When every flat ideal is projective

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Abstract. In this paper, we study the class of rings in which every flat ideal is projective. We investigate the stability of this property under homomorphic image, and its transfer to various contexts of constructions such as direct products, and trivial ring extensions. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

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1. Introduction

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary.

Let R be a ring and let M be an R-module. As usual, we use $pd_R(M)$ and $fd_R(M)$ to denote the usual projective and flat dimensions of M, respectively. If R is an integral domain, we denote its quotient field by $qf(R)$. In this paper, we are interested in those rings in which every flat ideal is projective and which will be called FP-rings. In particular, perfect rings, hereditary rings and Noetherian rings are examples of FP-rings. Also, every local ring (R, M) with $M^2 = 0$ is an FP-ring by [24, Lemma 2.1]. In [25, Corollary 4], the author showed that flat ideals in a Mori domain are invertible. So, Mori domains are FP-domains. In particular, Krull domains and UFDs are FP-domains.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A [2, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [2], [15], [18].

The purpose of this paper is to give some simple methods in order to construct FP-rings. For this, we investigate the stability of the FP-property under homomorphic image, and its transfer to various contexts of constructions such as direct products, pullback rings and trivial ring extensions. Our results generate original examples which enrich the current literature with new families of non-Noetherian rings satisfying the FP-property.

2. Main results

Recall that R is called semi-hereditary if every finitely generated ideal of R is projective and is said to has weak global dimension ≤ 1 if every finitely generated ideal of R is flat. A semi-hereditary ring R has $wdim(R) \leq 1$. In the domain context, all these forms coincide with the definition of a Prüfer domain. Glaz [14, Example 3.2.1] provides an example of non-semi-hereditary ring of $wdim \leq 1$. See for instance [3], [4], [14].

We start with examples of non-FP-rings.

Proposition 2.1. *Any non-hereditary ring R of $wdim(R) \leq 1$ is a non-FP-ring.*

Proof: Let R be a non-hereditary ring with $wdimR \leq 1$. Then, there exists an ideal I of R which is not projective. On the other hand, I is flat since $wdimR \leq 1$. Hence, R is a non-FP-ring, as desired. □

Now, we give a new class of FP-rings.

Recall that R is called FF-ring if every flat ideal of R is finitely generated. See for instance [12].

Proposition 2.2. *Any FF-ring is an FP-ring.*

Proof: By [12, Remark 2.4]. □

The converse does not generally hold as the following example shows.

Example 2.3. Let R be a non-Noetherian hereditary ring [6, Example 2.7]. Then:
(1) R is an FP-ring;
(2) R is not an FF-ring.

In the domain context, the FP notion coincides with the FF one since in an integral domain, a flat ideal is finitely generated if and only if it is projective.

The FP-property descends into a faithfully flat domain homomorphism.

Proposition 2.4. *Let R and S be two domains and $f : R \longrightarrow S$ be a ring homomorphism making S a faithfully flat R-module. If S is an FP-ring, then so is R.*

Proof: Let I be a flat ideal of R. Then, $I \otimes_R S = IS$ is a flat ideal of S and so $I \otimes_R S$ is a projective ideal of S since S is an FP-ring. Then, $I \otimes_R S$ is a finitely generated ideal of S since S is a domain. Hence, I is a finitely generated ideal of A (since S is faithfully flat R-module) and so I is a projective ideal of A (since A is a domain). Therefore, A is an FP-ring. □

We combine this proposition with [24, Theorem 4.1] to get the following corollary.

Corollary 2.5. *Let R be a domain and let X be an indeterminate over R. The following statements are equivalent.*
(1) *R is an FP-ring.*

(2) *$R[X]$ is an FP-ring.*
(3) *$R[[X]]$ is an FP-ring.*

Now, we study the transfer of a FP-property between a ring A and $A \propto E$, the trivial ring extension of A by E, where E is an A-module. The main result (Theorem 2.6) enriches the literature with original examples of FP-rings. Recall that if E is an A-module, then $Z(E) = \{a \in A$ such that $ae = 0$ for some $e(\neq 0) \in E\}$.

Theorem 2.6. *Let A be a ring, E be an A-module and let $R = A \propto E$ be the trivial ring extension of A by E.*

(1) *Assume that E is a flat A-module or an ideal of A. If R is an FP-ring, then so is A.*
(2) *Assume that A is a domain, E is a K-vector space, where $K = qf(A)$. Then R is an FP-ring if and only if so is A.*
(3) *Assume that (A, M) is a local ring such that $ME = 0$. Then $R = A \propto E$ is an FP-ring if and only if so is A.*

Before proving Theorem 2.6, we establish the following lemmas.

Lemma 2.7 ([1, Theorem 8])**.** *Let A be a ring, E be an A-module and let $R = A \propto E$ be the trivial ring extension of A by E.*

(1) *If $J = I \propto E$ (where I is a non-zero ideal of A) is a flat ideal of R, then I is a flat ideal of A. The converse is true if E is flat.*
(2) *If $J = I \propto E$ (where I is a non-zero ideal of A) is a projective ideal of R, then I is a projective ideal of A. The converse is true if E is flat.*

An R-module M is called P-flat if, for any $(s, x) \in R \times M$ such that $sx = 0$, $x \in (0 : s)M$. If M is flat, then M is naturally P-flat. In the domain case P-flat is equivalent to torsion-free and when R is an arithmetical ring (i.e., such that the lattice formed by its ideals is distributive), then any P-flat module is flat (by [7, p. 236]). Also, every P-flat cyclic module is flat (by [7, Proposition 1(2)]).

Before proving Theorem 2.6, we also need the following lemma of independent interest.

Lemma 2.8. *Let A be a domain, E be an A-module, $F(\neq 0)$ be a sub-module of E and $R = A \propto E$ be the trivial ring extension of A by E. Then $0 \propto F$ is not a P-flat R-module.*

Proof: Let $F(\neq 0)$ be a submodule of E. Two cases are possible then.

Case 1: $Z(F) = 0$. Let $(0, f)$ $(\neq (0, 0))$, $(0, e)$ $(\neq (0, 0)) \in 0 \propto F$. Then, $(0, f)(0, e) = (0, 0)$ and $(0 : (0, e)) = 0 \propto E$ since $Z(F) = 0$. Then $(0, f) \notin (0 : (0, e))(0 \propto F) = (0 \propto E)(0 \propto F) = 0$. Thus $0 \propto F$ is not a P-flat R-module.

Case 2: $Z(F) \neq 0$. Let $d(\neq 0) \in Z(F)$ and $f(\neq 0) \in F$ such that $df = 0$. Hence, $(d, 0)(0, f) = (0, 0)$ and $(0 : (d, 0)) \subseteq 0 \propto E$ and so $(0, f) \notin (0 : (d, 0))(0 \propto F) = 0$. Therefore, $0 \propto F$ is not a P-flat R-module, as desired. □

Proof of Theorem 2.6: 1) Assume that R is an FP-ring and E is an ideal of A or E is a flat A-module. Let I be a flat ideal of A. Then $I \otimes_A R = I \propto (I \otimes_A E) = I \propto IE$ is a flat ideal of R and so it is a projective since R is an FP-ring. Therefore, I is a projective ideal of A by Lemma 2.7 and so R is an FP-ring.

2) Assume that A is a domain and E is a K-vector space, where $K = qf(A)$. Let J be a nonzero flat ideal of R, we need to prove that J is projective. By [2, Corollary 3.4], $J = I \propto E$ or $J = 0 \propto E'$ for some ideal I of R or some submodule E' of E. As $0 \propto E'$ is not flat (since it is not P-flat by Lemma 2.8), then $J = I \propto E$. Hence, I is a flat ideal of A (by Lemma 2.7) and so it is projective (since A is an FP-ring). Therefore $I \propto E$ is projective, by Lemma 2.7. The converse holds by 1, as desired.

3) Let (A, M) be a local FP-ring, $R = A \propto E$ be the trivial ring extension of A by an A-module E such that $ME = 0$ and let J be a flat ideal of R. By [24, Lemma 2.1], we may assume that $J(M \propto E) = J$. Then $J = J(M \propto E) \subseteq (M \propto E)(M \propto E) = M^2 \propto 0$. Hence, $J = I \propto 0$ for some ideal I of A. We have $J \otimes_R A \cong J \otimes_R R/(0 \propto E) \cong J/J(0 \propto E) \cong I \propto 0/(I \propto 0)(0 \propto E) = I \propto 0$. So, I is a flat ideal of A since J is a flat ideal of R. Hence, I is a projective ideal of A since A is an FP-ring. We claim that J is a projective ideal of R.

Indeed, since I is a projective ideal of A and $J = (I \propto 0) \cong I$, then J is a projective A-module. Therefore, $J = (J \otimes_A R)$ is a projective ideal of R.

Conversely, let I be a flat ideal of A. Since $(I \propto 0) \cong I$, then J is a flat A-module. Hence, $J = (J \otimes_A R)$ is a flat ideal of R and so it is projective since R is an FP-ring. Therefore, I is projective by Lemma 2.7, and this completes the proof of Theorem 2.6. □

Theorem 2.6 generates new and original examples of FP-rings.

Example 2.9. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q} = qf(\mathbb{Z})$, $\mathbb{R}$ be the field of reals numbers, and let $S = \mathbb{Z} \propto \mathbb{R}$, and $T = \mathbb{Z} \propto \mathbb{Q}[X]$. Then

(1) S and T are FP-rings by Theorem 2.6;
(2) S and T are not coherent by [18, Theorem 2.8]. In particular S and T are not Noetherian.

Example 2.10. Let (A, M) be a local FP-ring and let $R = A \propto (A/M)^\Lambda$ be the trivial ring extension of A by the A-module $(A/M)^\Lambda$, where Λ is an infinite set. Then

(1) R is an FP-ring by Theorem 2.6;
(2) R is not coherent by [20, Theorem 2.1]. In particular R is non-Noetherian.

Now, we study the transfer of the FP-property in the $D + M$ constructions. We adopt the following notations: T is a ring of the form $T = K + M$, where K is a field and M is a nonzero maximal ideal of T, D is a subring of K such that $qf(D) = K$, and $R = D + M$. Then, $T = S^{-1}R$ with $S = D - \{0\}$ and R is a faithfully flat D-module.

Theorem 2.11. *Let T and R be as above such that T is an FP-ring. Then, R is an FP-ring if and only if for every flat ideal J of R, the D-module J/JM is projective.*

The proof of this theorem involves the following lemma.

Lemma 2.12 ([15, Theorem 5.1.1])**.** *Let $f : R \longrightarrow S$ be an injective ring homomorphism satisfying the fact that there is an ideal M of R such that $MS = M$. Then, an R-module E is projective if and only if $E \otimes_R S$ is a projective S-module and E/ME is a projective R/M-module.*

Proof of Theorem 2.11: Let J be a flat ideal of R. Then $J \otimes_R T = JT$ is a flat ideal of T and so $J \otimes_R T$ is a projective T-module since T is an FP-ring. By Lemma 2.12, J is a projective ideal of R, as desired. □

Proposition 2.13. *Let R be a ring. Then, if R is an FP-ring, then so is R_P for every $P \in Spec(R)$*

Proof: Let $P \in Spec(R)$ and let I_P be a flat ideal of R_P. Then I is a flat ideal of R and so is projective since R is an FP-ring. Hence, $I_P = I \otimes_R R_P$ is projective. Thus R_P is an FP-ring. □

The converse does not generally hold as the following example shows.

Example 2.14. Let R be a non-hereditary von Neumann regular ring. Then:

(1) R is not an FP-ring;
(2) R_P is an FP-ring for each prime ideal P of R (since R_P is a field).

Our next result establishes the transfer of FP property to a particular homomorphic image.

Proposition 2.15. *Let R be a ring and let I be a pure ideal of R. If R is an FP-ring, then so is R/I.*

Proof: Let R be an FP-ring and let J/I be a flat ideal of R/I. Then J is a flat ideal of R (using the exact sequence: $0 \longrightarrow I \longrightarrow J \longrightarrow J/I \longrightarrow 0$ where I and J/I are flat R-modules since I is a pure ideal of R). Therefore, J is a projective ideal of R and so $J \otimes_A R/I = J/I$ is a projective ideal of R/I. Hence, R/I is an FP-ring. □

Next, we study the transfer of the FP-property to direct products.

Theorem 2.16. *Let $(R_i)_{i=1,\dots,n}$ be a family of commutative rings. Then $R = \prod_{i=1}^{n} R_i$ is an FP-ring if and only if so is R_i for each $i = 1, \dots, n$.*

The proof of the theorem involves the following lemma.

Lemma 2.17. *Let $(R_i)_{i=1,2}$ be a family of rings and $(E_i)_{i=1,2}$ be an R_i module for $i = 1, 2$. Then*

(1) $fd_{R_1 \times R_2}(E_1 \times E_2) = \sup\{fd_{R_1}(E_1), fd_{R_2}(E_2)\}$;
(2) $pd_{R_1 \times R_2}(E_1 \times E_2) = \sup\{pd_{R_1}(E_1), pd_{R_2}(E_2)\}$.

Proof: By [19, Lemma 2.5]. □

Proof of Theorem 2.16: The proof is done by induction on n and it suffices to check it for $n = 2$. Assume that $(R_1 \times R_2)$ is an FP-ring and we must to show that R_i is an FP-ring for $i = 1, 2$.

Let I_1 be a flat ideal of R_1. Then, $fd_{R_1 \times R_2}(I_1 \times R_2) = fd_{R_1}(I_1) = 0$ (by Lemma 2.17) and so $I_1 \times R_2$ is a flat ideal of $R_1 \times R_2$ which is an FP-ring. Thus, $I_1 \times R_2$ is projective. We have $pd_{R_1}(I_1) = pd_{R_1 \times R_2}(I_1 \times R_2) = 0$ by Lemma 2.17. Then, I_1 is a projective ideal of R_1 and so R_1 is an FP-ring. We have also that R_2 is an FP-ring by the same argument.

Conversely, let R_1 and R_2 be two FP-rings and let $I = I_1 \times I_2$ be a flat ideal of $R_1 \times R_2$, where I_i is an ideal of R_i for each $i = 1, 2$. Hence, $fd_{R_i}(I_i) = 0$ and so $pd_{R_i}(I_i) = 0$ since R_i is an FP-ring. Therefore, $pd_{R_1 \times R_2}(I_1 \times I_2) = \sup\{pd_{R_i}(I_i), i = 1, 2\} = 0$ and this completes the proof of the theorem. □

Now, we construct a new example of non-FP-rings.

Example 2.18. Let R_1 be a non-FP-ring, R_2 be any ring and let $R = R_1 \times R_2$. Then R is not an FP-ring.

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