On star covering properties related to countable compactness and pseudocompactness

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Abstract. We prove a number of results on star covering properties which may be regarded as either generalizations or specializations of topological properties related to the ones mentioned in the title of the paper. For instance, we give a new, entirely combinatorial proof of the fact that Ψ-spaces constructed from infinite almost disjoint families are not star-compact. By going a little further we conclude that if X is a star-compact space within a certain class, then X is neither first countable nor separable. We also show that if a topological space is pseudonormal and has countable extent, then its Alexandroff duplicate satisfies property (a). A number of problems and questions are also presented.

Keywords: star-compact spaces; spaces star determined by a finite number of convergent sequences; (a)-spaces; selectively (a)-spaces

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1. Introduction

If X is a set, $\mathcal{U}$ is a family of subsets of X and $A \subseteq X$, then the *star of A with respect to $\mathcal{U}$* is the subset of X given by

$$\mathrm{St}(A,\mathcal{U}) := \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

In this paper we work with *star covering properties* — i.e., topological properties defined in terms of stars with respect to open covers. Indeed, several topological properties were defined and/or characterized in this way in the last 25 years (see, e.g., [6] and [13]).

More recently, van Mill, Tkachuk and Wilson ([14]) have introduced the notion of *star-$\mathcal{P}$* spaces, where $\mathcal{P}$ denotes a topological property.

Definition 1.1 ([14])**.** Let X be a topological space.

(i) Given an open cover $\mathcal{U}$ of X, a subset $A \subseteq X$ is said to be a **star kernel of $\mathcal{U}$** if $\mathrm{St}(A,\mathcal{U}) = X$.

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(ii) If $\mathcal{P}$ is a topological property, X is said to be a **star-$\mathcal{P}$ space** if every open cover $\mathcal{U}$ of X has a star kernel which satisfies $\mathcal{P}$. □

In this paper we will be interested on the investigation of star covering properties which may be regarded as either generalizations or specializations of topological properties related to the ones mentioned in the title of the paper; the well-known relationships between those star covering properties and the properties of countable compactness and pseudocompactness will be remembered in its due time. Besides of star-compact and related spaces, we will also investigate Matveev's *property* (a) ([12]), as well as its *selective version*, recently introduced by Caserta, Di Maio and Kočinac ([3]).

The cardinal functions we deal with within this paper are: the *extent* of X, denoted $e(X)$, which is the supremum of the cardinalities of all closed discrete subsets of X, provided this is an infinite cardinal, or is $\aleph_0$ otherwise; the *character* of X, denoted $\chi(X)$, which is the smallest infinite cardinal κ such that every point of the space has a local base of size not larger than κ; and the *pseudocharacter* of X, denoted $\psi(X)$, which is the smallest infinite cardinal κ such that for every point x of X there is a family $\mathcal{V}_x$ of open neighbourhoods of x satisfying $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \mathcal{V}_x = \{x\}$. More information on these and many other cardinal functions may be found in [8].

Throughout this paper, all spaces are assumed to be T_1 topological spaces. So, any open cover of any space in this paper has a kernel which is a closed discrete set.

Let us describe the organization of this paper. In Section 2 we give a new, entirely combinatorial proof that infinite Mrówka-Isbell spaces are not star-compact; our proof uses *dominating families* of functions. We go further and show that star-compact spaces in a certain class (which is strictly larger than the class of pseudocompact spaces constructed from almost disjoint families) are neither first countable nor separable; in particular, there are no second countable star-compact spaces in such class. In Section 3 we strengthen a result of Song by showing that if a topological space is pseudonormal and has countable extent then its Alexandroff duplicate satisfies property (a). In Section 4 we present some notes, questions and problems on (a)-spaces and selectively (a)-spaces.

2. Restrictions on the presence of star-compactness

2.1 Star-cs and related spaces. Throughout this paper, a subset S of a topological space X will be said to be a *convergent sequence* if S is of the form $S = \{x_n : n < \omega\} \cup \{x\}$ and the sequence $\langle x_n : n < \omega\rangle$ converges to x, meaning that for every open neighbourhood U of x the set $\{n < \omega : x_n \notin U\}$ is finite.

As it is (maybe unfortunately) usual in the literature of star covering properties, sometimes a certain property has received different names, accordingly to different authors. The following property was investigated by Song in [20] as *cs-starcompactness*, and was referred to in [14] as the property of *being star determined by a convergent sequence*. We will prefer the following terminology:

Definition 2.1. A topological space X is said to be **star-cs** if every open cover of X has a star kernel which is a convergent sequence.

It follows that every star-cs space is star-compact. A related notion is the following:

Definition 2.2. A topological space X is said to be **star-finite-cs**, or **star determined by a finite number of convergent sequences**, if every open cover of X has a star kernel which is a finite union of convergent sequences.

Star-finite-cs spaces are also star-compact, of course.

Properties which are versions of star-compactness are often viewed as intermediate "steps" between countable compactness and pseudocompactness (as previously discussed in [13] and [14]). Indeed, it is well-known that, for Hausdorff spaces, X is countably compact if and only if X is star-finite (see [7, Exercise 3.12.23(d)]), and, as mentioned in [14], it is a part of the folklore that, for Tychonoff spaces, a space is pseudocompact if and only if it is star-pseudocompact. So, for Tychonoff spaces we have the following:

$$\begin{gathered}\text{Countably compact} \iff \text{Star-finite} \Rightarrow \text{Star-cs} \Rightarrow \\ \text{Star-finite-cs} \Rightarrow \text{Star-compact} \Rightarrow \text{Pseudocompact.}\end{gathered}$$

In Section 3 of [14] one may find several examples showing that the above implications are not reversible.

2.2 Results on Ψ-spaces. A family $\mathcal{A}$ of infinite subsets of ω, *i.e.* $\mathcal{A} \subseteq [\omega]^\omega$, is said to be an *almost disjoint family* if for every distinct $A, B \in \mathcal{A}$ one has $|A \cap B| < \aleph_0$. A *MAD family* is an almost disjoint family which is maximal (in the sense of inclusion).

For an almost disjoint family $\mathcal{A}$, we consider the so-called *Mrówka-Isbell space*, or Ψ*-space*, denoted $\Psi(\mathcal{A})$, whose underlying set is $\omega \cup \mathcal{A}$ and we take as basic open sets the singletons of the elements of ω and all sets of the form $\{A\} \cup (A \setminus F)$, where $A \in \mathcal{A}$ and F is a finite subset of ω. Mrówka-Isbell spaces are classical objects from Set Theoretic Topology and have provided, since the 50's, several nice examples and counterexamples for a large number of topological properties (see, e.g., [5]). Just for mentioning, let us recall that an almost disjoint family $\mathcal{A}$ is maximal if and only if the corresponding $\Psi(\mathcal{A})$ is pseudocompact ([16]) — so infinite MAD families naturally provide examples of pseudocompact spaces which are neither normal nor countably compact.

As star-compactness implies pseudocompactness, we conclude that if we want to investigate questions such as "When is a Ψ-space star-compact? Star-finite-cs? Star-cs?" then we have to consider MAD families. It turned out that the question for star-compactness was already settled in [14]: in Proposition 3.4 op.cit., it is shown that a MAD family provides, via the Ψ-space construction, a Tychonoff first-countable pseudocompact space which is not star-compact (nor star determined by countably compact spaces). The proof given in [14], however, depends on some results presented in [6] which involves another star covering property, which we will not define here (namely, certain combinatorial results on "1-starcompact" spaces are used in the argument, together with some strictly topological facts; the interested reader may see Lemma 2.2.4 of [6]). We will present here an alternative, purely combinatorial proof of this result, using *dominating families* — and we believe that our alternative proof may bring some fresh techniques for the research on star covering properties, and that's the reason why we include such proof here.

Before presenting our proof, we remark that it is easy to check, for any almost disjoint family $\mathcal{A}$ and $K \subseteq \Psi(\mathcal{A})$, the equivalence of the statements "K is compact", "K is countably compact" and "K is a finite union of convergent sequences". Also, we remark that, in what follows, maximality of $\mathcal{A}$ need not to be hypothesized — only infiniteness.

Theorem 2.3. *Let $\mathcal{A}$ be an arbitrary almost disjoint family. If $\mathcal{A}$ is infinite, then $\Psi(\mathcal{A})$ is **not** star determined by a finite number of convergent sequences.*

Proof: Let $\mathcal{H} \subseteq \mathcal{A}$ be a finite subset of $\mathcal{A}$ (i.e., $\mathcal{H} \in [\mathcal{A}]^{<\omega}$), and consider some fixed $X \in \mathcal{A}$. Let $X_{\mathcal{H}}$ be the set

$$X_{\mathcal{H}} = \{0\} \cup \{\sup(X \cap H) : H \in \mathcal{H} \setminus \{X\}\}.$$

Notice that $X_{\mathcal{H}} = \{0\}$ if $X \cap H = \emptyset$ for all $H \in \mathcal{H} \setminus \{X\}$ (including the vacuously true case on which $\mathcal{H} = \{X\}$).

For every $\mathcal{H} \in [\mathcal{A}]^{<\omega}$ and $F \in [\omega]^{<\omega}$, let $f_{\mathcal{H},F} : \mathcal{A} \to \omega$ be defined as follows: for each $A \in \mathcal{A}$,

$$f_{\mathcal{H},F}(A) = \max(A_{\mathcal{H}}) + \sup(F) + 1.$$

Consider the family $\mathcal{F} = \{f_{\mathcal{H},F} \colon \mathcal{H} \in [\mathcal{A}]^{<\omega}, F \in [\omega]^{<\omega}\}$. We now show that, assuming that $\Psi(\mathcal{A})$ is star determined by a finite number of converging sequences, then $\mathcal{F}$ is a dominating family in the mod finite order $\langle {}^{\mathcal{A}}\omega, \leq^* \rangle$.

A family of functions is dominating if and only if it dominates all functions with unbounded image, so it suffices to check that $\mathcal{F}$ dominates any unbounded function. So, let $g \in {}^{\mathcal{A}}\omega$ be a function with unbounded image.

Consider the open cover of $\Psi(\mathcal{A})$ given by

$$\mathcal{U} = \big\{\{A\} \cup (A \setminus g(A)) : A \in \mathcal{A}\big\} \cup \{\{n\} : n \in \omega\}.$$

By our assumption on the Ψ-space, there are convergent sequences $S_1, S_2, \dots, S_n$ such that $\mathrm{St}(S_1 \cup S_2 \cup \cdots \cup S_n, \mathcal{U}) = \Psi(\mathcal{A})$. As we are assuming that $g \in {}^{\mathcal{A}}\omega$ has unbounded image, at least one of these sequences must have infinite image. By rearranging, if necessary, we may assume, without loss of generality, that all the sequences $S_1, S_2, \dots, S_n$ have infinite image with limits $A_1, A_2, \dots, A_n \in \mathcal{A}$, respectively.

Notice that, as $S_i \to A_i$ for $1 \le i \le n$, the set

$$S' = \bigcup_{i=1}^{n} \big((S_i \setminus \mathcal{A}) \setminus A_i\big)$$

is a finite union of finite sets, and so $S' \in [\omega]^{<\omega}$.

Consider $S = S_1 \cup S_2 \cup \cdots \cup S_n$, $\mathcal{H} = \{A_1, A_2, \ldots, A_n\}$ and $\mathcal{F} = S \cap \mathcal{A}$; notice that $\mathcal{F}$ is a finite set (since a convergent sequence cannot include an infinite set without accumulation points) and $\mathcal{H} \subseteq \mathcal{F}$. Let $B \in \mathcal{A} \setminus \mathcal{F}$. As S is the kernel of the cover, and there is only one open set in the cover which contains B, it follows that $S \cap (B \setminus g(B)) \neq \emptyset$, and therefore exists at least a sequence S_j, $1 \le j \le n$, with $S_j \to A_j$ and such that $S_j \cap (B \setminus g(B)) \neq \emptyset$. If $A_j \cap (B \setminus g(B)) \neq \emptyset$ then

$$g(B) < \sup(A_j \cap B) \le \max(B_{\mathcal{H}}) < f_{\mathcal{H},S'}(B).$$

If $S_j \cap (B \setminus g(B)) \neq \emptyset$ but $A_j \cap (B \setminus g(B)) = \emptyset$, then we necessarily have $S' \cap (B \setminus g(B)) \neq \emptyset$ and therefore

$$g(B) < \max(S' \cap B) \le \max(S') < f_{\mathcal{H},S'}(B).$$

In any case we conclude that $\{A \in \mathcal{A} : f_{\mathcal{H},S'}(A) < g(A)\} \subseteq \mathcal{F}$ and so $g \le^* f_{\mathcal{H},S'}$.

We have just proved that $\mathcal{F}$ is a dominating family of size $|\,[\mathcal{A}]^{<\omega} \times [\omega]^{<\omega}| = |\,\mathcal{A}|$ in the family of functions $\langle {}^{\mathcal{A}}\omega, \le^* \rangle$ — but, this is absurd, since there is no dominating family in $\langle {}^{\mathcal{A}}\omega, \le^* \rangle$ of size $|\mathcal{A}|$, by a standard diagonal argument[1]. $\square$

It is well-known (see, e.g., 11.2 of [5]) that Ψ-spaces are precisely the Hausdorff, first-countable, locally compact, separable spaces whose sets of non-isolated points are non-empty and discrete — meaning that for any topological space X with such a list of properties, there is an almost disjoint family $\mathcal{A}$ such that X and the corresponding Ψ-space are homeomorphic.

Considering the above mentioned list of properties which characterize the Mrówka-Isbell spaces, we are now able to prove a corollary which shows that star-compact spaces within a certain class — which is much more larger than the class of pseudocompact Ψ-spaces — are neither first countable nor separable.

In what follows, "regular" means, as usual, "Hausdorff + T_3". For a given topological space X, the *derived set* X' is the set of its non-isolated points. It is easy to see that, if X' is discrete, then it necessarily has empty interior, and in such case its complement — the set of isolated points $D = X \setminus X'$ — is a dense subset of X.

[1] For the convenience of the reader, we give here such diagonal argument. Let λ be any infinite cardinal. As the cofinalities of $\langle {}^{\lambda}\omega, \le^* \rangle$ and $\langle {}^{\lambda}\omega, \le \rangle$ coincide (by a very general result due to Comfort — see [4]), it suffices to show that there is no dominating family of size λ in $\langle {}^{\lambda}\omega, \le \rangle$. Let $\mathcal{F} = \{f_\alpha : \alpha < \lambda\} \subseteq \langle {}^{\lambda}\omega, \le \rangle$ be an arbitrary λ-sized family of functions from λ into ω. Then, the diagonally constructed function $g : \lambda \to \omega$ defined by $g(\alpha) = f_\alpha(\alpha) + 1$ shows that $\mathcal{F}$ is not a dominating family in the pointwise defined order.

Corollary 2.4. *Let $\mathcal{C}$ denote the class of all regular spaces X such that their derived sets X' are infinite, discrete and satisfy $|X| = |X'|$. If $X \in \mathcal{C}$ is a star-compact space, then X is neither first countable nor separable. In particular, there are no second countable star-compact spaces in the class $\mathcal{C}$.*

Proof: The desired will follow from three claims. For any $X \in \mathcal{C}$, we will use the notation $D = X \setminus X'$ for its dense set of isolated points. Notice that, as we are assuming that $|X| = |X'|$, one has $|D| \leq |X'|$.

Claim 1. If $X \in \mathcal{C}$ (regardless of any other assumption), then X is zero-dimensional and each point of X' has a local base of clopen neighbourhoods which are normal subsets of X.

Indeed, by discreteness of X' and regularity of X, every point $x \in X'$ has a base of closed neighbourhoods of the form $V = \{x\} \cup U$, where $U \subseteq D$. So, these neighbourhoods are clearly clopen, and such neighbourhoods are normal subspaces of X because, given any two disjoint closed subsets of them, at least one of these closed sets does not contain x — so it is also a clopen set.

Claim 2. If $X \in \mathcal{C}$ is star-compact, then X is not separable.

Suppose towards a contradiction that X is separable. As D is a dense set of isolated points, we deduce that D is countable. By star-compactness we get pseudocompactness, so those clopen neighbourhoods of points in X' which are normal subspaces of X are necessarily countably compact; but they are countable, and, therefore, X is, in fact, locally compact. As already remarked in the first claim, for a given $x \in X'$ such countable, compact neighbourhoods are of the form $V = \{x\} \cup U$, where $U \subseteq D$. It is clear that the pseudocharacter of x in X is the pseudocharacter of x in any such V, and it should also be clear that, as these neighbourhoods are countable sets, we are in a situation where all points have countable pseudocharacter. As the cardinal functions pseudocharacter and character coincide in the case of compact Hausdorff spaces (which is the case for those countable, compact neighbourhoods of points in X'), we conclude that X is first countable. But, summing up all conclusions made, we get that X is a star-compact, Hausdorff, first countable, locally compact, separable space with the set of non-isolated points infinite and discrete — and so it is homeomorphic to an infinite star-compact Ψ-space, which is an absurd by the preceding theorem.

Claim 3. If $X \in \mathcal{C}$ is star-compact, then it cannot satisfy the first axiom of countability.

Let us check what happens if one assumes that a star-compact $X \in \mathcal{C}$ is first-countable. In this case, D does not need to be a countable set, but, with the very same arguments of Claim 2 (by pseudocompactness, etc.), we conclude that X is locally countably compact. With the first axiom of countability at hand, we could fix, for every $x \in X'$, a decreasing local base of clopen, normal, countably compact neighbourhoods of x, say $\mathcal{V}_x = \{V_{x,n} : n < \omega\}$, where $V_{x,n} = \{x\} \cup A_{x,n}$,

$A_{x,n} \subseteq D$, for every $n < \omega$. But we clearly have, for any $x \in X'$,

$$A_{x,0} = \bigcup_{n<\omega} A_{x,n} \setminus A_{x,n+1},$$

and therefore each one of the sets $A_{x,0}$ is countable, since it is possible to write them as a countable union of finite sets — notice that $A_{x,n} \setminus A_{x,n+1} = V_{x,n} \setminus V_{x,n+1}$, and the latter is a closed discrete subset of a countably compact subspace of X. The same reasoning gives us that $A_{x,0}$ converges to x, and therefore $\mathcal{A} = \{A_{x,0} : x \in X'\}$ is an almost disjoint family of countable subsets of D. In this case, identifying x with $A_{x,0}$ for every $x \in X'$, we have that X is essentially a space of the form $\Psi(D, \mathcal{A})$, in the terminology of Section 11 of [5]; such spaces are very similar to the usual Ψ-spaces — points of D are isolated, basic neighbourhoods of points in $\mathcal{A}$ are defined in the expected way — but they are not separable if D is an uncountable set. Nevertheless, one can proceed with easy adaptations through the proofs of either Theorem 2.3 or of Proposition 3.4 of [14] to conclude that no pseudocompact $\Psi(D, \mathcal{A})$ space could satisfy star-compactness (for instance, one could check — arguing towards a contradiction — that compact subsets of a space of the form $\Psi(D, \mathcal{A})$ are finite unions of convergent sequences as well, and then proceed as in Theorem 2.3 to get a dominating family of size $|\mathcal{A}| = |\mathcal{A} \times [D]^{<\omega}|$ in $\langle {}^{\mathcal{A}}D, \leq^* \rangle$, and this would be an absurd since the very same diagonal argument applies). So, if $X \in \mathcal{C}$ is star-compact, then it cannot be first countable — because first countable star-compact spaces in the class $\mathcal{C}$ would be spaces of the form $\Psi(D, \mathcal{A})$, and these cannot satisfy star-compactness. □

3. On (a), selectively-(a) and related spaces

Property (a) was introduced by Matveev in the late 90's, and its selective version was more recently introduced by Caserta, Di Maio and Kočinac.

Definition 3.1 ([12]). A topological space X satisfies **property** (a) (or is said to be an (a)**-space**) if for every open cover $\mathcal{U}$ of X and for every dense set $D \subseteq X$ there is a set $F \subseteq D$ which is (i) closed and discrete in X; and (ii) a star kernel of $\mathcal{U}$.

Definition 3.2 ([3]). A topological space X is said to be a **selectively** (a)**-space** if for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers and for every dense set $D \subseteq X$ there is a sequence $\langle A_n : n < \omega \rangle$ of subsets of D which are closed and discrete in X and such that $\{\mathrm{St}(A_n, \mathcal{U}_n) : n < \omega\}$ covers X.

Property (a) was introduced by Matveev in [12] as a kind of generalization of a specialization of countable compactness. Let us explain this quaint turn of phrase: first, recall that countable compactness is equivalent to star-finiteness for Hausdorff spaces. In [11], Matveev defined the notion of *absolutely countably compact space* in the following way: X is an absolutely countably compact space if for every open cover $\mathcal{U}$ and for every dense set $D \subseteq X$ there is a finite $F \subseteq D$ such that $\mathrm{St}(F, \mathcal{U}) = X$. For T_1 countably compact spaces, "closed and discrete" means the same as "finite" — and in this way property (a) is exactly what we have to add to a T_1 countably compact space for it to become an absolutely countably compact space; the previous remark explains the choice of the "(a)" in this context (that is, (a) is for absoluteness).

The reader may find a collection of results on (a)-spaces and selectively (a)-spaces in [12], [9], [3] and [21]. It is worthwhile remarking that consistent combinatorial hypotheses provide the consistency of the non-existence of uncountable selectively (a)-spaces from almost disjoint families (see [18] and [15]; for related results on separable, locally compact selectively (a)-spaces in general, see [19]). A survey of recent results on *star selection principles* may be found in [10].

It is very usual, in the star covering properties literature, the research programme of looking for conditions (either in a topological space or in a combinatorial structure) under which, after carrying out a specific construction, we get examples of topological spaces satisfying a given topological star covering property (or, more recently, a star selection principle). In the case of almost disjoint families, Szeptycki and Vaughan investigated and characterized the almost disjoint families for which the corresponding Ψ-space satisfy property (a) ([22]), and the third author have did the same for the selective version of property (a) ([18]).

Besides of the Mrówka-Isbell spaces, another construction which is widely investigated in this context is that of *Alexandroff duplicates*. Given a topological space X, the classical and well-known construction of the Alexandroff duplicate, $\mathrm{AD}(X)$, proceeds as follows: the underlying set of the topological space $\mathrm{AD}(X)$ is $X\times\{0,1\}$. All points of $X\times\{1\}$ are declared isolated, and the basic neighborhoods of a point $\langle x,0\rangle \in X\times\{0\}$ are the sets of the form $(U\times\{0\})\cup((U\times\{1\})\setminus\{\langle x,1\rangle\})$, where U is an open neighbourhood of x in X.

In [21], it is shown that whenever X is a normal, selectively (a)-space with countable extent, then $\mathrm{AD}(X)$ is selectively (a) (Theorem 2.10). Here we present a considerable strengthening of such result. Recall that a space is *pseudonormal* if disjoint closed sets are separated by disjoint open sets, provided one of the closed sets is countable — or, equivalently, X is pseudonormal if countable closed sets have arbitrarily small closed neighbourhoods. One can easily check that pseudonormal spaces with countable extent satisfy the hypothesis $(*)$ of the following theorem:

Theorem 3.3. *Let X be a topological space and suppose X satisfies the following property.*

$(*)$ *If F is a closed discrete subset of X, then F admits a locally finite, disjoint open expansion.*

Under this assumption, $\mathrm{AD}(X)$ *satisfies property* (a).

In fact, for pseudonormal spaces with countable extent the locally finite, disjoint open expansion of $(*)$ can be taken to be a discrete one[2].

[2]In [5], page 155, it is shown that, for regular spaces, a countable closed discrete subset has arbitrarily small closed neighbourhoods if and only if there is a discrete family of open sets which

So, immediately from the statement of the previous theorem we deduce the following:

Corollary 3.4. *If X is pseudonormal and has countable extent, then* $\mathrm{AD}(X)$ *satisfies property* (a).

The following easy remark will be also useful for us:

Remark 3.5. If F is a closed discrete subset of a topological space X, then $F \times \{0\}$, $F \times \{1\}$ and $F \times 2$ are closed discrete subsets of $AD(X)$.

Notice that, as we are assuming that all spaces are T_1, the third statement above follows from the first two. Indeed, it is easily checked that X is T_1 if and only if $\mathrm{AD}(X)$ is T_1, and for T_1 spaces a finite union of closed discrete subsets is also a closed discrete subset.

Proof of Theorem 3.3: Let $2 = \{0,1\}$, $\mathrm{AD}(X) = X \times 2$ and $I_X = X \setminus X'$ be the set of all isolated points of X. Consider $D = I_X \times \{0\} \cup X \times \{1\}$; it is easy to see that D is a dense subset of $\mathrm{AD}(X)$ which is included in any other dense subset of $\mathrm{AD}(X)$. Let $\mathcal{C}$ be an arbitrary open cover of $\mathrm{AD}(X)$. We will show that there exists $\mathcal{C}'$ refinement of $\mathcal{C}$ and $F \subseteq D$ which is a closed discrete subset of X such that $\mathrm{St}(F, \mathcal{C}') = \mathrm{AD}(X)$.

In fact, it suffices for us to exhibit some $G \subseteq D$, with G being a closed discrete subset of $\mathrm{AD}(X)$, such that G satisfies $\mathrm{St}(G, \mathcal{C}') \supseteq X \times \{0\}$ — in this case, $G' = \mathrm{AD}(X) \setminus \mathrm{St}(G, \mathcal{C}')$ will be a closed subset of $\mathrm{AD}(X)$ included in $X \times \{1\}$ (therefore, closed and discrete in the duplicate), and then $F = G \cup G'$ will perform the desired job.

Let us construct our refinement. For each non-isolated $z \in X$, consider an open neighbourhood C_z of $\langle z, 0\rangle$ (in $\mathrm{AD}(X)$) with $C_z \in \mathcal{C}$, and fix an open neighbourhood V_z of z (in X) such that

$$\langle z, 0\rangle \in (V_z \times 2) \setminus \{\langle z, 1\rangle\} \subseteq C_z.$$

If z is an isolated point of X, fix $V_z = \{z\}$, and consider the refinement of $\mathcal{C}$ given by

$$\mathcal{C}' = \{(V_z \times 2) \setminus \{\langle z, 1\rangle\} : z \notin I_X\} \cup \{\{\langle z, 0\rangle\} : z \in I_X\} \cup \{\{\langle z, 1\rangle\} : z \in X\}.$$

As $\mathcal{V} = \{V_z : z \in X\}$ is open cover of X, then, by T_1, there exist $A \subseteq X$ closed and discrete such that $\mathrm{St}(A, \mathcal{V}) = X$.

We have already noticed that $X \times \{0\}$ is the "important" part to be covered by a closed discrete subset of $\mathrm{AD}(X)$ included in D. Regarding the set $A \times \{1\}$ — a subset of D which is closed and discrete in X, by our previous remark — we are in position of being quite specific about the points in $X \times \{0\}$ which are not covered by $\mathrm{St}(A \times \{1\}, \mathcal{C}')$:

separates its points. As pseudonormal spaces are clearly regular, the desired follows. Notice that, in particular, pseudonormal spaces are $\aleph_0$-collectionwise Hausdorff.

Claim. If y is a point of X such that $\langle y,0\rangle \notin \mathrm{St}(A\times\{1\},\mathcal{C}')$, then there is some $a\in A$ such that $y\in V_a$.

Indeed, if $y\in A$, obviously (for $y=a$) the statement holds, regardless of what happens to $\langle y,0\rangle$; this includes the cases where $y\in A\cap I_X$ and $V_y=\{y\}$. So, suppose $y\notin A$. Now, regardless of y being isolated or not, there should be some non-isolated $x\in X$ such that $y\in V_x$ and $V_x\cap A\neq\emptyset$. However, notice that if $x\notin A$ then $(V_x\times 2)\setminus\{\langle x,1\rangle\}$ would intersect $A\times\{1\}$ — but this contradicts $\langle y,0\rangle\notin \mathrm{St}(A\times\{1\},\mathcal{C}')$. So, *a fortiori* we have $x\in A$ and we can pick $a=x$.

Now, let

$$A'=\{z\in A:\ \exists y\in X[\langle y,0\rangle\in \mathrm{AD}(X)\setminus \mathrm{St}(A\times\{1\},\mathcal{C}')\ \wedge\ (y\in V_z)]\}.$$

A' is clearly closed discrete *in* X (since it is a subset of A). By $(*)$, A' admits a locally finite, disjoint open expansion $\{W_z : z\in A'\}$.

For $z\in A'$, we have two cases to be considered:

(1) if z is isolated in X, then $z\in A'\cap I_X$ — which is also a closed discrete subset of X — and we have $V_z=\{z\}=W_z$; and
(2) if z is not isolated in X, fix a point $b_z\in(W_z\setminus\{z\})\cap V_z$ and consider the set: $B=\{b_z : z\in A'\setminus I_X\}$. The local finiteness of $\{W_z : z\in A'\}$ gives us that B is closed discrete *in* X (recall that X is supposed to be T_1).

Now it is easy to check that our construction ensures that

$$X\times\{0\}\subseteq \mathrm{St}\Big(\big((A'\cap I_X)\times\{0\}\big)\cup\big((A\cup B)\times\{1\}\big),\mathcal{C}'\Big),$$

and so the closed discrete subset of D given by

$$G=\big((A'\cap I_X)\times\{0\}\big)\cup\big((A\cup B)\times\{1\}\big)$$

does precisely what suffices for us. □

4. Notes, questions and problems

In what follows, we propose a number of questions and problems about selectively (a)-spaces; some of them resemble previous questions and problems posed for (a)-spaces.

In [12] and [9], some conditions on X which imply that $X\times(\omega+1)$ is an (a)-space are considered. We propose the following problem:

Problem 4.1. Find conditions on X which imply that $X\times(\omega+1)$ is a selectively-(a) space.

Of course, the analogous question for $X\times K$, where K is compact and metrizable, could be also considered. Within this context, we ask:

Question 4.2. *Are the following statements equivalent for any topological space X?*

(i) $X\times(\omega+1)$ *is selectively* (a).
(ii) $X\times[0,1]$ *is selectively* (a).

(iii) *$X \times K$ is selectively (a) for any metrizable compact space K.*

The preceding question is a kind of "selective version" of Question 5 of [9], posed for (a)-spaces (and which remains unanswered, as far as our knowledge goes).

Let us turn to other direction. In [1] and [2], Bella, Bonanzinga, Matveev and Spadaro introduced and investigated some selective versions of separability, as follows:

- A topological space X is said to be *D-separable* if for every family of dense sets $\{E_n : n < \omega\}$ one can pick, for each $n < \omega$, a discrete set D_n satisfying $D_n \subseteq E_n$ and such that $\bigcup_{n<\omega} D_n$ is also a dense set;
- A topological space X is said to be *R-separable* if for every family of dense sets $\{D_n : n < \omega\}$ one can pick, for each $n < \omega$, $p_n \in D_n$ such that $\{p_n : n < \omega\}$ is also a dense set;
- A topological space X is said to be *M-separable* if for every family of dense sets $\{D_n : n < \omega\}$ one can pick, for each $n < \omega$, a finite $F_n \subseteq D_n$ such that $\bigcup_{n<\omega} F_n$ is also a dense set.

The following question was raised during a session of the Sao Paulo Topology Seminar, at USP.

Question 4.3. *Are there some non-trivial relationships between the property of being selectively (a) and any of the above mentioned selective versions of separability? The same question can be posed (possibly, in a more properly way) for spaces X satisfying the topological property $(\dagger)$ described below, obtained by modifying the definition of selectively (a) spaces and which appears very natural to be considered.*

$(\dagger)$ *For every sequence $\langle \mathcal{U}_n : n < \omega\rangle$ of open covers of X and for every sequence of dense sets $\langle D_n : n < \omega\rangle$ there is a sequence $\langle A_n : n < \omega\rangle$ of closed and discrete subsets of X satisfying $A_n \subseteq D_n$ for every $n < \omega$ and such that $\{St(A_n, \mathcal{U}_n) : n < \omega\}$ covers X.*

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