Ultracompanions of subsets of a group

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Abstract. Let G be a group, βG be the Stone-Čech compactification of G endowed with the structure of a right topological semigroup and $G^* = \beta G \setminus G$. Given any subset A of G and $p \in G^*$, we define the p-companion $\Delta_p(A) = A^* \cap Gp$ of A, and characterize the subsets with finite and discrete ultracompanions.

Keywords: Stone-Čech compactification; ultracompanion; sparse and discrete subsets of a group

Classification: 54D35, 22A15, 20F69

1. Introduction

Given a discrete space X, we take the points of βX, the Stone-Čech compactification of X, to be the ultrafilters on X, with the points of X identified with the principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters on X. The topology on βX can be defined by stating that the sets of the form $\overline{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, form a base for the open sets. We note the sets of this form are clopen and that for any $p \in \beta X$ and $A \subseteq X$, $A \in p$ if and only if $p \in \overline{A}$. For any $A \subseteq X$, we denote $A^* = \overline{A} \cap G^*$. The universal property of βX states that every mapping $f : X \to Y$, where Y is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \to Y$.

Now let G be a discrete group. Using the universal property of βG, we can extend the group multiplication from G to βG in two steps. Given $g \in G$, the mapping

$$x \mapsto gx : \ G \to \beta G$$

extends to the continuous mapping

$$q \mapsto gq : \ \beta G \to \beta G.$$

Then, for each $q \in \beta G$, we extend the mapping $g \mapsto gq$ defined from G into βG to the continuous mapping

$$p \mapsto pq : \ \beta G \to \beta G.$$

The product pq of the ultrafilters p, q can also be defined by the rule: given a subset $A \subseteq G$,

$$A \in pq \leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

To describe a base for pq, we take any element $P \in p$ and, for every $x \in P$, choose some element $Q_x \in q$. Then $\bigcup_{x\in P} xQ_x \in pq$, and the family of subsets of this form is a base for the ultrafilter pq.

By the construction, the binary operation $(p, q) \mapsto pq$ is associative, so βG is a semigroup, and G^* is a subsemigroup of βG. For each $q \in \beta G$, the right shift $x \mapsto xq$ is continuous, and the left shift $x \to gx$ is continuous for each $g \in G$.

For the structure of a compact right topological semigroup βG and plenty of its applications to combinatorics, topological algebra and functional analysis see [2], [4], [5], [19], [21].

Given a subset A of a group G and an ultrafilter $p \in G^*$ we define a *p-companion* of A by

$$\Delta_p(A) = A^* \cap Gp = \{gp : g \in G, A \in gp\},$$

and say that a subset S of G^* is an *ultracompanion* of A if $S = \Delta_p(A)$ for some $p \in G^*$.

Clearly, A is finite if and only if $\Delta_p(A) = \emptyset$ for every $p \in G^*$, and $\Delta_p(G) = Gp$ for each $p \in G^*$.

We say that a subset A of a group G is

- *sparse* if each ultracompanion of A is finite;
- *disparse* if each ultracompanion of A is discrete.

In fact, the sparse subsets were introduced in [3] with rather technical definition (see Proposition 5) in order to characterize strongly prime ultrafilters in G^*, the ultrafilters from $G^* \setminus \overline{G^*G^*}$.

In this paper we study the families of sparse and disparse subsets of a group, and characterize in terms of ultracompanions the subsets from the following basic classification.

A subset A of G is called

- *large* if $G = FA$ for some finite subset F of G;
- *thick* if, for every finite subset F of G, there exists $a \in A$ such that $Fa \subseteq A$;
- *prethick* if FA is thick for some finite subset F of G;
- *small* if $L \setminus A$ is large for every large subset L;
- *thin* if $gA \cap A$ is finite for each $g \in G \setminus \{e\}$, e is the identity of G.

In the dynamical terminology [5], the large and prethick subsets are called syndetic and piecewise syndetic respectively. For references on the subset combinatorics of groups see the survey [12].

We conclude the paper with discussions of some modifications of sparse subsets and a couple of open questions.

2. Characterizations

Proposition 1. *For a subset A of a group G and an ultrafilter $p \in G^*$, the following statements hold:*

(i) $\Delta_p(FA) = F\Delta_p(A)$ *for every finite subset F of G;*
(ii) $\Delta_p(Ah) = \Delta_{ph^{-1}}(A)$ *for every $h \in G$;*

(iii) $\Delta_p(A \cup B) = \Delta_p(A) \cup \Delta_p(B)$.

Proposition 2. *A subset A of a group G is large if and only if $\Delta_p(A) \neq \emptyset$ for every $p \in G^*$.*

Proof: Suppose that A is large and pick a finite subset F of G such that $G = FA$. We take an arbitrary $p \in G^*$ and choose $g \in F$ such that $gA \in p$ so $A \in g^{-1}p$ and $\Delta_p(A) \neq \emptyset$.

Assume that $\Delta_p(A) \neq \emptyset$ for each $p \in G^*$. Given any $p \in G^*$, we choose $g_p \in G$ such that $A \in g_p p$. Then we consider a covering of G^* by the subsets $\{g_p^{-1}A : p \in G^*\}$ and choose its finite subcovering $g_{p_1}^{-1}A, \dots, g_{p_n}^{-1}A$. Since $G \setminus (g_{p_1}^{-1}A \cup \dots \cup g_{p_n}^{-1}A)$ is finite, we see that A is large. □

Proposition 3. *For an infinite subset A of a group G the following statements hold:*

(i) *A is thick if and only if there exists $p \in G^*$ such that $\Delta_p(A) = Gp$;*
(ii) *A is prethick if and only if there exists $p \in G^*$ and a finite subset F of G such that $\Delta_p(FA) = Gp$;*
(iii) *A is small if and only if, for every $p \in G^*$ and each finite F of G, we have $\Delta_p(FA) \neq Gp$;*
(iv) *A is thin if and only if $|\Delta_p(A)| \leq 1$ for each $p \in G^*$.*

Proof: (i) We note that A is thick if and only if $G \setminus A$ is not large and apply Proposition 2.

(ii) follows from (i).

(iii) We note that A is small if and only if A is not prethick and apply (ii).

(iv) follows directly from the definitions of thin subsets and $\Delta_p(A)$. □

For $n \in \mathbb{N}$, a subset A of a group G is called n-thin if, for every finite subset F of G, there is a finite subset H of G such that $|Fg \cap A| \leq n$ for every $g \in G \setminus H$.

Proposition 4. *For a subset A of a group G, the following statements are equivalent:*

(i) *$|\Delta_p(A)| \leq n$ for each $p \in G^*$;*
(ii) *for every distinct $x_1, \dots, x_{n+1} \in G$, the set $x_1A \cap \dots \cap x_{n+1}A$ is finite;*
(iii) *A is n-thin.*

Proof: We note that $x_1A \cap \dots \cap x_{n+1}A$ is infinite if and only if there exists $p \in G^*$ such that $x_1^{-1}p, \dots, x_{n+1}^{-1}p \in A^*$. This observation proves the equivalence (i)⇔(ii).

(ii)⇒(iii) Assume that A is not thin. Then there are a finite subset F of G and an injective sequence $(g_m)_{m<\omega}$ in G such that $|Fg_m \cap A| > n$. Passing to subsequences of $(g_m)_{m<\omega}$, we may suppose that there exist distinct $x_1, \dots, x_{n+1} \in F$ such that $\{x_1, \dots, x_{n+1}\}g_m \subseteq A$ so $x_1^{-1}A \cap \dots \cap x_{n+1}^{-1}A$ is infinite.

(iii)⇒(i) Assume that $x_1A \cap \dots \cap x_{n+1}A$ is infinite for some distinct $x_1, \dots, x_{n+1} \in G$. Then there is an injective sequence $(g_m)_{m<\omega}$ in $x_1A \cap \dots \cap x_{n+1}A$ such that $\{x_1^{-1}, \dots, x_{n+1}^{-1}\}g_m \subset A$ so A is not n-thin. □

By [7], a subset A of a countable group G is n-thin if and only if A can be partitioned into $\leq n$ thin subsets. The following statements are from [15]. Every n-thin subset of an Abelian group of cardinality $\aleph_m$ can be partitioned into $\leq n^{m+1}$ thin subsets. For each $m \geq 2$ there exist a group G of cardinality $\aleph_n$, $n = \frac{m(m+1)}{2}$ and a 2-thin subset A of G which cannot be partitioned into m thin subsets. Moreover, there is a group G of cardinality $\aleph_\omega$ and a 2-thin subset A of G which cannot be finitely partitioned into thin subsets.

Recall that an ultrafilter $p \in G^*$ is strongly prime if $p \in G^* \setminus \overline{G^* G^*}$.

Proposition 5. *For a subset A of a group G, the following statements are equivalent:*

(i) *A is sparse;*
(ii) *every ultrafilter $p \in A^*$ is strongly prime;*
(iii) *for every infinite subset X of G, there exists a finite subset $F \subset X$ such that $\bigcap_{g \in F} gA$ is finite.*

Proof: The equivalence (ii)$\Leftrightarrow$(iii) was proved in [3, Theorem 9].

To prove (i)$\Leftrightarrow$(ii), it suffices to note that $\Delta_p(A)$ is infinite if and only if $\Delta_p(A)$ has a limit point qp, $q \in G^*$ in A^*. □

Proposition 6. *A subset A of a group G is sparse if and only if, for every countable subgroup H of G, $A \cap H$ is sparse in H.*

Proof: Assume that A is not sparse. By Proposition 5(iii), there is a countable subset $X = \{x_n : n < \omega\}$ of G such that for any $n < \omega$ $x_0 A \cap \dots \cap x_n A$ is infinite. For any $n < \omega$, we pick $a_n \in x_0 A \cap \dots \cap x_n A$, put $S = \{x_0^{-1} a_n, \dots, x_n^{-1} a_n : n < \omega\}$ and denote by H the subgroup of G generated by $S \cup X$. By Proposition 5(iii), $A \cap H$ is not sparse in H. □

A family $\mathcal{I}$ of subsets of a group G is called an ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, and $A \in \mathcal{I}$, $A' \subset A$ implies $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is left (right) translation invariant if $gA \in \mathcal{I}$ ($Ag \in \mathcal{I}$) for each $A \in \mathcal{I}$.

Proposition 7. *The family Sp_G of all sparse subsets of a group G is a left and right translation invariant ideal in $\mathcal{P}_G$.*

Proof: Apply Proposition 1. □

Proposition 8. *For a subset A of a group G, the following statements are equivalent:*

(i) *A is disparse;*
(ii) *if $p \in A^*$ then $p \notin G^* p$.*

Recall that an element s of a semigroup S is right cancelable if, for any $x, y \in S$, $xs = ys$ implies $x = y$.

Proposition 9. *A subset A of a countable group G is disparse if and only if each ultrafilter $p \in A^*$ is right cancelable in βG.*

Proof: By [5, Theorem 8.18], for a countable group G, an ultrafilter $p \in G^*$ is right cancelable in βG if and only if $p \notin G^* p$. Apply Proposition 8. $\square$

Proposition 10. *The family dSp_G of all disparse subsets of a group G is a left and right translation invariant ideal in $\mathcal{P}_G$.*

Proof: Assume that $A \cup B$ is not disparse and pick $p \in G^*$ such that $\Delta_P(A \cup B)$ has a non-isolated point gq. Then either $gp \in A^*$ or $gp \in B^*$ so gp is non-isolated either in $\Delta_p(A)$ or in $\Delta_p(B)$.

To see that dSp_G is translation invariant, we apply Proposition 1. $\square$

For an injective sequence $(a_n)_{n<\omega}$ in a group G, we denote

$$FP(a_n)_{n<\omega} = \{a_{i_1} a_{i_2} \dots a_{i_n} : i_1 < \dots < i_n < \omega\}.$$

Proposition 11. *For every disparse subset A of a group G, the following two equivalent statements hold:*

(i) *if q is an idempotent from G^* and $g \in G$ then $qg \notin A^*$;*
(ii) *for each injective sequence $(a_n)_{n<\omega}$ in G and each $g \in G$, $FP(a_n)_{n<\omega} g \setminus A$ is infinite.*

Proof: The equivalence (i)$\Leftrightarrow$(ii) follows from two well-known facts. By [5, Theorem 5.8], for every idempotent $q \in G^*$ and every $Q \in q$, there is an injective sequence $(a_n)_{n<\omega}$ in Q such that $FP(a_n)_{n<\omega} \subseteq Q$. By [5, Theorem 5.11], for every injective sequence $(a_n)_{n<\omega}$ in G, there is an idempotent $q \in G^*$ such that $FP(a_n)_{n<\omega} \in q$.

Assume that $qg \in A^*$. Then $q(qg) = qg$ so $qg \in G^* qg$ and, by Proposition 8, A is not disparse. $\square$

Proposition 12. *For every infinite group G, we have the following strong inclusions*

$$Sp_G \subset dSp_G \subset Sm_G,$$

where Sm_G is the ideal of all small subsets of G.

Proof: Clearly, $Sp_G \subseteq dSp_G$. To verify $dSp_G \subseteq Sm_G$, we assume that a subset A of G is not small. Then A is prethick and, by Proposition 2(ii), there exist $p \in G^*$ and a finite subset F of G such that $\Delta_p(FA) = Gp$. Hence, $G^* p \subseteq (FA)^*$. We take an arbitrary idempotent $q \in G^*$ and choose $g \in F$ such that $qp \in (gA)^*$. Since $q(qp) = qp$ so $q \in G^* qp$ and, by Proposition 8(ii), gA is not disparse. By Proposition 10 A is not disparse.

To prove that $dSp_G \setminus Sp_G \neq \emptyset$ and $Sm_G \setminus dSp_G \neq \emptyset$, we may suppose that G is countable. We put $F_0 = \{e\}$ and write G as an union of an increasing chain $\{F_n : n < \omega\}$ of finite subsets.

1. To find a subset $A \in dSp_G \setminus Sp_G$, we choose inductively two sequences $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ in G such that

(1) $F_n b_n \cap F_{n+1} b_{n+1} = \emptyset$, $n < \omega$;
(2) $F_i a_i b_j \cap F_k a_k b_m = \emptyset$, $0 \le i \le j < \omega$, $0 \le k \le m < \omega$, $(i, j) \neq (k, m)$.

We put $a_0 = b_0 = e$ and assume that $a_0, \dots, a_n$, $b_0, \dots, b_n$ have been chosen. We choose b_{n+1} to satisfy $F_{n+1}b_{n+1} \cap F_i b_i = \emptyset$, $i \le n$ and

$$\bigcup_{0\le i\le j<\omega} F_i a_a b_i \cap (\bigcup_{o\le i\le n} F_i a_i) b_{n+1} = \emptyset.$$

Then we pick a_{n+1} so that

$$F_{n+1}a_{n+1}b_{n+1} \cap (\bigcup_{0\le i\le j<\omega} F_i a_i b_j) = \emptyset,\ F_{n+1}a_{n+1}b_{n+1} \cap (\bigcup_{0\le i\le n} F_i a_i b_{n+1}) = \emptyset.$$

After ω steps, we put $A = \{a_i b_j : 0 \le i \le j < \omega\}$, choose two free ultrafilters p, q such that $\{a_i : i < \omega\} \in p$, $\{b_i : i < \omega\} \in q$ and note that $A \in pq$. By Proposition 5(ii), $A \notin Sp_G$.

To prove that $A \in dSp_G$, we fix $p \in G^*$ and take an arbitrary $q \in \Delta_p(A)$. For $n < \omega$, let $A_n = \{a_i b_j : 0 \le i \le n, i \le j < \omega\}$. By (1), the set $\{b_j : j < \omega\}$ is thin. Applying Proposition 2(iv) and Proposition 1, we see that A_n is sparse. Therefore, if $A_n \in q$ for some $n < \omega$ then q is isolated in $\Delta_p(A)$. Assume that $A_n \notin q$ for each $n < \omega$. We take an arbitrary $g \in G \setminus \{e\}$ and choose $m < \omega$ such that $g \in F_m$. By (2), $g(A \setminus A_m) \cap A = \emptyset$ so $gq \notin A^*$. Hence, $\Delta_p(A) = \{q\}$.

2. To find a subset $A \in Sm_G \setminus dSp_G$, we choose inductively two sequences $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ in G such that, for each $m < \omega$, the following statement holds:

(3) $b_m FP(a_n)_{n<\omega} \cap F_m(FP(a_n)_{n<\omega}) = \emptyset$.

We put $a_0 = e$ and take an arbitrary $g \in G \setminus \{e\}$. Suppose that $a_0, \dots, a_m$ and $b_0, \dots, b_m$ have been chosen. We pick b_{m+1} so that

$$b_{m+1} FP(a_n)_{n\le m} \cap F_{m+1}(FP(a_n)_{n\le m}) = \emptyset$$

and choose a_{n+1} such that

$$b_{m+1}(FP(a_n)_{n\le m})a_{n+1} \cap F_{m+1}(FP(a_n)_{n\le m}) = \emptyset,$$
$$b_{m+1}(FP(a_n)_{n\le m}) \cap F_{m+1}(FP(a_n)_{n\le m})a_{n+1} = \emptyset.$$

After ω steps, we put $A = FP(a_n)_{n<\omega}$. By Proposition 11, $A \notin dSp_G$. To see that $A \in Sm_G$, we use (3) and the following observation. A subset S of a group G is small if and only if $G \setminus FS$ is large for each finite subset F of G. $\square$

Proposition 13. *Let G be a direct product of some family $\{G_\alpha : \alpha < \kappa\}$ of countable groups. Then G can be partitioned into $\aleph_0$ disparse subsets.*

Proof: For each $\alpha < \kappa$, we fix some bijection $f_\alpha : G_\alpha \setminus \{e_\alpha\} \to \mathbb{N}$, where e_α is the identity of G_α. Each element $g \in G \setminus \{e\}$ has the unique representation

$$g = g_{\alpha_1} g_{\alpha_2} \dots g_{\alpha_n},\quad \alpha_1 < \alpha_2 < \dots < \alpha_n < \kappa,\quad g_{\alpha_i} \in G_{\alpha_i} \setminus \{e_{\alpha_i}\}.$$

We put $suptg = \{\alpha_1, \dots, \alpha_n\}$ and let $Seq_{\mathbb{N}}$ denote the set of all finite sequence in $\mathbb{N}$. We define a mapping $f : G \setminus \{e\} \to Seq_{\mathbb{N}}$ by

$$f(g) = (n, f_{\alpha_1}(g_{\alpha_1}), \dots, f_{\alpha_n}(g_{\alpha_n}))$$

and put $D_s = f^{-1}(s)$, $s \in Seq_{\mathbb{N}}$.

We fix some $s \in Seq_{\mathbb{N}}$ and take an arbitrary $p \in G^*$ such that $p \in D_s^*$. Let $s = \{n, m_1, \dots, m_n\}$, $g \in D_s$ and $i \in suptg$. It follows that, for each $i < \kappa$, there exists $x_i \in G_i$ such that $x_i H_i \in p$, where $H_i = \otimes\{G_i : j < \kappa, j \neq i\}$. We choose $i_1, \dots, i_k$, $k < n$ such that

$$\{i_1, \dots, i_k\} = \{i < \kappa : x_i H_i \in p,\ x_i \neq e_i\},$$

put $P = x_{i_1} H_{i_1} \cap \dots \cap x_{i_k} H_{i_k} \cap D_s$ and assume that $gp \in P^*$ for some $g \in G \setminus \{e\}$. Then $suptg \cap \{i_1, \dots, i_k\} = \emptyset$. Let $suptg = \{j_1, \dots, j_t\}$, $H = H_{j_1} \cap \dots \cap H_{j_t}$. Then $H \in p$ but $g(H \cap P) \cap D_s = \emptyset$ because $|suptgx| > n$ for each $x \in H \cap P$. In particular, $gp \notin P^*$. Hence, p is isolated in $\Delta_p(D_s)$. $\square$

By Proposition 13, every infinite group embeddable in a direct product of countable groups (in particular, every Abelian group) can be partitioned into $\aleph_0$ disparse subsets.

Question 1. *Can every infinite group be partitioned into $\aleph_0$ disparse subsets?*

By [9], every infinite group can be partitioned into $\aleph_0$ small subsets. For an infinite group G, $\eta(G)$ denotes the minimal cardinality κ such that G can be partitioned into $\eta(G)$ sparse subsets. By [11, Theorem 1], if $|G| > (\kappa^+)^{\aleph_0}$ then $\eta(G) > \kappa$, so Proposition 12 does not hold for partition of G into sparse subsets. For partitions of groups into thin subsets see [10].

3. Comments

1. A subset A of an amenable group G is called *absolute null* if $\mu(A) = 0$ for each Banach measure μ on G, i.e. finitely additive left invariant function $\mu : \mathcal{P}_G \to [0, 1]$. By [6, Theorem 5.1] and Proposition 5, every sparse subset of an amenable group G is absolute null.

Question 2. *Is every disparse subset of an amenable group G absolute null?*

To answer this question in affirmative, in view of Proposition 8, it would be enough to show that each ultrafilter $p \in G^*$ such that $p \notin G^*p$ has an absolute null member $P \in p$. But that is not true. We sketch a corresponding counterexample.

We put $G = \mathbb{Z}$ and choose inductively an injective sequence $(a_n)_{n<\omega}$ in $\mathbb{N}$ such that, for each $m < \omega$ and $i \in \{-(m+1), \dots, -1, 1, \dots, m+1\}$, the following statements hold:

$(*)$ $\quad (\bigcup_{n>m}(a_n + 2^{a_n}\mathbb{Z})) \cap (i + \bigcap_{n>m}(a_n + 2^{a_n}\mathbb{Z})) = \emptyset.$

Then we fix an arbitrary Banach measure μ on $\mathbb{Z}$ and choose an ultrafilter $q \in \mathbb{Z}^*$ such that $2^n\mathbb{Z} \in q$, $n \in \mathbb{N}$ and $\mu(Q) > 0$ for each $Q \in q$. Let $p \in G^*$ be a limit point of the set $\{a_n + q : n < \omega\}$. Clearly, $\mu(P) > 0$ for each $P \in p$. On the other hand, by $(*)$, the set $\mathbb{Z} + p$ is discrete so $p \notin \mathbb{Z}^* + p$.

In [18], for a group G, S. Solecki defined two functions $\sigma^R, \sigma^L : \mathcal{P}_G \to [0,1]$ by the formulas

$$\sigma^R(A) = \inf_F \sup_{g \in G} \frac{F \cap Ag}{|F|}, \quad \sigma^L(A) = \inf_F \sup_{g \in G} \frac{|F \cap gA|}{|F|},$$

where inf is taken over all finite subsets of G.

By [1] and [20], a subset A of an amenable group is absolute null if and only if $\sigma^R(A) = 0$.

Question 3. *Is $\sigma^R(A) = 0$ for every sparse subset A of a group G?*

To answer this question positively it suffices to prove that if $\sigma^R(A) > 0$ then there is $g \in G \setminus \{e\}$ such that $\sigma^R(A \cap gA) > 0$.

2. The origin of the following definition is in asymptology (see [16], [17]). A subset A of a group G is called *asymptotically scattered* if, for any infinite subset X of A, there is a finite subset H of G such that, for any finite subset F of G satisfying $F \cap H = \emptyset$, we can find a point $x \in X$ such that $Fx \cap A = \emptyset$. By [13, Theorem 13] and Propositions 5 and 6, a subset A is sparse if and only if A is asymptotically scattered.

We say that a subset A of G is *weakly asymptotically scattered* if, for any subset X of A, there is a finite subset H of G such that, for any finite subset F of G satisfying $F \cap H = \emptyset$, we can find a point $x \in X$ such that $Fx \cap X = \emptyset$.

Question 4. *Are there any relationships between disparse and weakly asymptotically scattered subsets?*

3. Let A be a subset of a group G such that each ultracompanion $\Delta_p(A)$ is compact. We show that A is sparse. In view of Proposition 6, we may suppose that G is countable. Assume the contrary: $\Delta_p(A)$ is infinite for some $p \in G^*$. On one hand, the countable compact space $\Delta_p(A)$ has an injective convergent sequence. On the other hand, G^* has no such a sequence.

4. Let X be a subset of a group G, $p \in G^*$. We say that the set Xp is uniformly discrete if there is $P \in p$ such that $xP^* \cap yP^* = \emptyset$ for all distinct $x, y \in X$.

Question 5. *Let A be a disparse subset of a group G. Is $\Delta_p(A)$ uniformly discrete for each $p \in G^*$?*

5. Let $\mathcal{F}$ be a family of subsets of a group G, A be a subset of G. We denote $\delta_{\mathcal{F}}(A) = \{g \in G : gA \cap A \in \mathcal{F}\}$. If $\mathcal{F}$ is the family of all infinite subsets of G, $\delta_{\mathcal{F}}(A)$ was introduced in [14] under the name combinatorial derivation of A. Now suppose that $\delta_p(\mathcal{F}) \neq \emptyset$, pick $q \in A^* \cap Gp$ and note that $\delta_p(A) = \delta_q(A)$. Then $\delta_q(A) = (\delta_q(A))^{-1}q$.

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(*Received* May 20, 2013, *revised* July 27, 2013)