ is an open neighborhood of $x$ such that $\varphi(N(x)) \subseteq \varphi(x) \subset V$, hence $\varphi$ is usc. Moreover for each $x \in X$ $\varphi(x)$ is closed and diam $(\varphi(x)) \leq 2\delta$ by construction, hence $\varphi$ is point-closed and $2\delta$-small/. □

**Example 5.** Let $X = \omega_1 \times (+1)$. This space is not orthocompact (see e.g. [1]) and does not have $2\delta$ closed usc extension property. For

$$\alpha \in (\omega_1 + 1), \text{ define } \quad \begin{cases} I(\alpha) = \{\alpha\} & \text{if } \alpha \text{ is isolated,} \\ I(\alpha) = [0, \alpha] & \text{if } \alpha \text{ is limit.} \end{cases}$$

For $z \in X, z = (\alpha, \beta)$ define $V_z : V_z = \big((\beta, \alpha] \times I(\alpha)\big) \times I(\beta)$ if $\alpha > \beta$, $V_z = I(\alpha) \times I(\beta)$ if $\alpha = \beta$, $V_z = I(\alpha) \times \big((\alpha, \beta] \cap I(\beta)\big)$ if $\alpha < \beta$. $\{V_z : z \in X\}$ is an open cover of $X$. We define a metric $d$ on $X$. Take $m > 0$. For $z \in X$ and $x \in V_z$, put $d(z, x) = m$. For other couples, we define the distance using a standard technique of chains, i.e.:

$$d(z, x) = \inf\{\Sigma\{d(a_i^c, a_{i+1}^c) : i \in I_c\} : a_0^c = z,\ a_{\mathrm{rend}\, x I_c}^c = x,\ d(a_i^c, a_{i+1}^c) = m\}.$$

If there is no such a chain, we put $d(z,x) = 10m$. (So we can associate in the above manner a metric $d(\mathcal{P},\varphi,m)$ on $X$ to any open cover $\mathcal{P}$ of $X$, any map $\varphi : X \to \mathcal{P}$ such that $x \in \varphi(x)$ and any real number $m > 0$.) If we take $\alpha$ isolated, $a < \alpha$ then $d((\alpha,\omega_1),(a,\alpha)) = 3m$ $(*)$. Define $f : X \to (X,d)$ by $f(x) = x$. Clearly, for any $\varepsilon > 0$, $f$ is $m(1+\varepsilon)$-continuous. Take some very small $\varepsilon$ (e.g. $\varepsilon < \frac{1}{4}$). Assume there is a multifunction/ $F : X \to (X,d)$ point-closed, $2m(1+\varepsilon)$-small and usc such that $f$ is a selection of it. As the topology of $(X,d)$ is discrete, $B_z = \{x \in X : F(x) \subseteq F(z)\}$ is open in $X$ for each $z \in X$. For each $z \in X$, choose a basic open set $W_z \subseteq B_z \cap V_z$. Notice that $W_z \subset F(z)$ as $f \subseteq F$. Take $\alpha \in \omega_1$ limit. Then $W_{(\alpha,\omega_1)} = (s(\alpha),\alpha] \times (t(\alpha),\omega_1]$. There is a stationary set $M \subset \omega_1$ such that $s(\alpha) = s$ for each $\alpha \in M$. Put $p = s+1$. Then $F(p,\omega_1) \subset F(\alpha,\omega_1)$ for each $\alpha \in M$. We know that $W_{(p,\omega_1)} = \{p\} \times (\gamma,\omega_1]$. When we take $\alpha \in M$ such that $\alpha > \gamma$ then $\operatorname{diam} F(\alpha,\omega_1) \geq 3m$ by $(*)$ (take an isolated $\beta, \beta \in (\gamma,\alpha)$ then $(\beta,\omega_1) \in F(\alpha,\omega_1), (\gamma,\beta) \in F(\alpha,\omega_1)$), a contradiction.

**Example 6.** Let $X = (\omega_1+1) \times \omega_1$. The topology on $X$ is defined as follows: all points of $\omega_1 \times \omega_1$ are isolated. A basic neighborhood $U(\beta,F)$ of $(\omega_1,\alpha)$ is given by $U(\beta,F) = \{\omega_1\} \times (\beta,\alpha] \bigcup \{\{\gamma\} \times (\beta,\alpha] : \gamma \in \omega_1 - F\}$, where $\beta < \alpha$ and $F \in [\omega_1]^{<\omega}$. It is shown in [3] that $X$ is not orthocompact though it is a continuous closed image of an orthocompact space (as shown in [1], even a perfect image of an orthocompact space). However $X$ has the $2\delta$ closed usc extension property. In fact let $f : X \to (Y,d)$ be $\delta$-continuous/. Then there is $Z \in [\omega_1]^{<\omega}$ and $\alpha < \omega_1$ such that $\operatorname{diam}\big(f(\omega_1+1-Z) \times (\alpha,\omega_1)\big) \leq 2\delta$ (if not, then we find $a_n, b_n \in \omega_1$ such that $d(f(a_n),f(b_n)) > 2\delta$ and $\{(a_n,b_n) : n \in \omega\}$ converges to some $(\omega_1,\gamma) \in X$, a contradiction). Put $M = (\omega_1+1-Z) \times (\alpha,\omega_1)$. Notice that $M$ is a clopen subset of $X$. For $x \in M$ define $F(x) = \operatorname{cl}_d f(M)$ for $X - M$ we can use the fact that $X - M$ is even paracompact.

**Remark 7.** If the space $X$ has the $2\delta$ closed usc extension property then let us take any open cover $\mathcal{P}$ of $X$. Pick up some $\varphi : X \to \mathcal{P}$ with $x \in \varphi(x)$. Let us consider $d(\varphi,1,\mathcal{P})$. Then for any $\varepsilon > 0, \mathrm{id} : X \to (X,d)$ is $(1+\varepsilon)$-continuous. So there is a point-closed usc $(2+2\varepsilon)$-small multifunction/ $F : X \to (X,d)$ such that $f$ is a selection of it. Using again that $(X,d)$ is discrete, we obtain for each $x \in X, W_x = \{z \in X : F(z) \subseteq F(x)\}$ is open. Put $\mathcal{W} = \{W_x : x \in X\}$. Clearly if $x \in \bigcap\{W_z : z \in M\}$ for some $M \subset X$ then $W_x \subseteq W_z$ for each $z \in M$, hence any subfamily of $\mathcal{W}$ has an open intersection. We can use the fact that $\operatorname{diam} F(x) \leq 2$ for each $x \in X$ to obtain properties weaker than orthocompactness, e.g.
(1) for every open cover $\mathcal{P}$ of $X$ there exists an open cover $\mathcal{W}$ of $X$ such that $\mathcal{W} \prec \{\mathrm{st}^2(x\mathcal{P}) : x \in X\}$ and it is closed under all intersections (of course, this property is quite far from orthocompactness, all countably compact spaces possess it);
(2) for every open cover $\mathcal{P}$ of $X$ there exists an open cover $\mathcal{W}$ of $X$ such that: $\mathcal{W}$ is closed under all intersections and $\mathcal{W} \prec \mathcal{Z}$ where $\mathcal{Z} = \{Z \subset X : Z \subset \mathrm{st}^2(z,\mathcal{P})$ for each $z \in Z\}$.
(3) the literal translation of our construction: for every open cover $\mathcal{P}$ of $X$ and $\varphi : X \to \mathcal{P}$ such that $x \in \varphi(x)$ there exists an open cover $\mathcal{W}$ of $X$ such that $\mathcal{W}$

is closed under all intersections and $\mathcal{W} \prec \mathcal{T}$ where $\mathcal{T} = \{T_x : x \in X\}$ and $T_x$ has the following property: $\widetilde{T}_x = \bigcup\{\varphi(z) : z \in X \text{ and } x \in \varphi(z)\} \cup \bigcup\{\varphi(u) : u \in X \text{ and there is } z \in \varphi(u) \text{ such that } x \in \varphi(z)\}$ then $T_x \subset \bigcap\{\widetilde{T}_z : z \in T_x\}$. We see that it leads to some kind of starwise version of orthocompactness. Nevertheless, we are not able to show that pointwise star-orthocompact spaces (see [3]) have the $2\delta$ closed usc extension property (clearly by [1, Theorem 3] and Theorem 4 it is the case if subparacompactness is assumed).

**Open problems.**
1. Does the $2\delta$ open lsc extension property imply collectionwise normality or normality?
2. Is the $2\delta$ closed usc extension property preserved by continuous closed or perfect maps?

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