The product of two ordinals is hereditarily dually discrete

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Abstract. In *Dually discrete spaces*, Topology Appl. **155** (2008), 1420–1425, Alas *et. al.* proved that ordinals are hereditarily dually discrete and asked whether the product of two ordinals has the same property. In *Products of certain dually discrete spaces*, Topology Appl. **156** (2009), 2832–2837, Peng proved a number of partial results and left open the question of whether the product of two stationary subsets of ω_1 is dually discrete. We answer the first question affirmatively and as a consequence also give a positive answer to the second.

Keywords: dually discrete spaces, stationary subsets, ordinal spaces

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1. Introduction

Among the many dual classes introduced by J. van Mill, V.V. Tkachuk and R.G. Wilson in [4], one of the most interesting, because of its relationship to the class of D-spaces, is the class of dually discrete spaces. There are many open questions regarding this class; one of them is whether or not every hereditarily Lindelöf space is dually discrete. The classes of D-spaces and dually discrete spaces are quite different; for instance, suborderable spaces are dually discrete but even ω_1 is not a D-space. There are examples of spaces which are not dually discrete; however, either they are not regular spaces or they have large size. We conjecture that at least consistently every space of size $\aleph_1$ is dually discrete. So we ask,

Question 1. *Is there a Tychonoff example of a non-dually discrete space of size $\aleph_1$?*

A similar question was raised by Buzyakova, Tkachuk and Wilson in [2] where it was asked whether there is a model of ZFC in which $\mathbb{R}^{\omega_1}$ is dually discrete. They showed that $\diamondsuit$ implies it is not. In [3], van Douwen and Pfeffer showed that $\mathbb{R}_\ell^n$ is a D-space for every $n \in \omega$. Here $\mathbb{R}_\ell$ is the Sorgenfrey line. They asked whether the countable power of $\mathbb{R}_\ell$ is a D-space as well. It seems that the following is also unknown:

Question 2. *Is the countable product of $\mathbb{R}_\ell$ a dually discrete space?*

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Alas, Junqueira and Wilson proved in [1] that products of certain types of ordinals are dually discrete, but they left open the general case. In [6], Peng showed that the product of any two ordinals is dually discrete and in [7] he asked whether the product of two stationary subsets of ordinals is dually discrete. In this paper we show that a product of two ordinals is hereditarily dually discrete thus answering affirmatively both of the above mentioned questions.

2. Definitions and preliminaries

We use standard notation and terminology, undefined terms can easily be found in any of the popular texts on general topology or set theory.

Let (X, τ) be a topological space. A *neighborhood assignment* is a function $\phi : X \to \tau$ such that $x \in \phi(x)$ for each $x \in X$.

Given a neighborhood assignment ϕ for a space X, a *kernel* for the assignment is a subset Y of X such that $X = \bigcup\{\phi(y) : y \in Y\}$.

Definition 1. A topological space X is *dually discrete* if for each neighborhood assignment ϕ there is a discrete kernel for ϕ. We say that X is *hereditarily dually discrete* if any subspace of X is dually discrete.

Observe that these properties are weakly hereditary; that is, hereditary to closed subspaces. A space X is a *D-space* if every neighborhood assignment has a closed discrete kernel.

Remember that if α is a regular ordinal and $C \subset \alpha$, we say that C is *closed unbounded* if C contains its limit points and is not bounded in α; also, $S \subset \alpha$ is *stationary* if $S \cap C \neq \emptyset$ for each C that is closed unbounded in α. Also recall that if $\{C_\beta : \beta < \alpha\}$ is a family of subsets such that C_β is a closed unbounded set for all $\beta < \alpha$, then the *diagonal intersection* of the family,

$$\Delta_{\beta<\alpha} C_\beta = \{\gamma \in \alpha : \gamma \in \bigcap_{\delta<\gamma} C_\delta\},$$

is a closed unbounded set. That is the main ingredient to prove the well known *Pressing Down Lemma*. For this, if α is an ordinal and $S \subset \alpha$, we say that $f : S \to \alpha$ is *regressive* if $f(s) < s$ for each $s \in S \setminus \{0\}$.

Theorem 2 (Fodor)**.** *If f is a regressive function on a stationary set $S \subset \alpha$, then there are a stationary subset T of S and a $\gamma < \alpha$ such that $f(t) = \gamma$ for any $t \in T$.*

Related to our question above, it is an easy elementary exercise to show the following proposition. As far as we know, even Martin's Axiom could settle our Question 1.

Proposition 3. *Every countable topological space is dually discrete.*

We mention before that the next result appeared in [1].

Theorem 4 (Alas, Junqueira, Wilson). *Every ordinal is hereditarily dually discrete.*

3. Main result

Our main result is the following theorem. L.X. Peng showed that the product of two ordinals is a dually discrete space and that subspaces of them that are either normal or of countable extent are dually discrete as well (such results appeared in [6], [7] and [5], respectively). We consider it appropriate to credit Peng for being able to isolate the main difficulty to establish the result. We first figured out how to solve Peng's problem asking whether or not the product of two disjoint stationary subsets of ω_1 is dually discrete and later we were able to adapt the proof to get our result.[1]

Theorem 5. *The product of two ordinals is hereditarily dually discrete.*

This theorem settles Question 3.6 from [1] which asked exactly that. An obvious consequence of this theorem solves Peng's question:

Corollary 6. *The product of two disjoint stationary subsets of ω_1 is a dually discrete space.*

In the next section we give the proof of Theorem 5. First some things that will be used. If X is a totally ordered set, then denote the next sets:

$$\begin{aligned}\Delta(X) &= \{\langle x, y\rangle \in X \times X : x = y\};\\ \Delta_{\downarrow}(X) &= \{\langle x, y\rangle \in X \times X : y < x\};\\ \Delta_{\uparrow}(X) &= \{\langle x, y\rangle \in X \times X : y > x\}.\end{aligned}$$

Also keep in mind an easy observation: If κ is a regular cardinal and X a non-stationary subset of κ, then there is an open and non-stationary subset U of κ such that $X \subset U$. The next theorem will be used in the proof of the main result.

Theorem 7 ([5]). *Let μ and ν be two ordinals. If $\mu \times \nu$ is not hereditarily dually discrete, and for each $\lambda < \mu$ (or $\delta < \nu$), the space $\lambda \times \nu$ (or $\mu \times \delta$) is hereditarily dually discrete, then μ and ν are uncountable regular cardinals and $\mu = \nu$.*

In fact, a stronger result appeared in [5], but for our purposes this formulation is enough.

[1]Independently, Liang-Xue Peng almost at the same time proved the same result. His proof is published in *The product of two ordinals is hereditarily dually discrete*, Topology Appl. **159** (2012), no. 1, 304–307.

4. The proof

We proceed by contradiction: suppose that there is an ordinal α such that $\alpha \times \alpha$ is not hereditarily dually discrete. Then by Theorem 7, we can assume that α is regular and minimal with respect to that property.

Fix an arbitrary subspace $X \subset \alpha \times \alpha$ and let ϕ be a neighborhood assignment for X. We shall show that there is a discrete kernel for ϕ, contradicting the choice of α. We work below the diagonal, for if $X \cap \Delta(\alpha) \neq \emptyset$, then as $\Delta(\alpha)$ is homeomorphic to α, by Theorem 4, there is a discrete kernel D for $X \cap \Delta(\alpha)$ and $X \setminus \bigcup\{\phi(d) : d \in D\}$ is the union of two disjoint closed subspaces of $X \cap \Delta_{\downarrow}(\alpha)$ and $X \cap \Delta_{\uparrow}(\alpha)$. Therefore, it will be dually discrete in case of these two subspaces are dually discrete. Thus, it suffices to show that $X_{\Delta_{\downarrow}} = X \cap \Delta_{\downarrow}(\alpha)$ is dually discrete since $X \cap \Delta_{\uparrow}(\alpha)$ will be dually discrete in an analogous way.

For $X_{\Delta_{\downarrow}}$ we can suppose that $\phi(v) \subset \Delta_{\downarrow}(\alpha)$ for each $v \in X_{\Delta_{\downarrow}}$ and, more precisely, that for each $\langle x, y\rangle \in X_{\Delta_{\downarrow}}$ we have that

$$\phi(\langle x, y\rangle) = (z_{\langle x,y\rangle}, x] \times (w_{\langle x,y\rangle}, y] \cap X_{\Delta_{\downarrow}},$$

where $z_{\langle x,y\rangle} < x$ and $w_{\langle x,y\rangle} < y$. We are left with the following cases:

• Case (1). For each $\beta < \alpha$ the set $A_\beta = \{x \in \alpha : \langle x, \beta\rangle \in X_{\Delta_{\downarrow}}\}$ is non-stationary. For each $\beta < \alpha$ let C_β be a closed unbounded set such that $C_\beta \cap A_\beta = \emptyset$ and consider $C = \Delta_{\beta<\alpha} C_\beta$. Let $\{c_\delta : \delta < \alpha\}$ be a continuous and increasing enumeration of C. We may assume that $c_0 = 0$.

We are working with $X_{\Delta_{\downarrow}}$; hence, by definition of C,

$$X_{\Delta_{\downarrow}} = \bigoplus\{X_{\Delta_{\downarrow}} \cap [(c_\delta, c_{\delta+1}] \times \alpha] : \delta < \alpha\}.$$

Then, by the minimality of α, each element of the partition that C defines is dually discrete and so is $X_{\Delta_{\downarrow}}$.

• Case (2). The set $B = \{y \in \alpha : \{x \in \alpha : \langle x, y\rangle \in X_{\Delta_{\downarrow}}\}$ is stationary$\}$ is non-stationary. Let $C_1 = \{c_\beta : \beta < \alpha\}$ be closed, unbounded and disjoint from B; suppose that $0 \in C_1$. Consider $Y = X_{\Delta_{\downarrow}} \cap [\alpha \times C_1]$, such Y is a closed subset of $X_{\Delta_{\downarrow}}$ and it is as in Case (1); let $D_1 \subset Y$ be a discrete kernel of Y. Let $F = X_{\Delta_{\downarrow}} \setminus \bigcup\{\phi(d) : d \in D_1\}$, and observe that $F = \bigoplus\{F \cap (\alpha \times (c_\beta, c_{\beta+1}]) : \beta < \alpha\}$ is closed in $X_{\Delta_{\downarrow}}$. Each $F \cap (\alpha \times (c_\beta, c_{\beta+1}])$ is dually discrete by minimality of α. For each $\beta < \alpha$, let $H_\beta \subset F \cap (\alpha \times (c_\beta, c_{\beta+1}])$ be a discrete kernel of it. So $D = D_1 \cup \bigcup\{H_\beta : \beta < \alpha\}$ is a discrete kernel of $X_{\Delta_{\downarrow}}$.

• Case (3). The set $B = \{y \in \alpha : \{x \in \alpha : \langle x, y\rangle \in X_{\Delta_{\downarrow}}\}$ is stationary$\}$ is stationary. For each $y < \alpha$ define $A_y = \{x \in \alpha : \langle x, y\rangle \in X_{\Delta_{\downarrow}}\}$. Note that for each $y \in B$ there are a stationary set $A'_y \subset A_y$ and a $w_y < y$ such that $w_{\langle x,y\rangle} = w_y$ for each $x \in A'_y$, because $|\{w_{\langle x,y\rangle} : x \in A_y\}| \leq |y| < \alpha$ and, hence, the set $\{x \in A_y : w_{\langle x,y\rangle} = w_y\}$ must be stationary for some w_y. Also note that the function that sends each $x \in A'_y$ to $z_{\langle x,y\rangle}$ is a regressive function. Then by

Fodor's Theorem there are a stationary subset $A''_y \subset A'_y$ and a $z_y < \alpha$ such that $z_{\langle x,y\rangle} = z_y$ for each $x \in A''_y$. We have defined a regressive function on B too (the function that sends each $y \in B$ to w_y), so by Fodor's Theorem there are a stationary subset $S \subset B$ and a $\gamma < \alpha$ such that $w_y = \gamma$ for each $y \in S$. Let $D_1 \subset S$ be a discrete set that is cofinal in α.

If $F_1 = X_{\Delta_\downarrow} \cap (\alpha \times [0,\gamma])$, then F_1 is a closed and open subset of $X_{\Delta_\downarrow}$ and by minimality of α it is dually discrete. Let $K_1 \subset F_1$ be a discrete kernel of F_1.

Consider now the set

$$F_2 = X_{\Delta_\downarrow} \setminus [\bigcup\{\phi(k) : k \in K_1\} \cup \bigcup\{\phi(d) : d \in \bigcup\{A''_y \times \{y\} : y \in D_1\}\}].$$

Observe that F_2 is a closed subset of $X_{\Delta_\downarrow}$ and $T = \{x \in \alpha : (\exists y)(\langle x,y\rangle \in F_2)\}$ is a non-stationary subset of α; because if T were stationary, then for each $x \in T$, there is a $y_x < x$ such that $\langle x, y_x\rangle \in F_2$ (this because $X_{\Delta_\downarrow} \subset \Delta_\downarrow(\alpha)$). This defines a regressive function on T and, by Fodor's Theorem there are a stationary subset T' of T and a $\delta < \alpha$ such that $\langle x, \delta\rangle \in F_2$ for any $x \in T'$. Let $d_0 \in D_1$ be such that $d_0 > \delta$; then as T' is unbounded in α we can find an $x_0 \in T'$ such that $x_0 > z_{d_0}$; so $\langle x_0, \delta\rangle \in \bigcup\{\phi(\langle x, d_0\rangle) : x \in A''_{d_0}\}$, which contradicts the definition of F_2 and, thus, T is non-stationary. Note that F_2 is as in Case (1) and so F_2 is dually discrete. Let K_2 be a discrete kernel of F_2.

As T is non-stationary, there is an open non-stationary $U \subset \alpha$ that contains T.

Now for each $d \in D_1$ let $H_d \subset (A''_d \setminus U)$ be a discrete subset that is cofinal in α. Define $K_3 = \bigcup\{H_d \times \{d\} : d \in D_1\}$. By definition, K_3 is discrete and, by construction we have that neighborhoods assigned to K_3 cover the same as those assigned to $\bigcup\{A''_d \times \{d\} : d \in D_1\}$. Then $K = K_1 \cup K_2 \cup K_3$ is a kernel of $X_{\Delta_\downarrow}$. To see that it is discrete, it will be enough to note that K_3 does not accumulate in K_2; however it is clear from the definition of K_3 because it is a subset of $(\alpha \times \alpha) \setminus (\alpha \times U)$ which is a closed subset disjoint with K_2. Thus the proof that $X_{\Delta_\downarrow}$ is dually discrete is complete.

A contradiction then follows, and we complete the proof of our main result.

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