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On extensions of bounded subgroups in Abelian groups

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Abstract. It is well-known that every bounded Abelian group is a direct sum of finite cyclic subgroups. We characterize those non-trivial bounded subgroups H of an infinite Abelian group G, for which there is an infinite subgroup G_0 of G containing H such that G_0 has a special decomposition into a direct sum which takes into account the properties of G, and which induces a natural decomposition of H into a direct sum of finite subgroups.

Keywords: Abelian group; bounded group; simple extension

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1. Introduction

Recall that an Abelian group G is *of finite exponent* or *bounded* if there exists a positive integer n such that $ng = 0$ for every $g \in G$. The minimal integer n with this property is called the *exponent* of G and is denoted by $\exp(G)$. When G is not bounded, we write $\exp(G) = \infty$ and say that G is *of infinite exponent* or *unbounded*.

The structure theory of infinite Abelian groups is sufficiently difficult and complicated. Fortunately, for a bounded Abelian group G there is a complete and clear description of its structure: G is a direct sum of finite cyclic subgroups. If G is not of finite exponent, G can even not be decomposable into a direct sum of two non-trivial subgroups.

Let now H be a bounded subgroup of an infinite Abelian group G. As simple examples show, even in the case H is finite and cyclic, H may not be a direct summand of G. So it is important to find a subgroup G_0 of G containing H such that G_0 has a decomposition into a direct sum of subgroups having simple forms which takes into account the properties of G (as $\exp(G)$), and which induces a decomposition of H into a direct sum of finite subgroups. The existence of such extensions of H plays an essential role in particular for constructing of Hausdorff group topologies on G having specific properties with respect to H. We demonstrate this by the following examples.

Let $G = \mathbb{Z}(3) \oplus \mathbb{Z}(2)^\omega$, $G_0 = \mathbb{Z}(2)^\omega$, H_1 is the first $\mathbb{Z}(2) \times \mathbb{Z}(2)$ in G and $H_2 = \mathbb{Z}(3)$. It is easy to see that G does not admit a connected Hausdorff group topology (see [4, §9]). On the other hand, Markov showed in [5] that there is a

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locally connected Hausdorff group topology τ on G such that G_0 is the connected component of (G, τ). So, algebraically H_1 can be extended to a subgroup G_0 which is connected. However, there is no Hausdorff group topology τ' on G in which H_2 is contained in a connected subgroup of (G, τ') because G_0 is clopen in any group topology on G [4, §9]. Further, it can be proved that there is a Hausdorff group topology ν on G such that H_1 is the von Neumann radical of (G, ν), but for H_2 such topologies do not exist (see [2]). Actually, these positive and negative results for H_1 and H_2 in G (and more generally, for subgroups of Abelian groups of finite exponent) depend on the possibility to extend them to an *infinite* subgroup G_0 (maybe of a big cardinality) such that G_0 is a direct sum of *finite* subgroups of *the same* exponent (see [3]). Between all infinite extensions of H_1 in G, which can be represented as a direct sum of finite subgroups of the same exponent, there is the smallest one by cardinality, for example $G_1 = \mathbb{Z}(2)^{(\omega)}$. So, the subgroup G_1 has the following properties: (1) G_1 is of finite exponent as G, (2) G_1/H_1 is countable, (3) $G_1 = \bigoplus_{i\in\omega} S_i$ with $\exp(G_i) = \exp(H_1)$ for all $i \in \omega$, and (4) this decomposition of G_1 induces a natural decomposition of H (see the conditions (2b) and (3) in the definition below).

Assume now that H is a finite non-trivial subgroup of an Abelian group G of infinite exponent. It is well-known that G contains a subgroup S which has one of the form $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ or $\bigoplus_{i\in\omega} S_i$ with $\exp(H) \leq \exp(S_0) < \exp(S_1) < \dots$. So it is quite natural to consider the subgroup $G_0 := S + H$. Then G_0 takes into account the properties of G and has infinite exponent as G, and G_0/H is countable.

For infinite bounded subgroups H of G the situation is more delicate, but these examples explain our definition of simple extension given below. We note that the main result of the article plays a crucial role for a description of bounded subgroups H of an Abelian non-torsion-free group G for which there exists a Hausdorff group topology τ such that H is the von Neumann radical of (G, τ) (see [3]).

Denote by $o(g)$ the order of an element g of an Abelian group G. The subgroup of G generated by a subset A is denoted by $\langle A\rangle$. We shall say that an Abelian group X *satisfies condition* (Λ) if X is a finite direct sum of groups of the form $\mathbb{Z}(p^a)^{(\kappa)}$, where p is prime, a is a natural number and the cardinal κ is infinite.

Definition 1. Let G be an infinite Abelian non-torsion-free group and H its non-zero bounded subgroup. We say that H has a *simple extension* in G if there is a subgroup G_0 of G which has a decomposition of the form

$$G_0 = X \oplus \bigoplus_{i\in\omega} S_i,$$

where:

(1) if $X \neq \{0\}$, then X is a subgroup of H satisfying condition (Λ);
(2) one of the following conditions holds:
   (a) $S_i = \{0\}$ for every $i \in \mathbb{N}$, and S_0 has one of the form $\mathbb{Z} \oplus H_0$ or $\mathbb{Z}(p^\infty) + H_0$, where H_0 is a finite (maybe trivial) subgroup of H;

(b) for every $i \in \omega$, S_i is a finite non-trivial subgroup of G such that either

$$\exp(H) \leq \exp(S_0) < \exp(S_1) < \dots, \quad \text{or}$$
$$\exp(H) = \exp(S_0) = \exp(S_1) = \dots;$$

(3) $H = X \oplus \bigoplus_{i\in\omega}(S_i \cap H)$.

Returning to the first above-mentioned example we see that H_1 has a simple extension (for instance, G_1), but H_2 does not have simple extensions in G.

The main goal of the article is to characterize all bounded subgroups of an infinite Abelian non-torsion-free group G which have a simple extension in G.

Theorem 2. *Let H be a non-zero bounded subgroup of an infinite Abelian group G. Then:*

(i) *if $\exp(G) = \infty$, then H has a simple extension in G;*
(ii) *if $\exp(G) < \infty$, then H has a simple extension in G if and only if G contains a subgroup of the form $\mathbb{Z}(\exp(H))^{(\omega)}$.*

In Theorems 9 and 10 below we prove more precise results.

2. The proof of Theorem 2

We shall use the following easy corollary of Prüfer-Baer's theorem [1, 11.2].

Lemma 3. *Let G be an infinite Abelian group of finite exponent. Then G is the direct sum $G = G_0 \oplus G_1$ of a finite (maybe trivial) subgroup G_0 and a subgroup G_1 satisfying condition (Λ).*

Let us recall that a subset X of an Abelian group G is called *independent* if for every finite sequence $x_1, \dots, x_n$ of pairwise distinct elements of X and each sequence $m_1, \dots, m_n$ of integers $m_1x_1 + \dots + m_nx_m = 0$ implies $m_ix_i = 0$ for all $i = 1, \dots, n$.

Proposition 4. *Let $G = \mathbb{Z}(p^\infty) + H$, where H is an infinite Abelian group of finite exponent. Then there is a finite (maybe trivial) subgroup H_0 of H and an infinite subgroup H_1 of H such that*

(1) *$H = H_0 \oplus H_1$;*
(2) *$G = (\mathbb{Z}(p^\infty) + H_0) \oplus H_1$;*
(3) *H_1 satisfies condition (Λ).*

Proof: By Prüfer-Baer's theorem [1, 11.2], H has a decomposition $H = \oplus_{i\in I}C_i$, where C_i are cyclic finite groups. As H is bounded, $\mathbb{Z}(p^\infty) \cap H$ is finite, so there exists a finite subset $J \subseteq I$ such that $\mathbb{Z}(p^\infty) \cap H \subseteq \oplus_{i\in J}C_i$.

We claim that the sum

$$G = \left(\mathbb{Z}(p^\infty) + \bigoplus_{i\in J} C_i\right) + \left(\bigoplus_{i\in I\setminus J} C_i\right)$$

is direct. Indeed, let $t = f+g \in (\mathbb{Z}(p^\infty)+\oplus_{i\in J}C_i)\cap(\oplus_{i\in I\setminus J}C_i)$, where $f \in \mathbb{Z}(p^\infty)$ and $g \in \oplus_{i\in J}C_i$. Then $f = t-g \in \oplus_{i\in J}C_i$ by the definition of J. Thus $t \in \oplus_{i\in J}C_i$. Since also $t \in \oplus_{i\in I\setminus J}C_i$, we obtain $t = 0$ and the sum is direct.

Using Lemma 3, decompose $\oplus_{i\in I\setminus J}C_i = H_0' \oplus H_1$, where H_0' is finite and H_1 satisfies condition (Λ). Put $H_0 = H_0' \oplus (\oplus_{i\in J}C_i)$. Then H_0 is a finite (maybe trivial) subgroup of H and H_1 is infinite. By construction and the claim, H_0 and H_1 satisfy conditions (1)–(3) of the proposition. □

The next proposition is not trivial only for uncountable subgroups and its proof essentially repeats the proof of Proposition 4.

Proposition 5. *Let an Abelian p-group G have the form $G = \langle A\rangle + H$, where H is an uncountable subgroup of G of finite exponent and $A = \{g_i\}_{i=1}^\infty$ is an independent sequence in G. Then there is a countable (maybe trivial) subgroup H_0 of H and an uncountable subgroup H_1 of H such that*

(1) *$H = H_0 \oplus H_1$;*
(2) *$G = (\langle A\rangle + H_0) \oplus H_1$;*
(3) *H_1 satisfies condition (Λ).*

Proof: By [1, 11.2], H has a decomposition $H = \oplus_{i\in I}C_i$, where C_i are cyclic finite groups. As $\langle A\rangle$ is countable, there exists a countable subset $J \subseteq I$ such that $\langle A\rangle \cap H \subseteq \oplus_{i\in J}C_i$. We claim that the sum

$$G = \left(\langle A\rangle + \bigoplus_{i\in J} C_i\right) + \left(\bigoplus_{i\in I\setminus J} C_i\right)$$

is direct. Indeed, let $t = f+g \in (\langle A\rangle+\oplus_{i\in J}C_i)\cap(\oplus_{i\in I\setminus J}C_i)$, where $f \in \langle A\rangle$ and $g \in \oplus_{i\in J}C_i$. Then $f = t-g \in \oplus_{i\in J}C_i$ by the definition of J. Thus $t \in \oplus_{i\in J}C_i$. Since also $t \in \oplus_{i\in I\setminus J}C_i$, we obtain $t = 0$ and the sum is direct.

Using Lemma 3, decompose $\oplus_{i\in I\setminus J}C_i = H_0' \oplus H_1$, where H_0' is finite and H_1 satisfies condition (Λ). Put $H_0 = H_0' \oplus (\oplus_{i\in J}C_i)$. Then H_0 is a countable (maybe trivial) subgroup of H and H_1 is infinite. By construction and the claim, H_0 and H_1 satisfy conditions (1)-(3) of the proposition. □

We omit the proof of the following simple lemma.

Lemma 6. *Let a sequence $\{b_n\}$ in an Abelian group G be independent and H be a finite subgroup of G. Then there is n_0 such that $H \cap \langle b_{n_0}, b_{n_0+1}, \dots\rangle = \{0\}$.*

We denote division by ":". In the next proposition we set $\infty - 1 = \infty$.

Proposition 7. *Let G be an Abelian p-group of the form $G = \langle A\rangle + H$, where H is a nonzero countable group of finite exponent and $A = \{g_i\}_{i=0}^\infty$ is an independent sequence such that either*

(a) *$\exp(H) \le N \le o(g_0) < o(g_1) < \dots$ for some natural number N, or*
(b) *$\exp(H) = o(g_i)$ for every $i \ge 0$.*

Then G has a subgroup G_0 of the form

$$G_0 = \bigoplus_{i=0}^{\infty}(H_i + \langle e_i \rangle),$$

where

(1) *the independent sequence $\{e_i\}$ satisfies the same condition* (a) *or* (b) *as the sequence $\{g_i\}$;*
(2) *there is $0 < M \le \infty$ such that H_j is a finite nonzero subgroup of G for every $0 \le j < M$, and, if $M < \infty$, $H_j = \{0\}$ for each $j \ge M$;*
(3) $H = \bigoplus_{i=0}^{\infty} H_i$.

Proof: We distinguish between two cases.

Case 1. *$\langle A \rangle \cap H$ is finite* (*maybe trivial*). By Lemma 6 we can choose $k \ge 0$ such that $(\langle A \rangle \cap H) \cap \langle g_k, g_{k+1}, \dots \rangle = \{0\}$. Then also $H \cap \langle g_k, g_{k+1}, \dots \rangle = \{0\}$. Set $e_i = g_{k+i}$, for every $i \ge 0$. Let $H = \bigoplus_{i=0}^{M-1} \langle h_i \rangle$, where $M \le \infty$ and $i \in \mathbb{N}$ [1, 11.2]. Set $G_0 = \langle e_0, e_1, \dots \rangle + H$. Then we have

$$G_0 = \bigoplus_{i=0}^{\infty}(H_i \oplus \langle e_i \rangle),$$

where $H_i = \langle h_i \rangle$ if $i < M$, and $H_i = 0$ for $i \ge M$. Then G_0 is as desired.

Case 2. *$\langle A \rangle \cap H$ is infinite*. Then H is countably infinite. Let $H = \bigoplus_{i=0}^{\infty} \langle h_i \rangle$ [1, 11.2]. We shall construct the sequences $\{H_n\}$ and $\{e_n\}$ by induction. Set

$$G^0 = G, \quad H^0 = H, \text{ and } g_j^0 = g_j, \forall j \ge 0.$$

Put $e_0 = g_0^0$. Choose the minimal index $\kappa_1 \ge 0$ such that

$$H^0 \cap \langle e_0 \rangle = \left(\bigoplus_{i=0}^{\kappa_1} \langle h_i \rangle \right) \cap \langle e_0 \rangle.$$

Set

$$Y_k^1 = \langle \{g_{k+i}^0\}_{i=1}^{\infty} \rangle, k \ge 0, \quad H_0 = \bigoplus_{i=0}^{\kappa_1} \langle h_i \rangle, \text{ and } X_1 = \bigoplus_{i=\kappa_1+1}^{\infty} \langle h_i \rangle.$$

Then $H_0 \neq 0$ and $H^0 = H_0 \oplus X_1$. We will need that

$$(H_0 + \langle e_0 \rangle) \cap X_1 = \{0\}. \tag{1}$$

Indeed, let $ae_0 + h_0 = x$, where a is integer, $h_0 \in H_0$ and $x \in X_1$. Then $ae_0 = x - h_0 \in H^0$ and hence $ae_0 \in H_0$. Thus $x = ae_0 + h_0 \in H_0 \cap X_1 = \{0\}$, and hence $x = 0$.

We distinguish between two subcases.

Subcase 2.1. *There is $k \ge 0$ such that*

$$\left(Y_k^1 + X_1\right) \cap (H_0 + \langle e_0 \rangle) = \{0\}.$$

Then we set

$$H^1 = X_1 = \bigoplus_{i=\kappa_1+1}^{\infty} \langle h_i \rangle, \quad g_j^1 = g_{k+1+j}^0, \forall j \ge 0, \text{ and } G^1 = \langle \{g_j^1\}_{j=0}^{\infty} \rangle + H^1.$$

So $H = H^0 = H_0 \oplus H^1$ and $(H_0 + \langle e_0 \rangle) \cap G^1 = \{0\}$, and we can proceed to the second step for G^1, H^1 and the independent sequence $\left\{g_j^1\right\}_{j=0}^{\infty}$ satisfying the same condition (a) or (b) as the sequence $\{g_j^0\}$.

Subcase 2.2. *For every $k \ge 0$,*

$$\left(Y_k^1 + X_1\right) \cap (H_0 + \langle e_0 \rangle) \ne \{0\}.$$

In this case, because of finiteness of $H_0 + \langle e_0 \rangle$ and since $\exp(X_1) < \infty$, we can choose the maximal natural number m satisfying the following condition:

($*$) there is a nonzero element $h \ne 0$ of $H_0 + \langle e_0 \rangle$ such that for infinitely many indices k, there are $y_k \in Y_k^1$ and $z_k \in X_1$ for which

$$y_k + z_k = h \quad \text{and} \quad o(y_k) = p^m.$$

Fix h satisfying ($*$) and choose the following:

(i) a sequence of indices of the form

$$0 < i_1^0 < \dots < i_{s_0}^0 < i_1^1 < \dots < i_{s_1}^1 < i_1^2 < \cdots ; \tag{2}$$

(ii) a sequence of integers $a_1^k, \dots, a_{s_k}^k$, where $(a_i^j, p) = 1$ for all i and j;

(iii) a sequence of natural numbers $r_1^k, \dots, r_{s_k}^k, \forall k \ge 0$; and

(iv) a sequence $z_0, z_1, \dots$ in X_1,

such that, for every $k \ge 0$,

$$0 \ne h = a_1^k p^{r_1^k} g_{i_1^k}^0 + \dots + a_{s_k}^k p^{r_{s_k}^k} g_{i_{s_k}^k}^0 + z_k \quad \text{and} \quad o(h - z_k) = p^m. \tag{3}$$

Set $t_k = \min\{r_1^k, \dots, r_{s_k}^k\}$ and

$$y_k' = a_1^k p^{r_1^k - t_k} g_{i_1^k}^0 + \dots + a_{s_k}^k p^{r_{s_k}^k - t_k} g_{i_{s_k}^k}^0, \ \forall k \ge 0.$$

So $o(p^{t_k} y_k') = p^m$ and $o(y_k') = p^{t_k + m}$ for all $k \ge 0$. By (2), the sequence $\{y_k'\}_{k=0}^{\infty}$ is independent and $p^{t_k} y_k' + z_k = h \in H_0 + \langle e_0 \rangle$ for every $k \ge 0$.

Subcase 2.2(a). Assume that $\exp(H) \le N \le o(g_0) < o(g_1) < \dots$. Then, by (2), $\exp(H) \le N \le o(y_0') < o(y_1') < \dots$, and hence $t_0 < t_1 < \dots$. Set

$$g_k' = p^{t_{2k+1} - t_{2k}} y_{2k+1}' - y_{2k}', \quad \forall k \ge 0.$$

Subcase 2.2(b). Assume that $\exp(H) = o(g_k), \forall k \geq 0$. Then $t_k = t_{k+1}$ and $p^{t_k+m} = \exp(H)$ for every $k \geq 0$. Put

$$g'_k = y'_{2k+1} - y'_{2k}, \quad \forall k \geq 0.$$

In both subcases 2.2(a) and 2.2(b) we have the following:

(α_1) the sequence $\{g'_j\}_{j=0}^{\infty}$ is independent by (2),
(α_2) the sequence $\{g'_j\}_{j=0}^{\infty}$ satisfies the same condition (a) or (b) as $\{g^0_j\}$,
(α_3) $o(g'_k) = o(y'_{2k}) = p^{t_{2k}+m}$, for every $k \geq 0$,
(α_4) $p^{t_{2k}} g'_k = p^{t_{2k+1}} y'_{2k+1} - p^{t_{2k}} y'_{2k} = z_{2k} - z_{2k+1} \in X_1$ by (3).

Set $Y'_k = \langle \{g'_j\}_{j=k}^{\infty} \rangle$, $k \geq 0$. Let us prove the following:

Claim. *There is $k \geq 0$ such that*

$$(Y'_k + X_1) \cap (H_0 + \langle e_0 \rangle) = \{0\}.$$

Proof of Claim: Assuming the converse we can find (as in (i)–(iv)) a nonzero element h' of $H_0 + \langle e_0 \rangle$, a sequence of indices of the form

$$1 < l^0_1 < \cdots < l^0_{q_0} < l^1_1 < \cdots < l^1_{q_1} < l^2_1 < \cdots,$$

a sequence of integers $b^k_1, \ldots, b^k_{q_k}$, $(b^j_i, p) = 1$, for all i and j, a sequence of natural numbers $w^k_1, \ldots, w^k_{q_k}, \forall k \geq 0$, and a sequence $x_0, x_1, \ldots$ in X_1, such that

$$0 \neq h' = b^k_1 p^{w^k_1} g'_{l^k_1} + \cdots + b^k_{q_k} p^{w^k_{q_k}} g'_{l^k_{q_k}} + x_k, \quad \forall k \geq 0.$$

Suppose there exists $k_0 \geq 0$ such that $w^k_i \geq t_{2l^k_i}$ for all $1 \leq i \leq l^k_{q_k}$ and for each $k \geq k_0$. Then, by (α_4),

$$0 \neq h' = b^k_1 p^{w^k_1 - t_{2l^k_1}} \left(p^{t_{2l^k_1}} g'_{l^k_1} \right) + \cdots + b^k_{q_k} p^{w^k_{q_k} - t_{2l^k_{q_k}}} \left(p^{t_{2l^k_{q_k}}} g'_{l^k_{q_k}} \right) + x_k \in X_1,$$

for every $k \geq k_0$. This contradicts (1) since $h' \in H_0 + \langle e_0 \rangle$.

So we can suppose that there is an infinite set I of indices such that for every $k \in I$ there exists an index $1 \leq \xi_k \leq q_k$ for which $w^k_{\xi_k} < t_{2\mu_k}$, where $\mu_k = l^k_{\xi_k}$. For every $k \in I$ set $\lambda_k = \min\{w^k_1, \ldots, w^k_{q_k}\}$ and

$$y''_k = b^k_1 p^{w^k_1 - \lambda_k} g'_{l^k_1} + \cdots + b^k_{q_k} p^{w^k_{q_k} - \lambda_k} g'_{l^k_{q_k}}.$$

Since $l^k_1 > k$ it follows that $y''_k \in Y^1_k$ for every $k \geq 0$. Thus, for all $k \in I$, we obtain the following:

- $y''_k \in Y^1_k$,
- $0 \neq p^{\lambda_k} y''_k + x_k = h' \in H_0 + \langle e_0 \rangle$,

- and, by (α_1) and (α_3),

$$\begin{aligned} o\left(p^{\lambda_k} y''_k\right) &= \max\left\{ o\left(y'_{2l_1^k}\right) : p^{w_1^k}, \ldots, o\left(y'_{2l_{q_k}^k}\right) : p^{w_{q_k}^k} \right\} \\ &\geq o\left(y'_{2\mu_k}\right) : p^{w_{\xi_k}^k} \quad (\text{since } w_{\xi_k}^k < t_{2\mu_k}) \\ &\geq o\left(y'_{2\mu_k}\right) : p^{t_{2\mu_k}-1} = (\text{by } (\alpha_3)) = p^{m+1}. \end{aligned}$$

Since I is infinite we obtained a contradiction to the choice of m (see condition $(*)$), thus proving the claim. □

By the claim we can choose k such that $(Y'_k + X_1) \cap (H_0 + \langle e_0 \rangle) = \{0\}$. Taking into account (α_1) and (α_2), we can put

$$H^1 = X_1, \quad g_j^1 = g'_{k+j}, \forall j \geq 0, \ \text{ and } G^1 = \langle \{g_j^1\}_{j=0}^{\infty} \rangle + H^1.$$

So $(H_0 + \langle e_0 \rangle) \cap G^1 = \{0\}$ and we proceed to the second step for G^1, H^1 and the independent sequence $\{g_j^1\}_{j=0}^{\infty}$ satisfying respectively one of the conditions (a) or (b) as $\{g_j^0\}$.

Iterating this process, we can find a sequence $\{H_i\}_{i=0}^{\infty}$ of finite nonzero subgroups of H and an independent sequence $\{e_i\}_{i=0}^{\infty}$ satisfying the same condition (a) or (b) as the sequence $\{g_i\}$ such that

$$H = \bigoplus_{i=0}^{\infty} H_i \ \text{ and } \ (H_k + \langle e_k \rangle) \cap \left(\sum_{i=k+1}^{\infty} (H_i + \langle e_i \rangle) \right) = \{0\}, \text{ for every } k \geq 0.$$

Hence the sum $G_0 := \sum_{i=0}^{\infty} (H_i + \langle e_i \rangle)$ is direct. Thus G_0 is as desired. This completes the proof of the proposition. □

In what follows we use the next well-known folklore lemma (the proof is similar to that of Lemma 4.2 of [6]):

Lemma 8. *Let G be an Abelian group of infinite exponent. Then one of the following assertions holds.*

(i) *G is not torsion. Then G has a subgroup $H \cong \mathbb{Z}$.*
(ii) *G is torsion but not reduced. Then G has a subgroup $H \cong \mathbb{Z}(p^\infty)$ for some prime p.*
(iii) *G is both torsion and reduced. Then G has a subgroup $H \cong \bigoplus_{i=0}^{\infty} \mathbb{Z}(n_i)$, where $n_0 < n_1 < \dots$.*

The next two theorems imply and make more precise Theorem 2.

Theorem 9. *Let G be an Abelian group of infinite exponent and H its nontrivial subgroup of finite exponent. Then at least one of the following assertions holds.*

(1) G contains an element g of infinite order. *If we set $G_0 = \langle g \rangle + H$, then $G_0 \cong (\mathbb{Z} \oplus H_0) \oplus X$, where*
   (a) *H_0 is a finite (maybe trivial) subgroup of H,*
   (b) *$H = H_0 \oplus X$,*

(c) $X \neq \{0\}$ *if and only if* H *is infinite. In this case* X *satisfies condition* (Λ).

(2) G contains a subgroup Y of the form $\mathbb{Z}(p^\infty)$. *If we set* $G_0 = Y + H$, *then* $G_0 \cong (\mathbb{Z}(p^\infty) + H_0) \oplus X$, *where*

(a) H_0 *is a finite (maybe trivial) subgroup of* H,

(b) $H = H_0 \oplus X$,

(c) $X \neq \{0\}$ *if and only if* H *is infinite. In this case* X *satisfies condition* (Λ).

(3) G is both torsion and reduced. *Then* G *has a subgroup* G_0 *of the form*

$$G_0 = X \oplus \bigoplus_{i=0}^{\infty} (H_i + \langle e_i \rangle),$$

where

(a) *the independent sequence* $\{e_i\}$ *satisfies the condition*

$$\exp(H) \leq o(e_0) < o(e_1) < \dots;$$

(b) *there is* $0 \leq M \leq \infty$ *such that* H_j *is a finite nonzero subgroup of* G *for every* $0 \leq j < M$, *and, if* $M < \infty$, $H_j = \{0\}$ *for each* $j \geq M$;

(c) $H = X \oplus \bigoplus_{i=0}^{\infty} H_i$;

(d) $X \neq \{0\}$ *if and only if* H *is uncountable. In this case* X *satisfies condition* (Λ).

PROOF: **(1)** *Let* G *contain an element* g *of infinite order*. It is clear that G_0 is a direct sum, i.e., $G_0 = \langle g \rangle \oplus H$.

If H is *infinite*, by Lemma 3, H can be represented in the form $H = H_0 \oplus X$, where H_0 is finite (maybe trivial) and X satisfies condition (Λ). So $G_0 \cong (\mathbb{Z} \oplus H_0) \oplus X$.

If H is *finite* we set $H_0 = H$. Then $G_0 \cong \mathbb{Z} \oplus H_0$.

(2) *Let* G *contains a subgroup* Y *of the form* $\mathbb{Z}(p^\infty)$.

If H is *infinite*, the assertion follows from Proposition 4.

If H is *finite*, it is enough to set $H_0 = H$ (and $X = 0$).

(3) *Let* G *be both torsion and reduced*. For a prime p, let H_p and G_p be the p-components of H and G respectively. Since H is of finite exponent, there are pairwise disjoint primes $p_1, \dots, p_n, p_{n+1}, \dots, p_N$, where $n < \infty$ and $n \leq N \leq \infty$, such that (see [1, Theorem 2.1])

$$H = \bigoplus_{i=1}^{n} H_{p_i} \text{ and } G = \bigoplus_{i=1}^{n} G_{p_i} \oplus G_1,$$

where $G_1 = \bigoplus_{i=n+1}^{N} G_{p_i}$ and all the groups H_{p_i} and G_{p_i} are nonzero.

We distinguish between the following two cases.

Case 1. $\exp(G_1) = \infty$. By Lemma 8, there is an independent sequence $\{e_n\}_{n=0}^{\infty}$ in G_1, where $\exp(H) \leq o(e_0) < o(e_1) < \dots$.

Subcase 1.1. Assume that H is *uncountable*. By Lemma 3, $H = H_0 \oplus X'$, where H_0 is finite (maybe trivial) and X' is an uncountable subgroup of H satisfying condition (Λ). Set $X = X'$.

If $H_0 \neq 0$, we set

$$G_0 = \left((H_0 \oplus \langle e_0 \rangle) \oplus \bigoplus_{i=1}^{\infty} \langle e_i \rangle \right) \oplus X, \text{ and } H_i = 0, \text{ for every } i \geq 1.$$

Then we obtain the desired (with $M = 1$).

If $H_0 = 0$ and hence $H = X$, we set

$$G_0 = \left(\bigoplus_{i=0}^{\infty} \langle e_i \rangle \right) \oplus X, \text{ and } H_i = 0, \text{ for every } i \geq 0.$$

Then we obtain the desired (with $M = 0$).

Subcase 1.2. Assume that H is *countably infinite*. By Lemma 3, $H = H_0 \oplus X'$, where H_0 is finite (maybe trivial) and X' is a countably infinite subgroup of H satisfying condition (Λ). By [1, 11.2] we have $X' = \bigoplus_{i=1}^{\infty} \langle h_i \rangle$. Set

$$G_0 = (H_0 \oplus \langle e_0 \rangle) \oplus \bigoplus_{i=1}^{\infty} (H_i \oplus \langle e_i \rangle), \text{ where } H_i = \langle h_i \rangle \text{ for every } i \geq 1.$$

Then we obtain the desired (in this case $X = 0$ and $M = \infty$).

Subcase 1.3. Assume that H is *finite and non-trivial*. In this case we set

$$H_0 = H, \; G_0 = (H_0 \oplus \langle e_0 \rangle) \oplus \bigoplus_{i=1}^{\infty} \langle e_i \rangle, \text{ and } H_i = 0, \text{ for every } i \geq 1.$$

Then we obtain the desired (in this case $X = 0$ and $M = 1$).

Case 2. $\exp(G_1) < \infty$. In this case there is $1 \leq l \leq n$ such that $\exp(G_{p_l}) = \infty$. If $\bigoplus_{i=1, i\neq l}^{n} H_{p_i}$ is finite, we set $H_0' := \bigoplus_{i=1, i\neq l}^{n} H_{p_i}$ and $X' = 0$. If $\bigoplus_{i=1, i\neq l}^{n} H_{p_i}$ is infinite, then, by Lemma 3, $\bigoplus_{i=1, i\neq l}^{n} H_{p_i} = H_0' \oplus X'$, where H_0' is finite (maybe trivial) and X' satisfies condition (Λ). Set $N = \exp(H)$.

Since G is both torsion and reduced, by Lemma 8, there is an independent sequence $\{g_i\}_{i=0}^{\infty}$ in G_{p_l} satisfying the condition $N \leq o(g_0) < o(g_1) < \dots$. Set $A := \{g_i\}_{i=0}^{\infty}$ and $Y := \langle A \rangle + H_{p_l}$. Note that H_{p_l} is nonzero by construction. If H_{p_l} is uncountable, we apply Proposition 5 to Y and H_{p_l}. If $H_0 \neq \{0\}$ in that Proposition 5 or in the case H_{p_l} is countable, we apply Proposition 7. So we can find a subgroup Y_0 of Y of the form

$$Y_0 = X'' \oplus \bigoplus_{i=0}^{\infty} (H_{p_l}^i + \langle e_i \rangle),$$

where

(a_1) the independent sequence $\{e_i\}$ satisfies the condition

$$N \le o(e_0) < o(e_1) < \dots;$$

(a_2) there is $0 \le M \le \infty$ such that $H^i_{p_l}$ is a finite nonzero subgroup of Y for every $0 \le i < M$, and, if $M < \infty$, $H^i_{p_l} = \{0\}$ for each $i \ge M$;

(a_3) $H_{p_l} = X'' \oplus \bigoplus_{i=0}^{\infty} H^i_{p_l}$;

(a_4) $X'' \neq \{0\}$ if and only if H_{p_l} is uncountable. In this case X'' satisfies condition (Λ).

Subcase 2.1. Assume that H is *uncountable*. Set $X = X' \oplus X''$. Then X is an uncountable subgroup of H satisfying the condition (Λ). Set

$$H^0 = H'_0 \oplus H^0_{p_l},\ H^i = H^i_{p_l} \text{ for } i \ge 1, \text{ and } G_0 = X \oplus \bigoplus_{i=0}^{\infty} (H^i + \langle e_i \rangle).$$

Since $H = X \oplus \bigoplus_{i=0}^{\infty} H^i$ we obtain the desired.

Subcase 2.2. Assume that H is *countably infinite*. Then $X'' = 0$, and X' is either trivial or $X' = \bigoplus_{i=1}^{\infty} H'_i$ by [1, 11.2], where H'_i is a finite (maybe trivial) cyclic group for every $i \ge 1$. Set $H^0 = H'_0 \oplus H^0_{p_l}$, and for every $i \ge 1$ put

$$H^i = H'_i \oplus H^i_{p_l} \text{ if } X' \neq 0, \text{ and } H^i = H^i_{p_l} \text{ if } X' = 0.$$

Then, by (a_2), H^i is a finite (maybe trivial) subgroup of H for every $i \ge 0$, and $H = \bigoplus_{i=0}^{\infty} H^i$ by (a_3). Setting

$$G_0 = \bigoplus_{i=0}^{\infty} (H^i + \langle e_i \rangle),$$

we obtain the desired by (a_1).

Subcase 2.3. Assume that H is *finite and non-trivial*. In this case we put $H^0 = H$. By Lemma 6 we can choose $k \ge 0$ such that $H^0 \cap \langle \{g_{k+i}\}_{i=0}^{\infty} \rangle = \{0\}$. Set $e_i = g_{k+i}$ for every $i \ge 0$. Putting

$$G_0 = \big(H^0 \oplus \langle e_0 \rangle\big) \oplus \bigoplus_{i=1}^{\infty} \langle e_i \rangle, \text{ and } H^i = 0, \text{ for every } i \ge 1,$$

we obtain the desired (in this case $X = 0$ and $M = 1$). $\square$

Theorem 10. *Let G be an Abelian group of finite exponent and H its nonzero subgroup. If G contains a subgroup of the form $\mathbb{Z}(\exp(H))^{(\omega)}$, then G has a subgroup G_0 of the form*

$$G_0 = X \oplus \bigoplus_{i=0}^{\infty} (H_i + \langle e_i \rangle),$$

where

(1) *the independent sequence* $\{e_i\}$ *satisfies the condition*

$$\exp(H) = o(e_0) = o(e_1) = \dots ;$$

(2) *there is* $0 < M \le \infty$ *such that* H_j *is a finite nonzero subgroup of* G *for every* $0 \le j < M$*, and, if* $M < \infty$*,* $H_j = \{0\}$ *for each* $j \ge M$*;*
(3) $H = X \oplus \bigoplus_{i=0}^{\infty} H_i$*;*
(4) $X \ne \{0\}$ *if and only if* H *is uncountable. In this case* X *satisfies condition* (Λ)*.*

Proof: For a prime p, let H'_p and G_p be the p-components of H and G respectively. Since G has finite exponent, by [1, 2.1] there are different primes $p_1, \dots, p_n$, $p_{n+1}, \dots, p_N$, where $1 \le n \le N < \infty$, such that

$$H = \bigoplus_{k=1}^{n} H'_{p_k} \quad \text{and} \quad G = \bigoplus_{k=1}^{n} G_{p_k} \oplus G_1,$$

where $G_1 = \bigoplus_{k=n+1}^{N} G_{p_k}$ and all the groups H'_{p_k} and G_{p_k} are nonzero.

By assumption, for every $1 \le k \le n$, G_{p_k} has a subgroup of the form $\mathbb{Z}(\exp(H'_{p_k}))^{(\omega)}$. Thus, for every $1 \le k \le n$, G_{p_k} has an independent sequence $A_k = \{g_i^k\}_{i=0}^{\infty}$ such that $o(g_i^k) = \exp(H'_{p^k})$ for every $i \ge 0$.

Fix arbitrarily k, $1 \le k \le n$, and consider the next two possible cases.

Case 1. H'_{p_k} is a (*nonzero*) *countable group*. So we can apply Proposition 7 to the group $\langle A_k\rangle + H'_{p_k} (\subseteq G_{p_k})$. Thus the group $\langle A_k\rangle + H'_{p_k}$ has a subgroup G_0^k of the form

$$G_0^k := \bigoplus_{i=0}^{\infty} \left(H_i^k + \langle e_i^k \rangle \right),$$

where

(a$_1$) the independent sequence $\{e_i^k\}$ satisfies the condition

$$\exp\left(H'_{p_k}\right) = o(e_1^k) = o(e_2^k) = \dots ;$$

(a$_2$) there is $0 < M_k \le \infty$ such that H_i^k is a finite nonzero subgroup of G_0^k for every $0 \le i < M_k$, and, if $M_k < \infty$, $H_i^k = \{0\}$ for each $i \ge M_k$;
(a$_3$) $H'_{p_k} = \bigoplus_{i=0}^{\infty} H_i^k$;

In this case we also put $X_k = \{0\}$.

Case 2. H'_{p_k} is an *uncountable group*. Applying Propositions 5 to the group $\langle A_k\rangle + H'_{p_k} (\subseteq G_{p_k})$, we can find a countable (maybe trivial) subgroup S'_k of H'_{p_k} and an uncountable subgroup S''_k of H'_{p_k} such that

(b$_1$) $H'_{p_k} = S'_k \oplus S''_k$;
(b$_2$) $\langle A_k\rangle + H'_{p_k} = (\langle A_k\rangle + S'_k) \oplus S''_k$;
(b$_3$) S''_k satisfies condition (Λ).

Represent S''_k in the form $S''_k = X_k \oplus (\bigoplus_{i=0}^{\infty} R_i^k)$, where

(c_1) R_i^k is nonzero and finite for every $i \geq 0$;
(c_2) $\exp(S'_k \oplus \bigoplus_{i=0}^{\infty} R_i^k) = \exp(H'_{p_k})$;
(c_3) X_k is uncountable and satisfies condition (Λ).

Now we can apply Proposition 7 to the group

$$(\langle A_k \rangle + S'_k) \oplus \bigoplus_{i=0}^{\infty} R_i^k = \langle A_k \rangle + \left(S'_k \oplus \bigoplus_{i=0}^{\infty} R_i^k \right).$$

Taking into account (b_1)–(b_3) and (c_1)–(c_3), we obtain that the group $\langle A_k \rangle + H'_{p_k} (\subseteq G_{p_k})$ has a subgroup G_0^k of the form

$$G_0^k := X_k \oplus \bigoplus_{i=0}^{\infty} \left(H_i^k \oplus \langle e_i^k \rangle \right),$$

where

(a_4) the independent sequence $\{e_i^k\}$ satisfies the condition

$$\exp\left(H'_{p_k}\right) = o(e_1^k) = o(e_2^k) = \dots;$$

(a_5) there is $0 < M_k \leq \infty$ such that H_i^k is a finite nonzero subgroup of G_0^k for every $0 \leq i < M_k$, and, if $M_k < \infty$, $H_i^k = \{0\}$ for each $i \geq M_k$;
(a_6) $H'_{p_k} = X_k \oplus \bigoplus_{i=0}^{\infty} H_i^k$;
(a_7) X_k is uncountable and satisfies condition (Λ).

Set $M = \max\{M_1, \dots, M_n\}$ and

$$G_0 = \bigoplus_{k=1}^{n} G_0^k,\ X = \bigoplus_{k=1}^{n} X_k,\ H_i = \bigoplus_{k=1}^{n} H_i^k \text{ and } e_i = e_i^1 + \dots + e_i^n \text{ for every } i \geq 0.$$

By (a_1)–(a_7), all the conditions (1)–(4) are fulfilled. The theorem is proved. □

Proof of Theorem 2: (i) immediately follows from Theorem 9.

(ii) If H has a simple extension in G, then G has a subgroup of the form $\mathbb{Z}(\exp(H))^{(\omega)}$ by item (2b) of the definition of simple extension. The converse follows from Theorem 10. □

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