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Jordan ideals and derivations in prime near-rings

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Abstract. In this paper we investigate 3-prime near-rings with derivations satisfying certain differential identities on Jordan ideals, and we provide examples to show that the assumed restrictions cannot be relaxed.

Keywords: prime near-rings; Jordan ideals; derivations; commutativity

Classification: 16N60, 16W25, 16Y30

1. Introduction

Throughout this paper N will be a zero-symmetric right near-ring, and usually N will be 3-prime, that is, will have the property that $xNy = 0$ for $x, y \in N$ implies $x = 0$ or $y = 0$. The symbol $Z(N)$ will denote the multiplicative center of N. A near-ring N is called zero-symmetric if $x0 = 0$, for all $x \in N$ (recall that right distributivity yields $0x = 0$). An additive mapping $d : N \longrightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [15], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. We will write for all $x, y \in N$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. Recall that N is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in N$.

Many results in literature indicate how the global structure of a ring is often tightly connected to the behavior of additive mappings defined on that ring. In this direction, several authors have studied the structure of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings. Moreover, many of the obtained results extend those proven previously just for the action of the considered mappings on the whole ring to their actions on appropriate subsets of the ring. Furthermore, many authors have proved analogous results for prime and semiprime near-rings (see [2], [3], [4], [7], [8], [15] etc). Recently, there has been a great deal of work concerning commutativity of prime rings with additive mappings satisfying certain differential identities involving Jordan ideals (see [11], [12], [13], [14], [16]). Here we continue this line of investigation and we study the structure of 3-prime near-rings in which derivations satisfy certain identities involving Jordan ideal. Indeed, motivated by the concepts of Jordan ideals on rings, here we initiate the concepts of Jordan ideals on near-rings as follows:

Definition 1. Let N be a near-ring. An additive subgroup J of N is said to be a Jordan ideal of N if $j \circ n \in J$ and $n \circ j \in J$ for all $j \in J$, $n \in N$.

Example 1. Define two operations "+" and "." on $\mathbb{Z}/4\mathbb{Z}$ by:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

and

.	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	1	2	3
3	1	0	3	2

It is easy to check that $(\mathbb{Z}/4\mathbb{Z}, +, .)$ is a right 3-prime near-ring. Moreover, if we set $J = \{0, 1\}$, then J is a Jordan ideal of $\mathbb{Z}/4\mathbb{Z}$.

2. Conditions on Jordan ideals

Our aim in this section is to prove that if a Jordan ideal satisfies suitable conditions, then the near-ring must be a commutative ring. We leave the proof of the following easy lemmas to the reader.

Lemma 1. *Let N be a 3-prime near-ring and J a nonzero Jordan ideal of N. If $Jx = \{0\}$, then $x = 0$.*

Lemma 2. *Let N be a 3-prime near ring and J a Jordan ideal of N. If $j^2 = 0$ for all $j \in J$, then $J = 0$.*

It is well known that a 2-torsion free 3-prime ring must be commutative if it admits a nonzero central Jordan ideal. The following lemma gives an analogous result for near-rings.

Lemma 3. *Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. If $J \subseteq Z(N)$, then N is a commutative ring.*

Proof: From

$$(x+y)(j+j) = (j+j)(x+y) \ \text{ for all } \ x, y \in N,\ j \in J$$

it follows

$$x(j+j) + y(j+j) = j(x+y) + j(x+y) \ \text{ for all } \ x, y \in N,\ j \in J$$

so that

$$(j+j)x + (j+j)y = (x+y)j + (x+y)j \ \text{ for all } \ x, y \in N,\ j \in J.$$

Hence

$$jx + jy = yj + xj \ \text{ for all } \ x, y \in N,\ j \in J$$

and therefore

$$j(x+y-x-y) = 0 \ \text{ for all } \ j \in J,\ x, y \in N.$$

By Lemma 1, we get

$$x + y = y + x \quad \text{for all} \quad x, y \in N$$

which proves that $(N, +)$ is an abelian group. On the other hand, we have

$$2mnj = m(j + j)n = m(j \circ n) = (jn + jn)m = 2nmj \quad \text{for all} \quad m, n \in N, \ j \in J$$

which, in light of 2-torsion freeness, yields

$$j(mn - nm) = 0 \quad \text{for all} \quad m, n \in N, \ j \in J.$$

Applying Lemma 1, we conclude that

$$mn = nm \quad \text{for all} \quad m, n \in N$$

proving that N is a commutative ring. □

Remark 1. In ring theory it is known that a 2-torsion free 3-prime ring must be commutative if it admits a nonzero commutative Jordan ideal. In the case of a 2-torsion free 3-prime near-ring N, the assumption that $[J, J] = 0$ yields $(i + j)(k + k) = (k + k)(i + j)$ so that

$$k((j + i) - (i + j)) = 0 \quad \text{for all} \quad i, j, k \in J$$

and application of Lemma 1 yields $j + i = i + j$ for all $i, j \in J$. It seems that the near-ring need not be a commutative ring but we are unable to construct a counter-example. Hence, we leave it as an open question.

Theorem 1. *Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. Then N must be a commutative ring if J satisfies one of the following conditions:*

(i) $i \circ j \in Z(N)$ *for all* $i, j \in J$,
(ii) $i \circ j \pm [i, j] \in Z(N)$ *for all* $i, j \in J$.

Proof: (i) Assume that

$$i \circ j \in Z(N) \quad \text{for all} \quad i, j \in J. \tag{1}$$

In particular for $i, j, k \in J$ we have $(i \circ j) \circ k \in Z(N)$ and then

$$(i \circ j)(k + k) \in Z(N) \quad \text{for all} \quad i, j, k \in J.$$

Hence

$$(i \circ j)(k + k)n = (i \circ j)n(k + k) \quad \text{for all} \quad i, j, k \in J, n \in N$$

implying that

$$(i \circ j)N[k + k, n] = \{0\} \quad \text{for all} \quad i, j, k \in J, n \in N. \tag{2}$$

In light of the 3-primeness of N, (2) implies

$$j^2 = 0 \text{ or } k + k \in Z(N) \text{ for all } j, k \in J. \tag{3}$$

Using Lemma 2, because of $J \neq \{0\}$, equation (3) reduces to $k + k \in Z(N)$ for all $k \in J$. Since

$$k \circ k = (k + k)k \in Z(N) \text{ for all } k \in J$$

then

$$(k + k)N[k, n] = \{0\} \text{ for all } k \in J, n \in N. \tag{4}$$

Once again using the 3-primeness of N, equation (4) implies that either $k \in Z(N)$ or $k = 0$ for all $k \in J$ and therefore $J \subseteq Z(N)$. According to Lemma 3, we conclude that N is a commutative ring.

(ii) Suppose that

$$i \circ j \pm [i, j] \in Z(N) \text{ for all } i, j \in J. \tag{5}$$

In particular, $2j^2 \in Z(N)$ and replacing j by $2j^2$ in (5), we find that

$$i \circ (2j^2) \in Z(N) \text{ for all } i, j \in J.$$

Therefore

$$(2j^2)(i + i) \in Z(N) \text{ for all } i, j \in J$$

which implies that

$$(2j^2)N[i + i, n] = \{0\} \text{ for all } i, j \in J, \ n \in N. \tag{6}$$

In light of the 3-primeness of N, equation (6) yields

$$j^2 = 0 \text{ or } 2i \in Z(N) \text{ for all } i, j \in J. \tag{7}$$

Since equation (7) is the same as equation (3), then arguing as in the first case we get the required result. □

3. Conditions with derivations

Theorem 2. *Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. If N admits a nonzero derivation d such that $[d(n), j] = 0$ for all $n \in N$, $j \in J$, then N is a commutative ring.*

Proof: We are given that

$$d(n)j = jd(n) \text{ for all } n \in N, \ j \in J.$$

Replacing n by $nd(t)$ in the last equation, in view of [5, Lemma 1.1], we get

$$d(n)d(t)j + nd^2(t)j = jd(n)d(t) + jnd^2(t) \text{ for all } n, t \in N, \ j \in J$$

and hence

$$nd^2(t)j = jnd^2(t) \text{ for all } n, t \in N,\ j \in J. \tag{8}$$

Substituting nm for n in (8) we obtain

$$jnmd^2(t) = nmd^2(t)j = njmd^2(t) \text{ for all } n, m, t \in N,\ j \in J$$

implying

$$(jn - nj)Nd^2(t) = \{0\} \text{ for all } n, t \in N,\ j \in J. \tag{9}$$

In light of the 3-primeness of N, equation (9) assures that

$$\text{either } d^2 = 0 \text{ or } J \subseteq Z(N).$$

If $d^2 = 0$, then [7, Lemma 3] forces $d = 0$, which contradicts our original assumption that $d \neq 0$. Consequently $J \subseteq Z(N)$ and Lemma 3 assures that N is a commutative ring. □

Corollary 1 ([5, Theorem 2.1])**.** *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d such that $d(N) \subset Z(N)$, then N is a commutative ring.*

The following example proves that the primeness hypothesis in Theorems 1 and 2 is not superfluous.

Example 2. Let S be a noncommutative ring and set $N = \{\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) \mid x, y \in S\}$. Let us consider $J = \left\{\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) \mid y \in S\right\}$ and $d\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$. It is straightforward to check that J is a Jordan ideal of N. Moreover, d is a nonzero derivation of N which satisfies the conditions:

$$A \circ B \in Z(N),\ A \circ B \pm [A, B] \in Z(N),\ [d(C), B] = 0 \text{ for all } A, B \in J, C \in N;$$

but N is not a commutative ring.

Theorem 3. *Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. If N admits a nonzero derivation d such that $d([n, j]) = 0$ for all $n \in N$, $j \in J$, then N is a commutative ring.*

Proof: Assume that

$$d([n, j]) = 0 \text{ for all } n \in N,\ j \in J. \tag{10}$$

Replacing n by nj in (10), because of $[nj, j] = [n, j]j$, we get

$$[n, j]d(j) = 0 \text{ for all } n \in N,\ j \in J,$$

that is, we have that

$$njd(j) = jnd(j) \text{ for all } n \in N,\ j \in J. \tag{11}$$

Substituting nt for n in (11) we obtain $[n,j]td(j)=0$ which yields

$$[n,j]Nd(j)=\{0\} \text{ for all } n\in N,\ j\in J. \tag{12}$$

Since N is 3-prime, then equation (12) forces

$$d(j)=0 \text{ or } j\in Z(N) \text{ for all } j\in J. \tag{13}$$

If there is an element $j_0\in J$ such that $d(j_0)=0$, then (10) assures that

$$d(n)j_0=j_0d(n) \text{ for all } n\in N. \tag{14}$$

Substituting $nd(m)$ for n in (14) and using [5, Lemma 1.1] we obtain

$$d(n)d(m)j_0+nd^2(m)j_0=j_0d(n)d(m)+j_0nd^2(m) \text{ for all } n,m\in N.$$

Applying (14), the last equation reduces to

$$nd^2(m)j_0=j_0nd^2(m) \text{ for all } n,m\in N. \tag{15}$$

Writing nr instead of n in (15) we find that

$$(nj_0-j_0n)rd^2(m)=0 \text{ for all } n,m,r\in N$$

and therefore

$$[n,j_0]Nd^2(m)=\{0\} \text{ for all } n,m\in N. \tag{16}$$

Since $d\neq 0$, [7, Lemma 3] yields $d^2\neq 0$ and equation (16) together with 3-primeness of N forces $j_0\in Z(N)$. Accordingly, (13) reduces to $J\subseteq Z(N)$ and Lemma 3 shows that N is a commutative ring. $\square$

Corollary 2 ([2, Theorem 4.1]). *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d such that $d([x,y])=0$ for all $x,y\in N$, then N is a commutative ring.*

The following example proves that the 3-primeness hypothesis in Theorem 3 is not superfluous.

Example 3. Let $N=\left\{\begin{pmatrix}0&x&y\\0&0&0\\0&0&z\end{pmatrix}\mid x,y,z\in S\right\}$ and $J=\left\{\begin{pmatrix}0&0&r\\0&0&0\\0&0&0\end{pmatrix}\mid r\in S\right\}$ where S is an arbitrary ring. Define a derivation d on N by setting $d\begin{pmatrix}0&x&y\\0&0&0\\0&0&z\end{pmatrix}=\begin{pmatrix}0&x&0\\0&0&0\\0&0&0\end{pmatrix}$. It is easy to verify that J is a Jordan ideal of N such that $d[A,j]=0$ for all $A\in N$, $j\in J$. However, N is not a commutative ring. $\square$

Theorem 4. *Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. Then N admits no nonzero derivation d such that*

(i) *$d(n\circ j)=n\circ j$ for all $n\in N$, $j\in J$*
or
(ii) *$d(n\circ j)=-(n\circ j)$ for all $n\in N$, $j\in J$.*

Proof: (i) Assume that N admits a nonzero derivation d such that

$$d(n \circ j) = n \circ j \quad \text{for all} \quad n \in N,\ j \in J. \tag{17}$$

Substituting nj for n in (17), because of $nj \circ j = (n \circ j)j$, we get

$$(n \circ j)d(j) = 0 \quad \text{for all} \quad n \in N,\ j \in J$$

and thus

$$njd(j) = (-j)nd(j) \quad \text{for all} \quad n \in N,\ j \in J. \tag{18}$$

Replacing n by nt in (18) we obtain

$$n(-j)td(j) = (-j)ntd(j) \quad \text{for all} \quad n, t \in N,\ j \in J$$

which implies that

$$[n, j]Nd(-j) = \{0\} \quad \text{for all} \quad n \in N,\ j \in J.$$

Using the 3-primeness of N, we get

$$d(j) = 0 \quad \text{or} \quad j \in Z(N) \quad \text{for all} \quad j \in J. \tag{19}$$

If there exists $j_0 \in J$ such that $d(j_0) = 0$, then from $d(j_0 \circ j_0) = j_0 \circ j_0$ it follows that $j_0^2 = 0$. Since $d(n \circ j_0) = n \circ j_0$, then replacing n by nj_0 in this equation we get

$$j_0 d(n) j_0 = j_0 n j_0 \quad \text{for all} \quad n \in N. \tag{20}$$

Substituting nj_0t for n in (20) we find that $j_0nj_0tj_0 = 0$ so that

$$j_0 N (j_0 t j_0) = \{0\} \quad \text{for all} \quad t \in N.$$

As N is 3-prime, we conclude that $j_0 = 0$. Accordingly, equation (19) reduces to $J \subseteq Z(N)$ and Lemma 3 assures that N is a commutative ring. Hence, by 2-torsion freeness, equation (17) becomes

$$d(nj) = nj \quad \text{for all} \quad n \in N,\ j \in J. \tag{21}$$

Replacing n by nm in (21), where $m \in N$, we get $d(n)mj = 0$ so that

$$d(n)Nj = \{0\} \quad \text{for all} \quad n \in N,\ j \in J$$

and 3-primeness of N forces $d = 0$; a contradiction.

(ii) Using similar arguments, we get the required result. $\square$

Theorem 5. *Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. Then N admits no nonzero derivation d such that*

(i) $d(n) \circ j = n \circ j$ *for all* $n \in N$, $j \in J$
or

(ii) $d(n) \circ j = -(n \circ j)$ *for all* $n \in N$, $j \in J$.

Proof: (i) Suppose N admits a nonzero derivation d such that

$$d(n) \circ j = n \circ j \quad \text{for all} \quad n \in N,\ j \in J. \tag{22}$$

Replacing n by nj in (22) and using [5, Lemma 1.1], we obtain

$$nd(j)j + jnd(j) = 0 \quad \text{for all} \quad j \in J,\ n \in N$$

and hence

$$nd(j)j = (-j)nd(j) \quad \text{for all} \quad j \in J,\ n \in N. \tag{23}$$

Substituting nt for n in (23) we get

$$njtd(-j) = (j)ntd(-j) \quad \text{for all} \quad j \in J,\ n,t \in N$$

and therefore

$$[n,j]Nd(-j) = \{0\} \quad \text{for all} \quad j \in J,\ n \in N.$$

Since N is 3-prime, this implies that

$$d(j) = 0 \quad \text{or} \quad j \in Z(N) \quad \text{for all} \quad j \in J.$$

Using similar arguments as used previously we arrive at $J \subseteq Z(N)$ and application of Lemma 3 implies that N is a commutative ring. Hence equation (22) together with 2-torsion freeness forces

$$d(n)j = nj \quad \text{for all} \quad n \in N,\ j \in J \tag{24}$$

which leads to $d = 0$; a contradiction.

(ii) Using similar techniques, we get the required result. □

Remark 2. The results in this paper remain true for left near-rings with the obvious changes.

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