Order boundedness and weak compactness of the set of quasi-measure extensions of a quasi-measure

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Abstract. Let $\mathfrak{M}$ and $\mathfrak{N}$ be algebras of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{N}$, and denote by $E(\mu)$ the set of all quasi-measure extensions of a given quasi-measure μ on $\mathfrak{M}$ to $\mathfrak{N}$. We give some criteria for order boundedness of $E(\mu)$ in $ba(\mathfrak{N})$, in the general case as well as for atomic μ. Order boundedness implies weak compactness of $E(\mu)$. We show that the converse implication holds under some assumptions on $\mathfrak{M}$, $\mathfrak{N}$ and μ or μ alone, but not in general.

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1. Introduction

This is a continuation of the author's many years' work devoted to the convex set $E(\mu)$ of all quasi-measure extensions of a given *quasi-measure* μ, i.e., a positive additive function on an algebra $\mathfrak{M}$ of subsets of a set Ω, to a larger algebra $\mathfrak{N}$ of subsets of Ω. Most of that work is summarized in a recent memoir [16]. Its Section 5 discusses weak compactness of $E(\mu)$ as a subset of the Banach lattice $ba(\mathfrak{N})$.

This paper is mainly concerned with order boundedness (from above) of $E(\mu)$, a property that implies weak compactness in our setting, in view of classical results (see Proposition 2(c) in Section 4 and the comments following it and Theorem 0 therein). The property in question has not been studied so far, with the exception of Theorem 1(a) in [8]. According to that result, $E(\mu)$ is order bounded provided $\mathfrak{N}$ is generated, as an algebra, by $\mathfrak{M}$ and a finite subfamily of $\mathfrak{N}$. See Corollaries 1 and 4 in Section 5 for generalizations.

In Sections 5 and 6, the main body of the paper, we present some criteria for order boundedness of $E(\mu)$ for arbitrary μ (Theorem 2) as well as for atomic μ (Theorem 3). They parallel the corresponding results on weak compactness of $E(\mu)$. In addition, some formulas for the infimum and supremum of $E(\mu)$ in $ba(\mathfrak{N})$ are given (Theorems 1–3 and Remark 2). We also deal with the coincidence of order boundedness and weak compactness of $E(\mu)$ under some assumptions on $\mathfrak{M}$, $\mathfrak{N}$ and μ or μ alone (Corollary 2 and Proposition 5, respectively). Moreover,

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we present some new results on weak compactness of $E(\mu)$ (Propositions 3 and 4), which are partial answers to a problem posed in [16].

In Section 7 we establish a general criterion for order boundedness in $ba(\mathfrak{M})$ (Theorem A1), which is an analogue of the corresponding results on relative weak compactness and norm boundedness from the literature (Theorems A2 and A3). Section 7 can be read independently of the previous sections, with the exception of Section 3.

Sections 2–4 are of introductory character. They explain the terminology and notation used in the paper. In addition, they contain a series of auxiliary results. The terminology and notation are mostly standard and they coincide with those of [16], which is also our main reference for the author's results. The original source, one of the papers [8]–[11], [13] and [14] is, nevertheless, usually given, too.

2. Remarks on order boundedness in linear lattices

The purpose of this section is to introduce some notation and terminology concerning linear lattices and to establish Proposition 1 below. This simple result shows that some assertions of Theorems 1 and 2 in Section 5 are instances of a more general phenomenon (see Remark 1 therein).

Let X be a real linear lattice (= Riesz space in the terminology of [1]), with the order and lattice operations denoted by $\leqslant$ and $\wedge$, $\vee$, respectively. As usual, X_+ stands for the positive cone of X.

Let V be a subset of X. We say that $x \in X$ is a *lower* [resp. *upper*] *bound* of V if $x \leqslant v$ [resp. $x \geqslant v$] for each $v \in V$. If V has both lower and upper bounds in X, it is called *order bounded*.

Lemma 1. *For a subset V of X we have*

(a) *V and* $\operatorname{conv} V$ *have the same lower and upper bounds;*
(b) *if τ is a linear topology on X with X_+ closed, then V and $\bar{V}^\tau$ have the same lower and upper bounds.*

The proof of Lemma 1 is straightforward.

Proposition 1. *Let τ be a locally convex topology on X with X_+ closed and let W be a compact convex subset of X. Then W and* $\operatorname{extr} W$ *have the same lower and upper bounds.*

In view of Lemma 1, this is a direct consequence of the Krein–Milman theorem.

3. Preliminaries on supermeasures and submeasures

Throughout the rest of the paper, Ω stands for a nonempty set and $\mathfrak{M}$ for an algebra of subsets of Ω.

We call $\alpha\colon \mathfrak{M} \to [0,\infty]$ a *supermeasure* [resp. *submeasure*] if the following conditions are satisfied: $\alpha(\varnothing) = 0$; $\alpha(M) \leqslant \alpha(N)$ whenever M, $N \in \mathfrak{M}$ and $M \subset N$; and

$$\alpha(M_1 \cup M_2) \geqslant \alpha(M_1) + \alpha(M_2) \quad [\text{resp. } \alpha(M_1 \cup M_2) \leqslant \alpha(M_1) + \alpha(M_2)]$$

whenever $M_1, M_2 \in \mathfrak{M}$ are disjoint. For submeasures we refer the reader to [6].

For $\alpha\colon \mathfrak{M} \to [0,\infty]$ and $M \in \mathfrak{M}$ we define $l(\alpha)(M)$ and $u(\alpha)(M)$ as

$$\inf\Big\{\sum_{i=1}^{n} \alpha(M_i) : M_1,\dots,M_n \in \mathfrak{M} \text{ are pairwise disjoint and } M = \bigcup_{i=1}^{n} M_i\Big\},$$
$$\sup\Big\{\sum_{i=1}^{n} \alpha(M_i) : M_1,\dots,M_n \in \mathfrak{M} \text{ are pairwise disjoint and } M = \bigcup_{i=1}^{n} M_i\Big\},$$

respectively.

We begin with a lemma which will be used in establishing the next one as well as Theorems 1 and 2 in Section 5. Its proof is simple and standard, and so left to the reader.

Lemma 2. (a) *If $\alpha\colon \mathfrak{M} \to [0,\infty]$ is a supermeasure, then $l(\alpha)$ is additive and $\alpha \geqslant l(\alpha)$. Moreover, for every additive $\beta\colon \mathfrak{M} \to [0,\infty]$ with $\alpha \geqslant \beta$ we have $l(\alpha) \geqslant \beta$.*

(b) *If $\alpha\colon \mathfrak{M} \to [0,\infty]$ is a submeasure, then $u(\alpha)$ is additive and $\alpha \leqslant u(\alpha)$. Moreover, for every additive $\beta\colon \mathfrak{M} \to [0,\infty]$ with $\alpha \leqslant \beta$ we have $u(\alpha) \leqslant \beta$.*

The following lemma is close to known results (see [2, Lemma 2.2] and [18, Corollary 2]). It will be used in the proofs of Theorem 2 in Section 5 and Theorem A1 in Section 7.

Lemma 3. *For a submeasure $\alpha\colon \mathfrak{M} \to [0,\infty)$ the following two conditions are equivalent:*

(i) *$\sum_{j=1}^{\infty} \alpha(M_j) < \infty$ for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \varnothing$ whenever $j \neq j'$;*

(ii) *$u(\alpha)(\Omega) < \infty$.*

Proof: (cf. [12, proof of Theorem 1, (ii) $\Rightarrow$ (i)]). Clearly, only the implication (i) $\Rightarrow$ (ii) needs a proof.

We first establish the following claim: given $M \in \mathfrak{M}$ with $u(\alpha)(M) = \infty$, there exists $N \in \mathfrak{M}$ such that

$$N \subset M, \quad u(\alpha)(N) = \infty \quad \text{and} \quad u(\alpha)(M \setminus N) > 1.$$

Indeed, there exist pairwise disjoint $N_1,\dots,N_n \in \mathfrak{M}$ such that

$$M = \bigcup_{i=1}^{n} N_i \quad \text{and} \quad \sum_{i=1}^{n} \alpha(N_i) > \alpha(M) + 1.$$

Hence $\sum_{i\neq j} \alpha(N_i) > 1$ for every $j = 1,\dots,n$. On the other hand, $u(\alpha)(N_{j_0}) = \infty$ for some j_0, by Lemma 2(b). Thus, it is enough to set $N = N_{j_0}$.

Suppose $u(\alpha)(\Omega) = \infty$. Using the claim, we can find pairwise disjoint $\Omega_1, \Omega_2, \dots$ in $\mathfrak{M}$ with $u(\alpha)(\Omega_j) > 1$ for each j. As easily seen, this contradicts (i). $\square$

Recall that a real-valued function α on $\mathfrak{M}$ is called *exhaustive* or *strongly bounded* if $\alpha(M_j) \to 0$ for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \varnothing$ whenever $j \neq j'$.

The following lemma will be used in the proofs of Propositions 4 and 5 in Section 6.

Lemma 4. *Let $\alpha\colon \mathfrak{M} \to [0,\infty)$ be an exhaustive submeasure with finite range. We then have $u(\alpha)(\Omega) < \infty$.*

Proof: Denote by A the quotient of $\mathfrak{M}$ by the ideal of α-null sets. The assumptions imply that every family of pairwise disjoint elements in the Boolean algebra A is finite, and so A is also finite. It follows that, given pairwise disjoint elements $M_1, \dots, M_{n+1}$ in $\mathfrak{M}$, where 2^n is the cardinality of A, we have $\alpha(M_i) = 0$ for some i. Therefore, $u(\alpha)(\Omega) \leqslant n\alpha(\Omega)$. $\square$

4. More notation and measure-theoretic preliminaries

For a set E we denote by $|E|$ the cardinality of E and by 2^E the family of all subsets of E.

The set of nonzero $\{0, 1\}$-valued additive functions on a Boolean algebra A is denoted by $ult(A)$.

Given $\mathfrak{E} \subset 2^\Omega$, we denote by $\mathfrak{E}_b$ the algebra of subsets of Ω generated by $\mathfrak{E}$.

We denote by $ba(\mathfrak{M})$ the Banach lattice of all real-valued bounded (= exhaustive) additive functions on $\mathfrak{M}$ (see [3, Section 2.2]). According to [3, Theorem 2.2.1(9)], $ba(\mathfrak{M})$ is Dedekind (= boundedly) complete. By definition, $\|\varphi\| = |\varphi|(\Omega)$ for $\varphi \in ba(\mathfrak{M})$, where $|\varphi|$ stands for the modulus of φ. In addition to the strong topology, $ba(\mathfrak{M})$ is equipped with its weak and weak* topologies; see [3, Section 4.7] for the canonical Banach-lattice predual of $ba(\mathfrak{M})$.

Let $\mu \in ba_+(\mathfrak{M})$. Adapting a general linear-lattice-theoretical terminology (see [1, p. 36]), we say that $\nu \in ba(\mathfrak{M})$ is a component of μ if

$$\nu \wedge (\mu - \nu) = 0.$$

Denote by $\mathcal{U}_\mu$ the set of all components of μ that take at most two values. As easily seen (cf. [3, Proposition 5.2.2]), for different $\nu_1, \nu_2 \in \mathcal{U}_\mu$ we have $\nu_1 \wedge \nu_2 = 0$. Therefore, $\mathcal{U}_\mu$ is countable. We say that μ is (*purely*) *atomic* if $\mu = \sum_{\nu \in \mathcal{U}_\mu} \nu$.

As usual, we associate with $\mu \in ba_+(\mathfrak{M})$ the inner and outer quasi-measures μ_* and μ^* defined, for all $E \subset \Omega$, by the formulas:

$$\mu_*(E) = \sup\{\mu(M) : E \supset M \in \mathfrak{M}\},$$

$$\mu^*(E) = \inf\{\mu(M) : E \subset M \in \mathfrak{M}\}.$$

It is well known that μ_* is a supermeasure and μ^* is a submeasure on 2^Ω, in our terminology.

The following lemma will be used in establishing Corollary 1 in Section 5.

Lemma 5. *Let $\mu \in ba_+(\mathfrak{M})$ and let $\mathfrak{N}$ be an algebra of subsets of Ω. We then have*

$$u(\mu^*|(\mathfrak{M} \cup \mathfrak{N})_b)(\Omega) = u(\mu^*|\mathfrak{N})(\Omega).$$

PROOF: Let $\{R_1, \dots, R_s\}$ be an $(\mathfrak{M} \cup \mathfrak{N})_b$-partition of Ω. Then there are an $\mathfrak{M}$-partition $\{M_1, \dots, M_k\}$ of Ω and an $\mathfrak{N}$-partition $\{N_1, \dots, N_l\}$ of Ω with the following property:

$$\{R_1, \dots, R_s\} \subset \{M_i \cap N_j : i = 1, \dots, k;\ j = 1, \dots, l\}_b.$$

It follows that

$$\sum_{r=1}^{s} \mu^*(R_r) \leqslant \sum_{j=1}^{l} \sum_{i=1}^{k} \mu^*(M_i \cap N_j) = \sum_{j=1}^{l} \mu^*(N_j) \leqslant u(\mu^*|\mathfrak{N})(\Omega).$$

This yields the nontrivial inequality of the asserted equality. □

Throughout the rest of the paper, $\mathfrak{R}$ stands for an algebra of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$. Given $\mu \in ba_+(\mathfrak{M})$, we set

$$E(\mu) = \{\varrho \in ba_+(\mathfrak{R}) : \varrho|\mathfrak{M} = \mu\}.$$

It is a classical result that $E(\mu)$ is always nonempty (see [3, Chapter 3]). Moreover, it is, clearly, convex.

One of the reasons of the author's interest in order boundedness of the set $E(\mu)$ is part (c) of the following proposition.

Proposition 2. *Let $\mu \in ba_+(\mathfrak{M})$. Then*

(a) *$E(\mu)$ is weak* compact;*
(b) *$E(\mu)$ is weakly closed;*
(c) *$E(\mu)$ is weakly compact if it is order bounded.*

Part (a) was first observed by D. Plachky (see [16, Proposition 4.4(a)] or [8, Proposition 1(a)]). Plainly, part (b) follows from (a), but it is, in fact, established in the course of the proof of (a). In view of (b), part (c) is a consequence of a classical theorem (see [1, Theorem 12.9] or [20, Theorem 1.12]), since the norm of the Banach lattice $ba(\mathfrak{R})$ is order continuous. In place of that theorem, one could make use of another standard result, Theorem A2 in Section 7. Theorem A2 is also one of the main ingredients of the proof of the following Theorem 0 (= [16, Theorem 5.1]), which will be frequently applied below.

Theorem 0. *For $\mu \in ba_+(\mathfrak{M})$ the following three conditions are equivalent:*

(i) *$E(\mu)$ is weakly compact;*
(ii) extr $E(\mu)$ *is relatively weakly compact;*
(iii) *$\mu^*|\mathfrak{R}$ is exhaustive.*

Example 1 in Section 5 shows that part (c) of Proposition 2 cannot be reversed, even if μ is atomic and $\mathfrak{R}$ is generated, as an algebra, by $\mathfrak{M}$ and a countable partition of Ω. Nevertheless, we reveal below two situations where order boundedness and weak compactness of $E(\mu)$ coincide (see Corollary 2 in Section 5 and Proposition 5 in Section 6).

The following results, which hold for arbitrary $\mu \in ba_+(\mathfrak{M})$, will be used, in particular, in the proofs of Theorems 1 and 2, respectively.

(C)$_*$ $\min\limits_{\varrho\in E(\mu)} \varrho(R) = \min\limits_{\pi\in \operatorname{extr} E(\mu)} \pi(R) = \mu_*(R)$ *for* $R \in \mathfrak{R}$;

(C)* $\max\limits_{\varrho\in E(\mu)} \varrho(R) = \max\limits_{\pi\in \operatorname{extr} E(\mu)} \pi(R) = \mu^*(R)$ *for* $R \in \mathfrak{R}$.

Their proofs are sketched in [16, p. 19]. We note that, in view of Proposition 2(a), the first equalities of (C)$_*$ and (C)* are also a consequence of [4, Chapter II, § 7, Proposition 1] applied to the affine mapping

$$E(\mu) \ni \varrho \longmapsto \varrho(R) \in \mathbb{R}, \quad \text{where } R \in \mathfrak{R},$$

which is weak* continuous.

Given $\mu \in ba_+(\mathfrak{M})$, we denote by μ^{m} the maximal $\nu \in ba_+(\mathfrak{M})$ such that $\nu \leqslant \mu$ and $E(\nu)$ is a singleton, and set $\mu^{\mathrm{a}} = \mu - \mu^{\mathrm{m}}$. (The details are explained in [16, p. 22]; see also [9, Lemma 1].) We call μ *antimonogenic* if $\mu = \mu^{\mathrm{a}}$. The use of μ^{m} and μ^{a} below is limited to the formulations of Theorem 1 and Corollary 3 in Section 5, respectively.

The next result will be used in the proof of Lemma 6 below.

(D)$'$ *For* $\mu \in ult(\mathfrak{M})$ *we have* $\operatorname{extr} E(\mu) = E(\mu) \cap ult(\mathfrak{R})$.

See [16, p. 19] or [9, p. 396].

We continue with two lemmas which will be used in the proof of Theorem 3 in Section 6.

Lemma 6. *For* $\mu \in ult(\mathfrak{M})$ *the following two conditions are equivalent:*

(i) $E(\mu)$ *is order bounded;*
(ii) $\operatorname{extr} E(\mu)$ *is finite.*

Under these conditions, we have

$$\sup E(\mu) = \sum_{\pi\in \operatorname{extr} E(\mu)} \pi.$$

Proof: In view of (D)$'$, $\pi_1 \wedge \pi_2 = 0$ whenever $\pi_1, \pi_2 \in \operatorname{extr} E(\mu)$ and $\pi_1 \neq \pi_2$. Therefore, (i) implies (ii) and the inequality "$\geqslant$" of the asserted formula. The converse implication and inequality follow from (C)*. □

Lemma 7. *Let* $\mu, \mu_j \in ba_+(\mathfrak{M})$ *be such that* $\sum_{j=1}^{\infty} \mu_j = \mu$ *and* $\mu_j \wedge \mu_{j'} = 0$ *whenever* $j \neq j'$. *Then the following two conditions are equivalent:*

(i) $E(\mu)$ *is order bounded;*
(ii) $E(\mu_j)$ *is order bounded for each* j *and* $\{\sup E(\mu_j) : j = 1, 2, \dots\}$ *is order bounded.*

Under these conditions, we have

$$\sup E(\mu) = \sum_{j=1}^{\infty} \sup E(\mu_j).$$

PROOF: We first note that if $\varrho_j \in E(\mu_j)$ and $\varrho_{j'} \in E(\mu_{j'})$, then $\varrho_j \wedge \varrho_{j'} = 0$ whenever $j \neq j'$. Indeed, we have $\mu_j \wedge \mu_{j'} = 0$, by assumption, and so it is enough to apply [3, Theorem 2.2.1(b)]. It follows that

$$(\sup E(\mu_j)) \wedge (\sup E(\mu_{j'})) = 0 \quad \text{whenever } j \neq j'$$

provided $E(\mu_j)$ and $E(\mu_{j'})$ are order bounded.

Suppose (i) holds. Then $\sup E(\mu)$ exists and is an upper bound of $E(\mu_j)$ for each j, by [16, Theorem 6.1(a)] or [8, Lemma 2(a)]. Thus, (ii) and the inequality "$\geqslant$" of the asserted formula hold.

Suppose (ii) holds, and let $\varrho \in E(\mu)$. In view of [16, Theorem 6.1(a)] or [8, Lemma 2(a)], there exist $\varrho_j \in E(\mu_j)$ with $\sum_{j=1}^{\infty} \varrho_j = \varrho$. Consequently,

$$\varrho \leqslant \sum_{j=1}^{\infty} \sup E(\mu_j).$$

This yields (i) and the inequality "$\leqslant$" of the asserted formula. □

5. $E(\mu)$ for arbitrary μ

We start by establishing some formulas for the common infimum of the sets $E(\mu)$ and $\operatorname{extr} E(\mu)$, in the general case. A formula for the common supremum of those sets is contained in the next theorem.

Theorem 1. *Let $\mu \in ba_+(\mathfrak{M})$. Then*

(a) $\inf E(\mu) = \inf \operatorname{extr} E(\mu) = l(\mu_* | \mathfrak{R})$;
(b) $(\inf E(\mu))|\mathfrak{M} = \mu^{\mathrm{m}}$, *whence* $\inf E(\mu) = 0$ *if and only if* μ *is antimonogenic.*

PROOF: Part (a) follows from $(\mathrm{C})_*$ and Lemma 2(a). Part (b) is a consequence of (a) and [19, Remark 2 on Theorem 2]. □

Conditions (i)–(iii) of the next theorem are analogous to the corresponding conditions of Theorem 0. The equivalence of (i) and (iii) is closely related to Theorem A1 in Section 7.

Theorem 2. *For $\mu \in ba_+(\mathfrak{M})$ the following four conditions are equivalent:*

(i) $E(\mu)$ *is order bounded;*
(ii) $\operatorname{extr} E(\mu)$ *is order bounded;*
(iii) $\sum_{j=1}^{\infty} \mu^*(R_j) < \infty$ *for every sequence (R_j) in $\mathfrak{R}$ with $R_j \cap R_{j'} = \varnothing$ whenever $j \neq j'$;*
(iv) $u(\mu^* | \mathfrak{R})(\Omega) < \infty$.

Under these conditions, we have

$$\sup E(\mu) = \sup \operatorname{extr} E(\mu) = u(\mu^*|\mathfrak{R}).$$

Proof: The equivalence of (i), (ii) and (iv) as well as the final formula follow from (C)* and Lemma 2(b). Clearly, (iv) implies (iii). The converse implication is, in view of Lemma 2(b), a consequence of Lemma 3. □

Remark 1. The equivalence of conditions (i) and (ii) of Theorem 2 as well as the first equalities in Theorems 1(a) and 2 also follow from Propositions 1 and 2(a).

The setting of our next three results, Corollaries 1–3, and that of Proposition 4 in Section 6 has been suggested by the theory of products of measurable spaces and measure spaces.

Corollary 1. *Let $\mathfrak{N}$ be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$ and let $\mu \in ba_+(\mathfrak{M})$. Then the following three conditions are equivalent:*

(i) *$E(\mu)$ is order bounded;*
(ii) *$\sum_{j=1}^{\infty} \mu^*(N_j) < \infty$ for every sequence (N_j) in $\mathfrak{N}$ with $N_j \cap N_{j'} = \varnothing$ whenever $j \neq j'$;*
(iii) *$u(\mu^*|\mathfrak{N})(\Omega) < \infty$.*

Proof: In view of Lemma 5, the equivalence of (i) and (iii) is a consequence of the corresponding equivalence of Theorem 2. The equivalence of (ii) and (iii) holds by Lemmas 2(b) and 3. □

Clearly, condition (iii) of Corollary 1 holds if $\mathfrak{N}$ is finite, and so $E(\mu)$ is then order bounded. This special case is essentially [8, Theorem 1(a)]. See also Corollary 4 in this section for a generalization.

It is an open problem whether a counterpart of Corollary 1, (i) $\Leftrightarrow$ (ii), for weak compactness is true (see [16, Problem 13.4]). For partial answers to this problem see Proposition 3 and Remark 3 in this section as well as Proposition 4 in the next section.

Condition $(*)$ assumed in Corollaries 2 and 3 below has already been used by the author in [16, Proposition 12.4], which coincides with [11, Proposition 2], and [15, Theorem 7]. It is intermediate between the condition of independence and that of almost independence of algebras of sets considered by E. Marczewski (see [17, p. 220] for definitions).

Corollary 2. *Let $\mathfrak{N}$ be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$ and let $\mu \in ba_+(\mathfrak{M})$. Suppose*

$(*)$ *$M \cap N \neq \varnothing$ for all $M \in \mathfrak{M}$ with $\mu(M) > 0$ and nonempty $N \in \mathfrak{N}$.*

Then the following three conditions are equivalent:

(i) *$E(\mu)$ is order bounded;*
(ii) *$E(\mu)$ is weakly compact;*
(iii) *$\mathfrak{N}$ is finite or $\mu = 0$.*

Proof: By Proposition 2(c), (i) implies (ii). That (iii) implies (i) is clear from Corollary 1. (Thus, neither of these implications requires condition $(*)$.)

Suppose (ii) holds and $\mu \neq 0$. By $(*)$, we have $\mu^*(N) = \mu(\Omega)$ whenever $N \in \mathfrak{N}$ is nonempty. Since $\mu^*|\mathfrak{N}$ is exhaustive, by Theorem 0, it follows that every family of pairwise disjoint sets in $\mathfrak{N}$ is finite. Therefore, $\mathfrak{N}$ is itself finite, and so (iii) holds. □

Clearly, condition $(*)$ is essential for the validity of the implication (i) $\Rightarrow$ (iii) of Corollary 2. In fact, even if $E(\mu)$ is a singleton, (iii) need not be satisfied. (This comment also applies to the implication (i) $\Rightarrow$ (ii) of the next corollary.) Similarly, (ii) does not imply (i), in general, as already noted in the discussion following the formulation of Theorem 0.

Corollary 3. *In the setting of Corollary 2 the following two conditions are equivalent:*

(i) *$E(\mu)$ is strongly compact;*
(ii) *$\mathfrak{N}$ is finite and μ^{a} is atomic or $\mu = 0$.*

Proof: That (i) implies (ii) is a consequence of Corollary 2, (ii) $\Rightarrow$ (iii), and [16, Theorem 10.2, (i) $\Rightarrow$ (ii)] or [13, Theorem 2, (i) $\Rightarrow$ (ii)].

It follows from (ii) that $E(\mu^{\mathrm{a}})$ is strongly compact (see [8, Theorem 2(b)]). Consequently, (i) holds, by [16, Corollary 6.2] or [10, p. 471, (T)]. (Thus, $(*)$ is not needed for the validity of the implication (ii) $\Rightarrow$ (i).) □

Corollary 4. *Let $\mathfrak{E}$ be a family of pairwise disjoint subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{E})_b$ and let $\mu \in ba_+(\mathfrak{M})$. Then the following two conditions are equivalent:*

(i) *$E(\mu)$ is order bounded;*
(ii) $\sum_{E\in\mathfrak{E}} \mu^*(E) < \infty$.

Proof: Let $\mathfrak{F}$ be a finite subfamily of $\mathfrak{E}$. We have

$$\sum_{E\in\mathfrak{F}} \mu^*(E) + \mu^*(\Omega \setminus \bigcup_{E\in\mathfrak{F}} E) \leqslant \sum_{E\in\mathfrak{E}} \mu^*(E) + \mu(\Omega).$$

Set $\mathfrak{N} = \mathfrak{E}_b$. It follows that

$$\sum_{E\in\mathfrak{E}} \mu^*(E) \leqslant u(\mu^*|\mathfrak{N})(\Omega) \leqslant \sum_{E\in\mathfrak{E}} \mu^*(E) + \mu(\Omega).$$

Therefore, condition (ii) is equivalent to $u(\mu^*|\mathfrak{N})(\Omega) < \infty$. An application of Corollary 1, (i) $\Leftrightarrow$ (iii), completes the proof. □

Remark 2. We shall give a formula for $\sup E(\mu)$, provided it exists, in the setting of Corollary 4. In the case where $\mathfrak{E}$ is a finite partition of Ω, this formula is implicit in the proof of [8, Theorem 1(a)]. (We leave it to the reader to establish an analogous formula for $\inf E(\mu)$ in the setting of Corollary 4.) Set

$$\sigma_E(R) = \mu^*(R \cap E) \quad \text{for all } R \in \mathfrak{R} \text{ and } E \in \mathfrak{E}.$$

Moreover, set

$$\sigma_\infty(R) = \inf\{\mu^*(R \cap (\Omega \setminus \bigcup_{E \in \mathfrak{F}} E)) : \mathfrak{F} \subset \mathfrak{E} \text{ and } \mathfrak{F} \text{ is finite}\} \quad \text{for all } R \in \mathfrak{R}.$$

As easily seen, σ_E, σ_∞ are in $ba_+(\mathfrak{R})$. Moreover, we have $\sigma_E \wedge \sigma_F = 0$ whenever E, $F \in \mathfrak{E}$ and $E \neq F$. Also, $\sigma_E \wedge \sigma_\infty = 0$ whenever $E \in \mathfrak{E}$. Set

$$\sigma = \sum_{E \in \mathfrak{E}} \sigma_E + \sigma_\infty.$$

In view of condition (ii) of Corollary 4, it follows that $\sigma \in ba_+(\mathfrak{R})$. We claim that $\sigma = \sup E(\mu)$. Indeed, given $\varrho \in E(\mu)$ and $R \in \mathfrak{R}$, for every finite $\mathfrak{F} \subset \mathfrak{E}$ we have

$$\varrho(R) \leqslant \sum_{E \in \mathfrak{F}} \mu^*(R \cap E) + \mu^*(R \cap (\Omega \setminus \bigcup_{E \in \mathfrak{F}} E)).$$

Hence $\varrho(R) \leqslant \sigma(R)$. This yields the inequality "$\geqslant$" of the claim. To establish the converse inequality, consider $\tau \in ba_+(\mathfrak{R})$ with $\tau \geqslant \varrho$ for all $\varrho \in E(\mu)$. Then $\tau \geqslant \mu^*|\mathfrak{R}$, by (C)*. It follows that $\tau \geqslant \sigma_E$ for all $E \in \mathfrak{E}$ and $\tau \geqslant \sigma_\infty$. Hence $\tau \geqslant \sigma$, and we are done.

The next result is a version of Corollary 4 for weak compactness.

Proposition 3. *Let $\mathfrak{E}$ be a family of pairwise disjoint subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{E})_b$ and let $\mu \in ba_+(\mathfrak{M})$. Then the following two conditions are equivalent:*

(i) *$E(\mu)$ is weakly compact;*

(ii) *for every $\varepsilon > 0$ there exists a finite $\mathfrak{E}_0 \subset \mathfrak{E}$ such that $\mu^*(\bigcup_{E \in \mathfrak{F}} E) < \varepsilon$ whenever $\mathfrak{F} \subset \mathfrak{E} \setminus \mathfrak{E}_0$ is finite.*

Proof: According to Theorem 0, condition (i) is equivalent to the following one: $\mu^*|\mathfrak{R}$ is exhaustive. It is plain that the latter condition implies (ii). We shall establish the converse implication. To this end, fix $\varepsilon > 0$ and pairwise disjoint $R_k \in \mathfrak{R}$, $k = 1, 2, \ldots$. Choose $E_1, E_2, \ldots$ in $\mathfrak{E} \cup \{\varnothing\}$ so that

$$R_k \in (\mathfrak{M} \cup \{E_1, E_2, \ldots\})_b, \quad k = 1, 2, \ldots;$$
$$E_i \neq \varnothing \text{ implies } E_i \neq E_{i'} \quad \text{whenever } i \neq i'.$$

By (ii), there exists n_0 such that

$$\mu^*(\bigcup_{i=n_0+1}^{m} E_i) < \varepsilon/3 \quad \text{whenever } m \geqslant n_0 + 1.$$

Moreover, there exists k_1 such that

$$\mu^*(R_k \cap \bigcup_{i=1}^{n_0} E_i) < \varepsilon/3 \quad \text{whenever } k \geqslant k_1$$

(see the proof of [16, Corollary 5.4]). Choose $n_0 < m_1 < m_2 < \dots$ with

$$R_k \in (\mathfrak{M} \cup \{E_1, \dots, E_{m_k}\})_b, \quad k = 1, 2, \dots,$$

and set $F_k = \Omega \setminus \bigcup_{i=1}^{m_k} E_i$. Then $R_k \cap F_k = M_k \cap F_k$ for some $M_k \in \mathfrak{M}$, $k = 1, 2, \dots$. Define

$$\tilde{M}_1 = M_1 \text{ and } \tilde{M}_{k+1} = M_{k+1} \setminus \bigcup_{i=1}^{k} M_i, \quad k = 1, 2, \dots.$$

As easily seen, we then have $R_k \cap F_k = \tilde{M}_k \cap F_k$, $k = 1, 2, \dots$. Choose k_2 with $\mu(\tilde{M}_k) < \varepsilon/3$ whenever $k \geqslant k_2$. Since

$$R_k = (R_k \cap \bigcup_{i=1}^{n_0} E_i) \cup (R_k \cap \bigcup_{i=n_0+1}^{m_k} E_k) \cup (R_k \cap F_k),$$

it follows that $\mu^*(R_k) < \varepsilon$ for $k \geqslant k_1, k_2$. Thus, $\mu^*|\mathfrak{R}$ is exhaustive. □

Remark 3. Clearly, condition (ii) of Proposition 3 can be reformulated as follows: $\mu^*|\mathfrak{E}_b$ is exhaustive. Therefore, Proposition 3 gives a partial positive answer to [16, Problem 13.4]. For another partial answer to that problem see Proposition 4 in the next section.

The following example is already mentioned in the discussion after the formulation of Theorem 0.

Example 1. Set $\Omega = \mathbb{N}$, and let $\{M_1, M_2, \dots\}$ be a partition of Ω with $|M_i| = 2^i$ for each i. Define

$$\mathfrak{M} = \{M_1, M_2, \dots\}_b, \quad \mathfrak{E} = \{\{n\} : n \in \Omega\}, \quad \mathfrak{R} = \mathfrak{E}_b.$$

Let $\mu \in ba_+(\mathfrak{M})$ satisfy $\mu(M_i) = 2^{-i}$ for each i and $\mu(\Omega) = 1$. For $R \subset \Omega$ with $R \cap \bigcup_{i=1}^{k} M_i = \varnothing$ we then have $\mu^*(R) \leqslant 1 - \sum_{i=1}^{k} 2^{-i}$. Therefore, μ^* is exhaustive. Theorem 0, (iii) $\Rightarrow$ (i), now shows that $E(\mu)$ is weakly compact. However, $E(\mu)$ is not order bounded, in view of Corollary 4 (cf. also Theorem 3 in the next section). Indeed, we have $\sum_{n \in M_i} \mu^*(\{n\}) = 1$ for each i, and so $\sum_{n \in \Omega} \mu^*(\{n\}) = \infty$.

6. $E(\mu)$ for atomic μ

In accordance with Remark 3, we now present another partial positive answer to [16, Problem 13.4].

Proposition 4. *Let $\mathfrak{N}$ be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$ and let $\mu \in ba_+(\mathfrak{M})$ be atomic. If $\mu^*|\mathfrak{N}$ is exhaustive, then $E(\mu)$ is weakly compact.*

Proof: Let $\nu \in \mathcal{U}_\mu$. By assumption, $\nu^*|\mathfrak{N}$ is exhaustive, and so $\nu^*|\mathfrak{R}$ is exhaustive, as follows from Lemmas 4 and 5. Since $\mu^* = \sum_{\nu \in \mathcal{U}_\mu} \nu^*$, it follows that $\mu^*|\mathfrak{R}$

is also exhaustive. Therefore, $E(\mu)$ is weakly compact, according to Theorem 0, (iii) $\Rightarrow$ (i). □

We now present another situation where order boundedness and weak compactness of $E(\mu)$ are equivalent properties (cf. Corollary 2).

Proposition 5. *Let $\mu \in ba_+(\mathfrak{M})$ have finite range. Then the following three conditions are equivalent:*

(i) *$E(\mu)$ is order bounded;*
(ii) *$E(\mu)$ is weakly compact;*
(iii) *$\operatorname{extr} E(\mu)$ is finite.*

Proof: The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) hold for arbitrary $\mu \in ba_+(\mathfrak{M})$, by Proposition 2(c) and Theorem 2, (ii) $\Rightarrow$ (i), respectively. Moreover, according to [9, Theorem 3(b), (ii) $\Leftrightarrow$ (iii)], conditions (ii) and (iii) are equivalent for $\mu \in ba_+(\mathfrak{M})$ with finite range.

Observe that the implication (ii) $\Rightarrow$ (i) also follows by a joint application of Theorem 0, (i) $\Rightarrow$ (iii), Lemma 4 and Theorem 2, (iv) $\Rightarrow$ (i). □

In connection with the assumption of Proposition 5, we note that condition (iii) thereof implies that μ^{a} has finite range (see [16, Theorem 11.6] or [10, Theorem 5, (i) $\Rightarrow$ (ii)]).

We shall need the following notation (see [16, p. 18]). Given $\mu \in ba_+(\mathfrak{M})$, we set

$$\mathfrak{J}_\mu = \{R \in \mathfrak{R} : \text{there exists } M \in \mathfrak{M} \text{ with } R \subset M \text{ and } \mu(M) = 0\}.$$

Clearly, $\mathfrak{J}_\mu$ is an ideal in $\mathfrak{R}$.

We proceed with a characterization of those atomic quasi-measures μ on $\mathfrak{M}$ for which $E(\mu)$ is order bounded.

Theorem 3. *Let $\mu \in ba_+(\mathfrak{M})$ be atomic. Then the following two conditions are equivalent:*

(i) *$E(\mu)$ is order bounded;*
(ii) *$\mathfrak{R}/\mathfrak{J}_\nu$ is finite for each $\nu \in \mathcal{U}_\mu$ and*

$$\sum_{\nu \in \mathcal{U}_\mu} \nu(\Omega)|ult(\mathfrak{R}/\mathfrak{J}_\nu)| < \infty.$$

Under these conditions, we have

$$\sup E(\mu) = \sum_{\nu \in \mathcal{U}_\mu} \sum_{\pi \in \operatorname{extr} E(\nu)} \pi.$$

Proof: According to [16, Proposition 7.1, 4^0] or [11, Proposition 1], we have

$$|\operatorname{extr} E(\nu)| = |ult(\mathfrak{R}/\mathfrak{J}_\nu)| \quad \text{for each } \nu \in \mathcal{U}_\mu.$$

Moreover, as is well known, a Boolean algebra A is finite if and only if $ult(A)$ is finite. In view of these assertions, Lemmas 6 and 7 yield the equivalence of (i) and (ii) along with the accompanying formula. □

For atomic $\mu \in ba_+(\mathfrak{M})$ the first part of condition (ii) of Theorem 3 is equivalent to each of the following ones:

(ii)′ *$E(\mu)$ is weakly compact;*
(ii)″ *$E(\mu)$ is strongly compact.*

(See [16, Theorem 7.7] or [13, Lemma 1]). Thus, it is easy to construct examples of sets $E(\mu)$ which are weakly compact, but not order bounded (see Example 1). On the other hand, for every (atomic) $\mu \in ba_+(\mathfrak{M})$ satisfying (ii)′ and (ii)″, we can find a set Ω', two algebras $\mathfrak{M}'$ and $\mathfrak{R}'$ of subsets of Ω' with $\mathfrak{M}' \subset \mathfrak{R}'$, and an atomic $\mu' \in ba_+(\mathfrak{M}')$ such that $E(\mu')$ is order bounded and $E(\mu')$ and $E(\mu)$ are affinely homeomorphic when equipped with their strong topologies. This is seen from the following example and [16, Theorem 7.7] or [13, Theorem 2, (i) ⇔ (iv)].

Example 2. Let $S_1, S_2, \ldots$ be finite-dimensional simplices, and choose pairwise disjoint sets $\Omega_1, \Omega_2, \ldots$ with $|\Omega_j| = |\operatorname{extr} S_j|$ for each j. Define

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \quad \mathfrak{M} = \{\Omega_1, \Omega_2, \ldots\}_b, \quad \mathfrak{R} = \{\{\omega\} : \omega \in \Omega\}_b.$$

Moreover, set

$$\nu_j(M) = 2^{-j} \frac{|M \cap \Omega_j|}{|\Omega_j|^2} \quad \text{for } M \in \mathfrak{M},\ j = 1, 2, \ldots.$$

We then have $\nu_j \in ba_+(\mathfrak{M})$ and $E(\nu_j)$ is affinely isomorphic to S_j for each j. Finally, set $\mu = \sum_{j=1}^{\infty} \nu_j$. Clearly, $\mu \in ba_+(\mathfrak{M})$, and $E(\mu)$ is order bounded, by Theorem 3. Moreover, $E(\mu)$ equipped with its strong topology is affinely homeomorphic to $S_1 \times S_2 \times \ldots$, by [16, Theorem 6.1(b)] or [9, Theorem 1(b)].

Problem. Let $E(\mu)$, where $\mu \in ba_+(\mathfrak{M})$, be weakly compact. Can we find a set Ω', two algebras $\mathfrak{M}'$ and $\mathfrak{R}'$ of subsets of Ω' with $\mathfrak{M}' \subset \mathfrak{R}'$, and $\mu' \in ba_+(\mathfrak{M}')$ such that $E(\mu')$ is order bounded and $E(\mu')$ and $E(\mu)$ are affinely homeomorphic when equipped with their weak topologies?

As follows from the discussion introducing Example 2, the answer is affirmative provided μ is atomic.

7. Appendix. Order boundedness, relative weak compactness and norm boundedness in $ba(\mathfrak{M})$

Throughout this section, we continue to denote by $\mathfrak{M}$ an algebra of subsets of a set Ω.

We shall present a criterion for order boundedness in $ba(\mathfrak{M})$, which is closely related to Theorem 2, (i) ⇔ (iii). For comparison we shall then formulate some analogous criteria for relative weak compactness and norm boundedness in $ba(\mathfrak{M})$ from the literature.

Theorem A1. *For $K \subset ba(\mathfrak{M})$ the following two conditions are equivalent:*

(i) *K is order bounded;*

(ii) *$\sum_{j=1}^{\infty} \sup_{\varphi\in K} |\varphi(M_j)| < \infty$ for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \varnothing$ whenever $j \neq j'$.*

Proof: The implication (i) $\Rightarrow$ (ii) is straightforward. Suppose (ii) holds. Set, for $\varphi \in K$ and $M \in \mathfrak{M}$,

$$\bar{\varphi}(M) = \sup\{|\varphi(N)| : N \in \mathfrak{M} \text{ and } N \subset M\}.$$

It follows from (ii) that, for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \varnothing$ whenever $j \neq j'$, we have

$$\sum_{j=1}^{\infty} \sup_{\varphi\in K} \bar{\varphi}(M_j) < \infty.$$

Set $\alpha(M) = \sup_{\varphi\in K} \bar{\varphi}(M)$ for $M \in \mathfrak{M}$. Then α is a finite submeasure on $\mathfrak{M}$, and so (i) is a consequence of Lemmas 3 and 2(b). $\square$

For the following criterion see [5, Theorem] or [7, Theorem 2]. It is one of the main ingredients of the proof of Theorem 0.

Theorem A2. *For $K \subset ba(\mathfrak{M})$ the following two conditions are equivalent:*

(i) *K is relatively weakly compact;*

(ii) *K is norm bounded and uniformly exhaustive, i.e.,* $\sup_{\varphi\in K} |\varphi(M_j)| \to 0$ *for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \varnothing$ whenever $j \neq j'$.*

We close with the following specialization of [18, Theorem 1].

Theorem A3. *For $K \subset ba(\mathfrak{M})$ the following two conditions are equivalent:*

(i) *K is norm bounded;*

(ii) $\sup_{\varphi\in K} \sum_{j=1}^{\infty} |\varphi(M_j)| < \infty$ *for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \varnothing$ whenever $j \neq j'$.*

Postscript. More results on order-theoretic properties of the sets $E(\mu)$ and $\operatorname{extr} E(\mu)$ will be presented in another paper by the author (in preparation).

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