Boundedness of one-sided fractional integrals in the one-sided Calderón-Hardy spaces

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Abstract. In this paper we study the mapping properties of the one-sided fractional integrals in the Calderón-Hardy spaces $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$ for $0 < p \leq 1$, $0 < \alpha < \infty$ and $1 < q < \infty$. Specifically, we show that, for suitable values of p, q, γ, α and s, if $\omega \in A^+_s$ (Sawyer's classes of weights) then the one-sided fractional integral I^+_γ can be extended to a bounded operator from $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$ to $\mathcal{H}^{p,+}_{q,\alpha+\gamma}(\omega)$. The result is a consequence of the pointwise inequality

$$N^+_{q,\alpha+\gamma}\left(I^+_\gamma F; x\right) \leq C_{\alpha,\gamma} N^+_{q,\alpha}\left(F; x\right),$$

where $N^+_{q,\alpha}(F;x)$ denotes the Calderón maximal function.

Keywords: fractional integral, maximal, one-sided Calderón-Hardy, one-sided weights spaces

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1. Introduction

The purpose of this paper is to show that we can extend the fractional integral to a bounded operator between Calderón-Hardy spaces. For $0 < \gamma < 1$, we denote by $I_\gamma f$ the fractional integral defined by

$$I_\gamma f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\gamma}}\, dy$$

when this integral exists. The classical results of boundedness of the fractional integral are well known. One of them ensures that if $0 < \gamma < n$, $1 < q < r < \infty$, $\frac{1}{r} = \frac{1}{q} - \gamma$ and $f \in L^q(\mathbb{R}^n)$ then

$$\|I_\gamma f\|_{L^r(\mathbb{R}^n)} \leq C_{r,q} \|f\|_{L^q(\mathbb{R}^n)}. \tag{1.1}$$

A proof can be found e.g. in [3] or [13]. Another classical result affirms that if $f \in \Lambda_\alpha$, $0 < \alpha < 1$, then $I_\gamma f \in \Lambda_\beta$, $\beta = \alpha + \gamma$. A more general version of this result can be seen in [4].

We will study the behaviour of the operator I^+_γ, $0 < \gamma < 1$, defined by

$$I^+_\gamma f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}\, dy,$$

in the one-sided Calderón-Hardy spaces that we will define below. In 1982, Gatto, Jiménez and Segovia studied in [2], the Calderón-Hardy spaces in order to characterize the solutions of $\Delta^m F = f$, $m \in \mathbb{N}$, for distributions f in the Hardy spaces H^p. They proved that the operator Δ^m is a bijective mapping from the Calderón-Hardy spaces to H^p. Later, in 2001, Ombrosi studied in [7] a more general weighted version of these spaces. Ombrosi proved that the fractional integral I_γ^+ can be extended to a bounded operator from the one-sided Hardy spaces into the Calderón-Hardy spaces. To generalize these spaces Ombrosi used a one-sided version of the Calderón maximal function, denoted by $N_{q,\alpha}^+(F,x)$. To obtain our result, the key will be to prove a pointwise estimate for $N_{q,\alpha}^+(I_\gamma^+ F, x)$. Furthermore, this estimate will allow us to give another proof of the classical result of boundedness of I_γ^+ between Lipschitz spaces.

A weight ω is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its ω-measure by $\omega(E) = \int_E \omega(t)\,dt$. A function $f(x)$ belongs to $L^p(\omega)$, $0 < p < \infty$, if $\|f\|_{L^p(\omega)} = (\int_{-\infty}^{\infty} |f(x)|^p \omega(x)\,dx)^{1/p}$ is finite.

The classes A_s^+, $1 \le s \le \infty$, were defined by E. Sawyer in [12] (see also [6]). A weight ω belongs to the class A_s^+, $1 < s < \infty$, if there exists a constant C such that

$$\left(\frac{1}{h}\int_{x-h}^{x} \omega(t)\,dt\right)\left(\frac{1}{h}\int_{x}^{x+h} \omega(t)^{-\frac{1}{s-1}}\,dt\right)^{s-1} \le C, \tag{1.2}$$

for almost all real numbers x.

In the limit case of $s = 1$ we say that ω belongs to the class A_1^+ if $M^-\omega(x) \le C\omega(x)$ a.e. $x \in \mathbb{R}$, where $M^- f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f(t)|\,dt$. In a similar way, Sawyer defined that a weight ω belongs to the class A_s^-, $1 < s < \infty$, if there exists a constant C such that

$$\left(\frac{1}{h}\int_{x}^{x+h} \omega(t)\,dt\right)\left(\frac{1}{h}\int_{x-h}^{x} \omega(t)^{-\frac{1}{s-1}}\,dt\right)^{s-1} \le C, \tag{1.3}$$

for almost all numbers x. For $s = 1$ we say that ω belongs to the class A_1^- if $M^+\omega(x) \le C\omega(x)$ a.e. $x \in \mathbb{R}$, where $M^+ f(x) = \sup_{h>0} \frac{1}{h}\int_{x}^{x+h} |f(t)|\,dt$. The properties of one-sided weights which we will use in this paper can be seen in [5], [8] and [12].

Let us fix $w \in A_s^+$. Then there exists $x_{-\infty}$ such that $w(x) = 0$ if $x < x_{-\infty}$ and $w(x) > 0$ if $x > x_{-\infty}$ (see [8] for details). We denote by $L^q_{\mathrm{loc}}(x_{-\infty}, \infty)$, with $1 < q < \infty$, the space of the real valued functions $f(x)$ on $\mathbb{R}$ that belong locally to L^q for compact subsets of $(x_{-\infty}, \infty)$. We endow $L^q_{\mathrm{loc}}(x_{-\infty}, \infty)$ with the topology generated by the seminorms

$$|f|_{q,I} = \left(|I|^{-1}\int_I |f(y)|^q\,dy\right)^{1/q},$$

where $I=(a,b)$ is an interval in $(x_{-\infty},\infty)$ and $|I|=b-a$.

Let $f\in L^q_{\mathrm{loc}}(x_{-\infty},\infty)$ and let α be a real positive number. We define the maximal function $n^+_{q,\alpha}(f;x)$ by

$$n^+_{q,\alpha}(f;x)=\sup_{\rho>0}\rho^{-\alpha}|f|_{q,[x,x+\rho]}.$$

Let N be a non-negative integer and $\mathcal{P}_N$ the subspace of $L^q_{\mathrm{loc}}(x_{-\infty},\infty)$ formed by all the polynomials of degree at most N. This subspace is of finite dimension and therefore a closed subspace of $L^q_{\mathrm{loc}}(x_{-\infty},\infty)$. We denote by E^q_N the quotient space of $L^q_{\mathrm{loc}}(x_{-\infty},\infty)$ by $\mathcal{P}_N$. If $F\in E^q_N$, we define the seminorms

$$\|F\|_{q,I}=\inf_{f\in F}\left\{|f|_{q,I}\right\}.$$

The family of all such seminorms induces on E^q_N the quotient topology.

Given a positive real number α, we can write it as $\alpha=N+\beta$, where N is a non-negative integer and $0<\beta\le 1$. We fix $\alpha>0$ and its decomposition $\alpha=N+\beta$ in the previous conditions.

For $F\in E^q_N$, we define the maximal function $N^+_{q,\alpha}(F;x)$ as

$$N^+_{q,\alpha}(F;x)=\inf_{f\in F}\left\{n^+_{q,\alpha}(f;x)\right\}.$$

This type of maximal function was introduced by Calderón in [1].

Now we are ready to present the one-sided Calderón-Hardy spaces, $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$, defined by S. Ombrosi in [7]. The case $\omega=1$ and $\alpha\in\mathbb{N}$ has been studied by A. Gatto, J. Jiménez and C. Segovia in [2]. If $F\in E^q_N$, we say that F belongs to $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$, $0<p\le 1$, $1<q<\infty$, if the maximal function $N^+_{q,\alpha}(F;x)\in L^p(\omega)$. This means

$$\int_{x_{-\infty}}^{\infty}N^+_{q,\alpha}(F;x)^p\omega(x)\,dx<\infty.$$

The norm of F in $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$ is given by $\|F\|_{\mathcal{H}^{p,+}_{q,\alpha}(\omega)}=\|N^+_{q,\alpha}(F;x)\|_{L^p(\omega)}$.

We say that a class $A\in E^N_q$ is a p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$ if there exists a representative $a(y)$ of A and an interval I such that

(i) $\mathrm{supp}(a)\subset I\subset(x_{-\infty},\infty)$, $\omega(I)<\infty$,
(ii) $N^+_{q,\alpha}(A;x)\le\omega(I)^{\frac{-1}{p}}$ for all $x\in(x_{-\infty},\infty)$.

From the definition of a p-atom, the condition $\omega(I)<\infty$, $w\in A^+_s$ does not assure that I is bounded, nevertheless, given the properties of one-sided weights (see Lemma 1.1.7 and page 9 in [8]), I cannot be of type (a,∞); thus, if I is not bounded we have that $x_{-\infty}=-\infty$ and so $I=(-\infty,b)$, $b<\infty$.

As before, let $\alpha=N+\beta$ where $0<\beta\le 1$. The class $F\in E^N_q$ belongs to $\Lambda_\alpha(x_{-\infty},\infty)$ if $f\in F$ is such that $f\in C^N(x_{-\infty},\infty)$, and there exists a constant C such that the derivative D^Nf satisfies for every $x,x'\in(x_{-\infty},\infty)$ the Lipschitz condition

$$\left|D^N f(x) - D^N f(x')\right| \leq C \ |x - x'|^{\beta}.$$

We observe that to say $F \in \Lambda_\alpha(x_{-\infty}, \infty)$ is equivalent to saying that all their representatives belong to $\Lambda_\alpha(x_{-\infty}, \infty)$. To simplify notation, we write Λ_α instead of $\Lambda_\alpha(x_{-\infty}, \infty)$.

With the notation and definitions given above we can state the main results of this paper.

Theorem 1.1. *Let $0 < p \leq 1$, $0 < \beta < 1$, $\alpha = N + \beta$ with N an integer, $0 < \gamma + \beta < 1$, $1 < q < \frac{1}{\gamma}$ and $w \in A_s^+$ where $(\alpha + \frac{1}{q})p \geq s > 1$ or $(\alpha + \frac{1}{q})p > 1$ if $s = 1$. Let $\overline{I_\gamma^+}$ be the extension of the one-sided fractional integral given by (3.17). Then $\overline{I_\gamma^+}$ can be extended to a bounded operator from $\mathcal{H}_{q,\alpha}^{p,+}(\omega)$ into $\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)$.*

This theorem is a consequence of the following key result.

Theorem 1.2. *Let $F \in \Lambda_\alpha$, $\alpha = N + \beta$ with N an integer, $0 < \beta < 1$, $0 < \gamma + \beta < 1$ and $1 < q < \frac{1}{\gamma}$. Let $\overline{I_\gamma^+}$ be the extension of the one-sided fractional integral given by (3.17). Then*

$$N_{q,\alpha+\gamma}^{+}(\overline{I_\gamma^+}F; x) \leq C_{\alpha,\gamma} N_{q,\alpha}^{+}(F; x), \quad x \in (x_{-\infty}, \infty),$$

where $C_{\alpha,\gamma}$ does not depend on F.

The paper is organized as follows. In Section 2 we will present some auxiliary lemmas that we will need later in Section 3 and Section 4. In Section 3, we will prove the existence of the extension of the one-sided fractional integral to the classes $\mathcal{H}_{q,\alpha}^{p,+}(\omega) \cap \Lambda_\alpha$. In Section 4, we will prove the main results of this work, Theorem 1.1 and Theorem 1.2. In the last section we will give some remarks about the extension defined in Section 3.

2. Auxiliary lemmas

The following results establish some properties of the maximal function $N_{q,\alpha}^{+}(F, x)$ and the spaces $\mathcal{H}_{q,\alpha}^{p,+}(\omega)$.

First we observe that if $\chi_I(x)$ is the characteristic function of the interval $I = (a, b)$ and we denote $I^- = (a - |I|, a)$, it is not difficult to prove that

$$M^+ \chi_I(x) \geq \frac{1}{2}, \quad x \in I^- \cup I. \tag{2.1}$$

Lemma 2.1. *Let $F \in E_N^q$.*

(1) *Let f_1, f_2 be two representatives of F and $P = f_1 - f_2$. Then there exists a constant c_k such that for every x_1, x_2 and y in $(x_{-\infty}, \infty)$ the inequality*

$$\left|\left(\frac{d}{dy}\right)^k P(y)\right| \leq c_k \ \left(n_{q,\alpha}^{+}(f_1; x_1) + n_{q,\alpha}^{+}(f_2; x_2)\right) \left(|x_1 - y| + |x_2 - y|\right)^{\alpha - k}$$

holds.

(2) *If* $N^+_{q,\alpha}(F, x_0)$ *is finite for some* x_0 *then there exists a unique* $f \in F$ *such that* $n^+_{q,\alpha}(f; x_0) < \infty$ *and, therefore,* $N^+_{q,\alpha}(F; x_0) = n^+_{q,\alpha}(f; x_0)$.

(3) *If* $N^+_{q,\alpha}(F; x)$ *is finite,* f *is a representative of* F *and we denote by* $P(x, y)$ *the unique polynomial of degree at most* N *such that* $n^+_{q,\alpha}(f(y) - P(x, y); x) = N^+_{q,\alpha}(F; x)$, *then* $f(x) = P(x, x)$ *for almost every* x *such that* $N^+_{q,\alpha}(F; x)$ *is finite.*

(4) *Assume that* $N^+_{q,\alpha}(F, x) \leq t$ *for all* x *belonging to a set* $E \subset (x_{-\infty}, \infty)$. *Let* f *be a representative of* F *and let* $P(x, y)$ *be the unique polynomial in* $\mathcal{P}_N$, *such that* $N^+_{q,\alpha}(F; x) = n^+_{q,\alpha}(f(y) - P(y, x); x)$. *Then there exists* $c > 0$ *such that*

$$\left| A_k(x) - \sum \frac{1}{i!}(x - \overline{x})^i A_{k+i}(\overline{x}) \right| \leq c\, t\ |x - \overline{x}|^{\alpha - k},$$

for all x *and* $\overline{x}$ *in* E, *where* $A_k(x) = D^k_y P(x, y)\big|_{y=x}$.

(5) F *belongs to* Λ_α *if and only if there exists a finite constant* C *such that* $N^+_{q,\alpha}(F, x) \leq C$ *for all* $x \in (x_{-\infty}, \infty)$.

(6) *If* $F \in \Lambda_\alpha$, $x_1 \in (x_{-\infty,\infty})$ *and* f *is the representative of* F *such that* $N^+_{q,\alpha}(F, x_1) = n^+_{q,\alpha}(f, x_1)$, *then*

$$\left| D^i f(y) \right| \leq C \left\| N_{q,\alpha}(F; .) \right\|_\infty |y - x_1|^{\alpha - i}$$

holds for $i = 0, 1, \ldots, N$ *and* $y \in (x_{-\infty}, \infty)$.

The proof of (1) can be found in [7]. The proof of (2) is similar to the one of Lemma 3 in [2]. The proof of (3) can be seen in [8]. Proceeding as in the proof of Lemma 5 in [1] we obtain the proof of (4), also we can find a complete proof in [8]. Part (5) is Lemma 3.10 in [7]. The details of the proof of (6) can be seen in [8].

Remark 2.2. Given a representative $f \in F$, if for each x we have $N^+_{q,\alpha}(F; x) < \infty$ by Lemma 2.1(2), there exists a unique representative of F that realizes the maximal function $N^+_{q,\alpha}(F; x) < \infty$. We denote this representative by $f(y) - P(x, y)$, where $P(x, y)$ is a polynomial of degree less than or equal to N.

Lemma 2.3. *Let* $0 < p \leq 1$ *and* $w \in A^+_s$ *where* $(\alpha + \frac{1}{q})p \geq s > 1$ *or* $(\alpha + \frac{1}{q})p > 1$ *if* $s = 1$. *The space* $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$ *is complete.*

The proof of this result is similar to that of Corollary 2 in [2], see also [8].

The following result is fundamental for the proof of Theorem 1.1 in Section 4.

Theorem 2.4. *Let* $0 < p \leq 1$ *and* $w \in A^+_s$ *where* $(\alpha + \frac{1}{q})p \geq s > 1$ *or* $(\alpha + \frac{1}{q})p > 1$ *if* $s = 1$. *The set of classes* $\mathcal{H}^{p,+}_{q,\alpha}(\omega) \cap \Lambda_\alpha$ *is dense in* $\mathcal{H}^{p,+}_{q,\alpha}(\omega)$.

The proof of this result is due to Ombrosi [8], who used to prove it a one-sided version of the Calderón-Zygmund decomposition.

3. Extension of the one-sided fractional integral to the classes $\mathcal{H}^{p,+}_{q,\alpha}(w) \cap \Lambda_\alpha$

Let $0 < \gamma < 1$. Given a measurable function in $\mathbb{R}$, the one-sided fractional integral of order γ is defined by

$$I^+_\gamma f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}\, dy, \quad x \in (x_{-\infty}, \infty),$$

provided the integral exists.

Remark 3.1. If we consider the kernel $K(x) = \chi_{(-\infty,0)}|x|^{\gamma-1}$, we can write the one-sided fractional integral as a convolution product as follows

$$I^+_\gamma f(x) = (K * f)(x).$$

It is simple to prove that $K(x) \in L^1_{\text{loc}}(\mathbb{R} - \{0\})$ and K satisfies for $1 \le i \le n$,

$$|D^i K(x)| \le C_{\gamma,i}|x|^{\gamma-1-i}, \quad x \in (-\infty, 0). \tag{3.1}$$

From the definition it is trivial to prove that, for $f \ge 0$,

$$I^+_\gamma f(x) \le I_\gamma f(x), x \in (x_{-\infty}, \infty) \tag{3.2}$$

and

$$I^+_{\gamma_1} \circ I^+_{\gamma_2} f(x) = I^+_{\gamma_1+\gamma 2} f(x), x \in (x_{-\infty}, \infty). \tag{3.3}$$

In what follows we suppose that $\omega \in A^+_s$ where $(\alpha+\frac{1}{q})p \ge s > 1$ or $(\alpha+\frac{1}{q})p > 1$ if $s = 1$. Furthermore we consider the number $x_{-\infty}$ associated with $\omega \in A^+_s$ such that $x_{-\infty} < 0$.

Let us fix a function $\phi \in C^\infty_0$, $0 \le \phi(y) \le 1$, $\text{supp}(\phi) \subset [-2, 2]$, and such that $\phi(y) \equiv 1$ in $[-1, 1]$. Let $r > 0$ and $x_1 \in \mathbb{R}$. We denote

$$\phi_{x_1,r}(y) = \phi\left(\frac{y - x_1}{r}\right). \tag{3.4}$$

Then the support of $\phi_{x_1,r}(y)$ is contained in $[x_1 - 2r, x_1 + 2r]$ and $\phi(y) \equiv 1$ in $[x_1 - r, x_1 + r]$. Moreover, we have that

$$\left|D^i(\phi_{x_1,r})(y)\right| \le C_i r^{-i}, \tag{3.5}$$

for every non-negative integer i, where C_i is $\|D^i\phi\|_\infty$. If $x_1 = 0$, we denote $\phi_{0,r}(y)$ by $\phi_r(y)$.

Unless we state something different, we consider $\alpha > 0$, $\alpha \notin \mathbb{N}$ where α can be represented by $\alpha = N + \beta$ with $0 < \beta < 1$.

Lemma 3.2. *Let $F \in \Lambda_\alpha$ and let $f(y)$ be the representative of F such that $n^+_{q,\alpha}(f;0) = N^+_{q,\alpha}(F;0)$. If we define*

$$
\begin{aligned}
g_j(x) &= \int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}\phi_j(y)\,dy \\
&\quad - \sum_{i=0}^{N}\int_x^\infty D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(0) f(y)(\phi_j(y)-\phi_1(y))\,dy\frac{x^i}{i!}\,,
\end{aligned}
\tag{3.6}
$$

where $\phi_j(y)$ and $\phi_1(y)$ are given as in (3.4), then there exists $\lim_{j\to\infty} g_j$ in $L^q_{\mathrm{loc}}(x_{-\infty},\infty)$.

Proof: We fix $I = [a,b] \subset \mathbb{R}$, and we consider a natural number l such that $I \subset [-l/2, l/2]$. Then for all $x \in I$ and $j > l$ we can write $g_j(x)$ as

$$
\begin{aligned}
g_j(x) = g_l(x) + \int_x^\infty &\left[\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(0)\frac{x^i}{i!}\right] \\
&\times f(y)(\phi_j(y)-\phi_l(y))\,dy.
\end{aligned}
\tag{3.7}
$$

Now we prove that there exists the limit of the second term of (3.7) when $j \to \infty$.

Since $\phi_j(y) - \phi_l(y) \le 1 - \phi_l(y)$ and $\operatorname{supp}(1-\phi_l) \subset \{|y| \ge l\}$ it follows that

$$
\operatorname{supp}(\phi_j - \phi_l) \subset \operatorname{supp}(1-\phi_l) \subset \{|y| \ge l\}
$$

and so the second term of (3.7) can be estimated as

$$
\begin{aligned}
&\int_x^\infty \left|\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(0)\frac{x^i}{i!}\right| |f(y)|\,|\phi_j(y)-\phi_l(y)|\,dy \\
&\le \int_{\{|y|>l\}\cap(x,\infty)} \left|\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(0)\frac{x^i}{i!}\right| \\
&\quad\times |f(y)|\,|1-\phi_l(y))|\,dy.
\end{aligned}
\tag{3.8}
$$

We observe that if $x \in I \subset [-l/2, l/2]$, and $y \in A = \{|y| \ge l\} \cap (x,\infty)$, for $0 < \xi < 1$ we have $|\xi x - y|| \ge |y|/2$. In effect,

$$
|\xi x - y| = |y - \xi x| \ge |y| - |\xi x| \ge |y| - |\xi|\frac{l}{2} \ge |y| - \frac{l}{2} \ge |y| - \frac{|y|}{2} = \frac{|y|}{2}\,.
$$

Then by the Taylor's Formula and Lemma 2.1(6), we have

$$\begin{aligned}
&\int_{\{|y|>l\}\cap(x,\infty)} \left| \frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right| \\
&\quad \times |f(y)|(1-\phi_l(y))\, dy \\
&\le C_{\gamma,N} \int_{\{|y|>l\}\cap(x,\infty)} |\xi x - y|^{\gamma-(N+2)} |f(y)|\, dy\ |x|^{N+1} \\
&\le C_{\gamma,N} 2^{(N+2)} \int_{\{|y|>l\}\cap(x,\infty)} |y|^{\gamma-(N+2)} |f(y)|\, dy\ |x|^{N+1} \\
&\le C_{\gamma,N} 2^{(N+2)} \left\| N_{q,\alpha}(F;.) \right\|_\infty \int_{\{|y|>l\}\cap(x,\infty)} |y|^{\gamma-(N+2)} |y|^{N+\beta}\, dy\ |x|^{N+1}
\end{aligned} \tag{3.9}$$

and since the last integral is convergent for $0<\gamma+\beta<1$, it follows

$$\begin{aligned}
&\int_{\{|y|>l\}\cap(x,\infty)} \left| \frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right| |f(y)|\,(1-\phi_l(y))\, dy \\
&\le C_{\beta,\gamma,N,l} \left\| N_{q,\alpha}(F;.) \right\|_\infty < \infty.
\end{aligned}$$

Therefore,

$$\left(\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right) f(y)(1-\phi_l(y)) \in L^1((x_{-\infty},\infty))$$

and by the Dominated Convergence Theorem, the second term of (3.7) converges to

$$\int_x^\infty \left[\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^{N} D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right] f(y)(1-\phi_l(y))\, dy.$$

Then there exists $\lim_{j\to\infty} g_j(x)$ in $L^\infty_{\mathrm{loc}}(x_{-\infty},\infty)$, and, consequently, pointwise and in $L^q_{\mathrm{loc}}(x_{-\infty},\infty)$. $\square$

Taking into account the notation of Lemma 3.2 we define

$$\begin{aligned}
I_\gamma^{+,0} f(x) &= \lim_{j\to\infty} g_j(x) \\
&= \lim_{j\to\infty} \int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}} \phi_j(y)\, dy \\
&\quad - \sum_{i=0}^{N} \int_x^\infty D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) f(y)(\phi_j(y)-\phi_1(y))\, dy \frac{x^i}{i!}
\end{aligned} \tag{3.10}$$

where the limit is taken in the sense of $L^\infty_{\mathrm{loc}}(x_{-\infty},\infty)$.

In Lemma 3.2 we have proved that for $x \in I = [a,b] \subset [-l/2, l/2]$,

$$
\begin{aligned}
I_\gamma^{+,0} f(x) &= \lim_{j\to\infty} g_j(x) \\
&= g_l(x) + \int_x^\infty \left[\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^N D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right] \\
&\quad \times f(y)(1-\phi_l(y))\, dy,
\end{aligned} \tag{3.11}
$$

where

$$
\begin{aligned}
g_l(x) = &\int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}} \phi_l(y)\, dy \\
&- \sum_{i=0}^N \int_x^\infty D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) f(y)(\phi_l(y) - \phi_1(y))\, dy \ \frac{x^i}{i!}.
\end{aligned}
$$

Summarizing, if $F \in \Lambda_\alpha$ we have chosen a representative f, in particular f is such that $n^+_{q,\alpha}(f;0) = N^+_{q,\alpha}(F;0)$, and for this f we define an extension of the fractional integral operator I_γ^+, denoted by $I_\gamma^{+,0}$, such that $I_\gamma^{+,0} f$ belongs to $L^\infty_{\mathrm{loc}}(x_{-\infty}, \infty)$. The following step is to prove that if f is a polynomial of degree less than or equal to N then $I_\gamma^{+,0} f$ is also a polynomial of degree at most N, which shows that the extension does not depend on the representative f.

Lemma 3.3. *Let $P(y)$ be a polynomial of degree at most N. Then $I_\gamma^{+,0} P(x)$ (defined by (3.10)) coincides with a polynomial of degree at most N in $(x_{-\infty}, \infty)$.*

Proof: Without loss of generality, we can assume that $P(y) = y^n$ where $0 \le n \le N$. Let us fix $l \in \mathbb{N}$ and $x \in [-l/2, l/2]$. Then from (3.11), we have that

$$
\begin{aligned}
I_\gamma^{+,0} P(x) = &\int_x^\infty \frac{y^n}{(y-x)^{1-\gamma}} \phi_l(y)\, dy \\
&+ \int_x^\infty \left[\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^N D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right] y^n (1-\phi_l(y))\, dy \\
&- \sum_{i=0}^N \int_x^\infty D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(0)\ y^n (\phi_l(y) - \phi_1(y))\, dy \frac{x^i}{i!} \\
= &P_1(x) + P_2(x) + Q(x).
\end{aligned} \tag{3.12}
$$

It is enough to prove that $D^{N+1}(I_\gamma^{+,0} P) \equiv 0$. Since $Q(x)$ is a polynomial of degree at most N, we have that $D^{N+1}(Q)(x) = 0$.

Then, the only thing to prove is that

$$
D^{N+1}(P_2)(x) = -D^{N+1}(P_1)(x). \tag{3.13}
$$

We consider $\eta(y) = y^n \phi_l(y)$. Since $\phi_l(y) \in C_0^\infty(\mathbb{R})$, we have $\eta(y) \in C_0^\infty(\mathbb{R})$. By the change of variable $z = y - x$ we can write $P_1(x)$ as

$$P_1(x) = \int_0^\infty \frac{(x+z)^n \phi_l(x+z)}{z^{1-\gamma}}\, dz = \int_0^\infty \frac{\eta(x+z)}{z^{1-\gamma}}\, dz.$$

Given that η and its derivatives are compactly supported, by the standard theorem of derivation under the integral sign we have that $P_1(x)$ admits derivatives until order $N+1$ and

$$D^{N+1}(P_1)(x) = \int_0^\infty \frac{1}{z^{1-\gamma}} D^{N+1}\eta(x+z)\, dz. \tag{3.14}$$

Now we want to differentiate $P_2(x)$ until order $N+1$. For $s = 0, 1, \ldots, N+1$, $|y| > l$ and $x \in [-l/2, l/2]$ using Taylor's Formula and (3.1) we have

$$\begin{aligned} &\left| D_x^s \left[\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^N D^i \left(\frac{1}{|. - y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right] \right| \\ &\leq C D^{N+1} \left(\frac{1}{|. - y|^{1-\gamma}} \right)(\xi x) |x|^{N+1-s} \leq C_l |y|^{\gamma - N - 2}, \end{aligned}$$

hence,

$$\begin{aligned} &\int \left| D_x^s \left[\frac{1}{(y-x)^{1-\gamma}} - \sum_{i=0}^N D^i \left(\frac{1}{|. - y|^{1-\gamma}} \right)(0) \frac{x^i}{i!} \right] \right| |y^n (1 - \phi_l(y))|\, dy \\ &\leq C_l \int_{|y|>l} |y|^{n+\gamma-N-2}\, dy < \infty \end{aligned}$$

and so we obtain

$$\begin{aligned} D^{N+1}(P_2)(x) &= \int_x^\infty D_x^{N+1} \left(\frac{1}{(y-x)^{1-\gamma}} \right) y^n (1 - \phi_l(y))\, dy \\ &= (-1)^{N+1} \int_x^\infty D_y^{N+1} \left(\frac{1}{(y-x)^{1-\gamma}} \right) y^n (1 - \phi_l(y))\, dy. \end{aligned} \tag{3.15}$$

Applying integration by parts in (3.15) and changing variables $y = x + z$ we have that

$$\begin{aligned} &(-1)^{N+1} \int_x^\infty D_y^{N+1} \left(\frac{1}{(y-x)^{1-\gamma}} \right) y^n (1 - \phi_l(y))\, dy \\ &= (-1)^{2N+1} \int_x^\infty \frac{1}{(y-x)^{1-\gamma}} D_y^{N+1} (y^n \phi_l)(y)\, dy \\ &= -\int_0^\infty \frac{1}{z^{1-\gamma}} D_x^{N+1} \eta(x+z)\, dz \\ &= -D^{N+1}(P_1)(x). \end{aligned} \tag{3.16}$$

By (3.15) and (3.16), we have proved (3.13) which finishes the proof. □

Definition 3.4. Let $F \in \Lambda_\alpha$ and $f(y)$ be a representative of F. We define $\overline{I_\gamma^+}F$ as the class in E_N^q of the function in $L^q_{\text{loc}}(x_{-\infty},\infty)$ given by

$$I_\gamma^{+,0}f(x) = \lim_{j\to\infty}\left[\int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}\phi_j(y)\,dy - \sum_{i=0}^N \int_x^\infty D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(0)f(y)(\phi_j-\phi_1)(y)\,dy\frac{x^i}{i!}\right]. \tag{3.17}$$

This definition makes sense, since by Lemma 3.2 we have that for each representative of F the limit in (3.17) exists in the sense of $L^q_{\text{loc}}(x_{-\infty},\infty)$ and by Lemma 3.3 the class $\overline{I_\gamma^+}F$ does not depend on the representative f of F.

Furthermore, if we fix $x_0 \in (x_{-\infty},\infty)$, and define

$$I_\gamma^{+,x_0}f(x) = \lim_{j\to\infty}\left[\int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}\phi_{x_0,j}(y)\,dy - \sum_{i=0}^N \int_x^\infty D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)f(y)\left(\phi_{x_0,j}(y)-\phi_{x_0,1}(y)\right)dy\frac{(x-x_0)^i}{i!}\right],$$

where f is a representative of F, similar computations show that $I_\gamma^{+,x_0}f(x)$ differs from $I_\gamma^{+,0}f(x)$ by a polynomial of degree at most N and therefore $\overline{I_\gamma^+}F$ is also the class of $I_\gamma^{+,x_0}f(x)$.

4. Proofs of the main results

Proof of Theorem 1.2: Let $x_0 \in (x_{-\infty},\infty)$ and let $f(y)$ be the representative of F such that $n^+_{q,\alpha}(f;x_0) = N^+_{q,\alpha}(F;x_0)$.

We know that a representative of $\overline{I_\gamma^+}F$ is

$$I_\gamma^{+,x_0}f(x) = \lim_{j\to\infty}\left[\int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}\phi_{x_0,j}(y)\,dy - \sum_{i=0}^N \int_x^\infty D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)f(y)(\phi_{x_0,j}(y)-\phi_{x_0,1}(y))\,dy\frac{(x-x_0)^i}{i!}\right].$$

Let $\rho>0$ and $x\in[x_0,x_0+\rho/4]$. Our goal is to prove the following estimates

$$\left|I_\gamma^{+,x_0}\left(f(1-\phi_{x_0,\rho})\right)(x) - Q(x_0,x)\right| \le C_{\gamma,\alpha}N^+_{q,\alpha}(F;x_0)\rho^{\alpha+\gamma} \tag{4.1}$$

and

$$\left(\int_{x_0}^{x_0+\rho/4}\left|I_\gamma^{+,x_0}\left(f\phi_{x_0,\rho}\right)(x)\right|^q\,dx\right)^{\frac{1}{q}} \le C_{\gamma,\alpha}N^+_{q,\alpha}(F;x_0)\rho^{(\alpha+\gamma)+\frac{1}{q}}, \tag{4.2}$$

where

$$Q(x_0,x)=\sum_{i=0}^{N}\int_x^{\infty}D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)f(y)\phi_{x_0,1}(y)\,dy\frac{(x-x_0)^i}{i!}\,.$$

Let us see first that, if (4.1) and (4.2) hold, then we obtain the desired estimate:

$$\begin{aligned}
&\left(\int_{x_0}^{x_0+\rho/4}\left|I_\gamma^{+,x_0}f(x)-Q(x_0,x)\right|^q\,dx\right)^{\frac{1}{q}}\\
&=\left(\int_{x_0}^{x_0+\rho/4}\left|I_\gamma^{+,x_0}f(x)-I_\gamma^{+,x_0}\left(f\phi_{x_0,\rho}\right)(x)+I_\gamma^{+,x_0}\left(f\phi_{x_0,\rho}\right)(x)-Q(x_0,x)\right|^q dx\right)^{\frac{1}{q}}\\
&\le\left(\int_{x_0}^{x_0+\rho/4}\left|I_\gamma^{+,x_0}f\left(1-\phi_{x_0,\rho}\right)(x)-Q(x_0,x)\right|^q\,dx\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{x_0}^{x_0+\rho/4}\left|I_\gamma^{+,x_0}\left(f\phi_{x_0,\rho}\right)(x)\right|^q\,dx\right)^{\frac{1}{q}}\\
&\le C_{\gamma,\alpha}N_{q,\alpha}^+(F;x_0)\rho^{\alpha+\gamma}\left(\frac{\rho}{4}\right)^{\frac{1}{q}}+C_{\gamma,\alpha}N_{q,\alpha}^+(F;x_0)\rho^{\alpha+\gamma+\frac{1}{q}}\\
&=C_{\gamma,\alpha}N_{q,\alpha}^+(F;x_0)\rho^{\alpha+\gamma+\frac{1}{q}}\,.
\end{aligned}$$

Then for $\rho>0$

$$\frac{1}{\rho^{\alpha+\gamma}}\left(\frac{1}{\rho}\int_{x_0}^{x_0+\rho/4}\left|I_\gamma^{+,x_0}f(x)-Q(x_0,x)\right|^q\,dx\right)^{\frac{1}{q}}\le C_{\alpha,\gamma}N_{q,\alpha}^+(F,x_0),$$

and taking supremum for $\rho>0$ we have

$$n_{q,\alpha+\gamma}^+\left(I_\gamma^{+,x_0}f(x)-Q(x_0,x);x_0\right)\le C_{\alpha,\gamma}N_{q,\alpha}^+(F;x_0).$$

Since $I_\gamma^{+,x_0}f(x)-Q(x_0,x)\in\overline{I_\gamma^+}F$, we have

$$N_{q,\alpha+\gamma}^+\left(\overline{I_\gamma{}^+}F;x_0\right)\le C_{\alpha,\gamma}N_{q,\alpha}^+(F;x_0),\quad x_0\in(x_{-\infty},\infty).$$

Now we prove (4.1) and (4.2).

For (4.1) we have

$$
\begin{aligned}
I_\gamma^{+,x_0}\left(f\left(1-\phi_{x_0,\rho}\right)\right)(x) &= \lim_{j\to\infty}\left[\int_x^\infty \frac{f(y)(1-\phi_{x_0,\rho}(y))\phi_{x_0,j}(y)}{(y-x)^{1-\gamma}}\,dy\right.\\
&-\sum_{i=0}^N\int_x^\infty D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)f(y)(1-\phi_{x_0,\rho}(y))(\phi_{x_0,j}(y)-\phi_{x_0,1}(y))\,dy\\
&\left.\times\frac{(x-x_0)^i}{i!}\right].
\end{aligned}
\tag{4.3}
$$

Subtracting and adding up $\frac{f(y)}{(y-x)^{1-\gamma}}(1-\phi_{x_0,\rho}(y))\phi_{x_0,1}(y)$ in the first integral and associating we have

$$
\begin{aligned}
I_\gamma^{+,x_0}(f(1-\phi_{x_0,\rho}))(x) &= \int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}}(1-\phi_{x_0,\rho}(y))\phi_{x_0,1}(y)\,dy\\
&+\lim_{j\to\infty}\int_x^\infty\left[\frac{1}{(x-y)^{1-\gamma}}-\sum_{i=0}^N D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\frac{(x-x_0)^i}{i!}\right]\\
&\times f(y)(1-\phi_{x_0,\rho}(y))(\phi_{x_0,j}(y)-\phi_{x_0,1}(y))\,dy.
\end{aligned}
\tag{4.4}
$$

Writing

$$
\begin{aligned}
\frac{1}{(x-y)^{1-\gamma}} &= \sum_{i=0}^N D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\frac{(x-x_0)^i}{i!}\\
&+\left[\frac{1}{(x-y)^{1-\gamma}}-\sum_{i=0}^N D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\frac{(x-x_0)^i}{i!}\right]
\end{aligned}
\tag{4.5}
$$

and replacing (4.5) in the first integral of (4.4), we have

(4.6)

$$
\begin{aligned}
I_\gamma^{+,x_0}(f(1-\phi_{x_0,\rho}))(x) &= \sum_{i=0}^N\int_x^\infty f(y)D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\phi_{x_0,1}(y)\,dy\frac{(x-x_0)^i}{i!}\\
&-\sum_{i=0}^N\int_x^\infty f(y)D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\phi_{x_0,1}(y)\phi_{x_0,\rho}(y)\,dy\frac{(x-x_0)^i}{i!}\\
&+\lim_{j\to\infty}\int_x^\infty\left[\frac{1}{(y-x)^{1-\gamma}}-\sum_{i=0}^N D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\frac{(x-x_0)^i}{i!}\right]\\
&\times f(y)(1-\phi_{x_0,\rho}(y))\phi_{x_0,j}(y)\,dy\\
&= I_1(x)+I_2(x)+I_3(x).
\end{aligned}
$$

Let us estimate $I_1(x), I_2(x)$ and $I_3(x)$. Since $F \in \Lambda_\alpha$ and f is a representative such that $n^+_{q,\alpha}(f;x_0) = N^+_{q,\alpha}(F;x_0)$ then by Lemma 2.1(6) for $i=0$, we have

$$|f(y)| \leq C \left\| N^+_{q,\alpha}(F;.)\right\|_\infty |y-x_0|^\alpha .$$

Then from the last estimate and (3.1) we obtain that each integral in I_1 is bounded by

$$\begin{aligned}
&\int_x^\infty \left| f(y) D^i \left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\right| \phi_{x_0,1}(y)\, dy \\
&\leq C \left\| N^+_{q,\alpha}(F;.)\right\|_\infty \int_{x_0}^{x_0+2} \left| D^i \left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)\right| |y-x_0|^\alpha \, dy \\
&\leq C_{\gamma,i} \left\| N^+_{q,\alpha}(F;.)\right\|_\infty \int_{x_0}^{x_0+2} |y-x_0|^{\gamma-i-1} |y-x_0|^\alpha \, dy \\
&= C_{\gamma,i} \left\| N^+_{q,\alpha}(F;.)\right\|_\infty \int_{x_0}^{x_0+2} |y-x_0|^{\gamma-i-1+\alpha} \, dy < \infty,
\end{aligned}$$

since $\gamma+\alpha-i-1 = \gamma+\beta+N-i-1 > N-1-i \geq -1$ if $0 \leq i \leq N$. Then $I_1(x)$ is a polynomial of degree at most N, denoted by $Q(x_0,x)$, that is

$$Q(x_0,x) = \sum_{i=0}^N \int_x^\infty f(y) D^i \left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0) \phi_{x_0,1}(y)\, dy \frac{(x-x_0)^i}{i!} . \tag{4.7}$$

Now we estimate each term of I_2. Since $\operatorname{supp}(\frac{1}{|x_0-y|^{1-\gamma}} \phi_{x_0,\rho}(y)) \subset [x_0, x_0+2\rho]$ and using the condition (3.1) we have

$$\begin{aligned}
&\left| \int_x^\infty f(y) D^i \left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0) \phi_{x_0,1}(y)\phi_{x_0,\rho}(y)\, dy \frac{(x-x_0)^i}{i!} \right| \\
&\leq C_{\gamma,i} \int_{x_0}^{x_0+2\rho} |x_0-y|^{\gamma-1-i} |f(y)|\, dy \frac{|x-x_0|^i}{i!} \\
&\leq C_{\gamma,i} \int_{x_0}^{x_0+2\rho} |x_0-y|^{\gamma-1-i} |f(y)|\, dy \frac{\rho^i}{i!} \\
&\leq C_{\gamma,i} \rho^i \sum_{j=0}^\infty \int_{x_0+2^{-j}\rho}^{x_0+2^{-j+1}\rho} |x_0-y|^{\gamma-1-i} |f(y)|\, dy \\
&\leq C_{\gamma,i} \rho^i \sum_{j=0}^\infty \frac{1}{(2^{-j}\rho)^{1+i-\gamma}} \int_{x_0+2^{-j}\rho}^{x_0+2^{-j+1}\rho} |f(y)|\, dy \\
&\leq C_{\gamma,i} \rho^i \sum_{j=0}^\infty \frac{1}{(2^{-j}\rho)^{1+i-\gamma}} \int_{x_0}^{x_0+2^{-j+1}} |f(y)|\, dy
\end{aligned}$$

$$
\begin{aligned}
&\le C_{\gamma,i}\rho^i \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\rho)^{1+i-\gamma}} \left(\int_{x_0}^{x_0+2^{-j+1}\rho} |f(y)|^q\, dy \right)^{\frac{1}{q}} (2^{-j+1}\rho)^{\frac{1}{q'}} \\
&= C_{\gamma,i}\rho^i \sum_{j=0}^{\infty} \frac{(2^{-j+1}\rho)^{\frac{1}{q}}}{(2^{-j}\rho)^{1+i-\gamma}} \left(\frac{1}{2^{-j+1}\rho} \int_{x_0}^{x_0+2^{-j+1}\rho} |f(y)|^q\, dy \right)^{\frac{1}{q}} (2^{-j+1}\rho)^{\frac{1}{q'}} \\
&\le C_{\gamma,i}\rho^i \sum_{j=0}^{\infty} \frac{(2^{-j+1}\rho)}{(2^{-j}\rho)^{1+i-\gamma}} (2^{-j+1}\rho)^{\alpha} N^+_{q,\alpha}(F;x_0) \\
&\le C_{\gamma,i,\alpha}\rho^{\alpha+\gamma} N^+_{q,\alpha}(F;x_0) \sum_{j=0}^{\infty} 2^{-j(-i+\alpha+\gamma)}.
\end{aligned}
$$

Since $-i+\alpha+\gamma \ge \beta+\gamma > 0$, we have that the series $\sum_{j=0}^{\infty} 2^{-j(-i+\alpha+\gamma)}$ is convergent and then each term of $I_2(x)$ is controlled by

$$
\left| \int_x^{\infty} f(y) D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(x_0)\phi_{x_0,1}(y)\phi_{x_0,\rho}(y)\, dy \frac{(x-x_0)^i}{i!} \right| \le C_{\gamma,i,\alpha} N^+_{q,\alpha}(F;x_0)\rho^{\alpha+\gamma}.
$$

Then for $x \in [x_0, x_0+\frac{\rho}{4}]$,

$$
|I_2(x)| \le C_{\gamma,\alpha} N^+_{q,\alpha}(F;x_0)\rho^{\alpha+\gamma}. \tag{4.8}
$$

We estimate $I_3(x)$. Supposing that $x \in [x_0, x_0+\rho/4]$, $y \notin [x_0, x_0+\rho]$ and $0<\theta<1$ we have

$$
|x_0+\theta(x-x_0)-y| \ge |y-x_0| - |x-x_0| > |y-x_0| - \frac{\rho}{4} > \frac{3}{4}|y-x_0|.
$$

By the Mean Value Theorem, the condition (3.1) and the Taylor's Formula we have that $I_3(x)$ is estimated by

$$
\begin{aligned}
&\left| \int_x^{\infty} \left[\frac{1}{(x-y)^{1-\gamma}} - \sum_{i=0}^{N} D^i \left(\frac{1}{|.-y|^{1-\gamma}} \right)(x_0) \frac{(x-x_0)^i}{i!} \right] \right. \\
&\quad \times f(y)(1-\phi_{x_0,\rho}(y))\phi_{x_0,j}\, dy| \\
&\le \left| \int_x^{\infty} D^{N+1} \left(\frac{1}{|x_0+\theta(x-x_0)-y|^{1-\gamma}} \right) f(y)(1-\phi_{x_0,\rho}(y))\phi_{x_0 j}(y)\, dy \right. \\
&\quad \left. \times \frac{(x-x_0)^{N+1}}{(N+1)!} \right| \\
&\le C_{\gamma,N}\rho^{N+1} \int_{x_0+\rho}^{\infty} |x_0+\theta(x-x_0)-y|^{\gamma-1-(N+1)} |f(y)|\, dy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_{\gamma,N}\rho^{N+1}\int_{x_0+\rho}^{\infty}\frac{|f(y)|}{|y-x_0|^{-\gamma+N+2}}\,dy\\
&= C_{\gamma,N}\rho^{N+1}\sum_{j=0}^{\infty}\int_{x_0+2^j\rho}^{x_0+2^{j+1}\rho}\frac{|f(y)|}{|y-x_0|^{-\gamma+N+2}}\,dy\\
&\leq C_{\gamma,N}\rho^{\gamma-1}\sum_{j=0}^{\infty}\frac{1}{(2^j)^{-\gamma+N+2}}\left(\int_{x_0+2^j\rho}^{x_0+2^{j+1}\rho}|f(y)|\,dy\right)\\
&\leq C_{\gamma,N}\rho^{\gamma-1}\sum_{j=0}^{\infty}\frac{(2^j\rho)^{\frac{1}{q}}}{(2^j)^{-\gamma+N+2}}\left(\frac{1}{2^j\rho}\int_{x_0+2^j\rho}^{x_0+2^{j+1}\rho}|f(y)|^q\,dy\right)^{\frac{1}{q}}(2^j\rho)^{\frac{1}{q'}}\\
&= C_{\gamma,N}\rho^{\gamma-1}\sum_{j=0}^{\infty}\frac{2^j\rho}{(2^j)^{-\gamma+N+2}}\left(\frac{1}{2^j\rho}\int_{x_0+2^j\rho}^{x_0+2^{j+1}\rho}|f(y)|^q\,dy\right)^{\frac{1}{q}}\\
&\leq C_{\gamma,N}\rho^{\gamma+\alpha}N^+_{q,\alpha}(F,x_0)\sum_{j=0}^{\infty}\frac{2^{(j)(\alpha+1)}}{(2^j)^{-\gamma+N+2}}\\
&= C_{\gamma,N}\rho^{\gamma+\alpha}N^+_{q,\alpha}(F,x_0)\sum_{j=0}^{\infty}2^{j(\gamma-N-2+\alpha+1)}.
\end{aligned}
$$

Since $\gamma-N-2+\alpha+1<0$, we have that the series is convergent and

$$
|I_3(x)|\leq C_{\gamma,\alpha}N^+_{q,\alpha}(F,x_0)\rho^{\alpha+\gamma}. \tag{4.9}
$$

From the identity (4.6), the estimations (4.7), (4.8) and (4.9) give (4.1).

Now we have to prove (4.2):

(4.10)

$$
\begin{aligned}
I_\gamma^{+,x_0}(f\phi_{x_0,\rho})(x) &= \lim_{j\to\infty}\int_x^{\infty}\frac{f(y)}{(y-x)^{1-\gamma}}\phi_{x_0,\rho}(y)\phi_{x_0,j}(y)\,dy\\
&\quad-\sum_{i=0}^{N}\int_x^{\infty}D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)f(y)\phi_{x_0,\rho}(y)(\phi_{x_0,j}(y)-\phi_{x_0,1}(y))\,dy\frac{(x-x_0)^i}{i!}\\
&=\int_x^{\infty}\frac{f(y)}{(y-x)^{1-\gamma}}\phi_{x_0,\rho}(y)\,dy\\
&\quad-\sum_{i=0}^{N}\int_x^{\infty}D^i\left(\frac{1}{|.-y|^{1-\gamma}}\right)(x_0)f(y)\phi_{x_0,\rho}(y)(1-\phi_{x_0,1}(y))\,dy\frac{(x-x_0)^i}{i!}\\
&= J_1(x)+J_2(x).
\end{aligned}
$$

Arguing as in the proof of (4.8), we have

$$
|J_2(x)|\leq C_{\gamma,\alpha}N^+_{q,\alpha}(F;x_0)\rho^{\alpha+\gamma}\quad\text{for all}\quad x\in(x_0,x_0+\rho/4). \tag{4.11}
$$

In order to estimate the L^q norm of J_1, we use Hölder inequality and (1.1). In effect, if $r = \frac{q}{1-q\gamma}$, we have

$$\begin{aligned}
&\int_{x_0}^{x_0+\rho/4} \left| \int_x^\infty \frac{f(y)\phi_{x_0,\rho}(y)}{(y-x)^{1-\gamma}}\, dy \right|^q dx \\
&= \int_{x_0}^{x_0+\rho/4} \left| I_\gamma^+ \left(f\phi_{x_0,\rho} \right)(x) \right|^q dx \\
&\leq \left[\int_{x_0}^{x_0+\rho/4} \left(\left| I_\gamma^+ \left(f\phi_{x_0,\rho} \right)(x) \right|^q \right)^{\frac{r}{q}} dx \right]^{\frac{q}{r}} \left[\int_{x_0}^{x_0+\rho/4} 1^{\frac{r}{r-q}}\, dx \right]^{\frac{r-q}{r}} \\
&= C \| I_\gamma^+ (f\phi_{x_0,\rho}) \|_r^q \rho^{1-\frac{q}{r}} \\
&\leq C_{r,q}^q \| f\phi_{x_0,\rho} \|_q^q \rho^{1-\frac{q}{r}} \\
&= C_{r,q}^q \rho^{2-\frac{q}{r}} \left(\frac{1}{\rho} \int_{x_0}^{x_0+\rho/4} |f\phi_{x_0,\rho}(x)|^q\, dx \right) \\
&\leq C_{r,q}^q [N_{q,\alpha}^+(F,x_0)]^q \rho^{(\alpha+\gamma)q+1}.
\end{aligned}$$

Then from the estimations of $J_1(x)$ and $J_2(x)$ we have proved (4.2). □

Proof of Theorem 1.1: From Theorem 1.2 and by a standard argument we obtain Theorem 1.1. Anyway, for the sake of completeness we will do the proof. In effect, for $F \in \mathcal{H}_{q,\alpha}^{p,+}(\omega)$ we want to prove that there exists a constant $C_{\gamma,\alpha}$ such that

$$\left\| \overline{I_\gamma^+} F \right\|_{\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)} \leq C_{\gamma,\alpha} \|F\|_{\mathcal{H}_{q,\alpha}^{p,+}(\omega)}. \tag{4.12}$$

By Lemma 2.4 we have that there exists a sequence $F_j \in \mathcal{H}_{q,\alpha}^{p,+}(\omega) \cap \Lambda_\alpha$ such that $F_j \to F$ in $\mathcal{H}_{q,\alpha}^{p,+}(\omega)$.

Since $F_j \in \mathcal{H}_{q,\alpha}^{p,+}(\omega) \cap \Lambda_\alpha \subset \Lambda_\alpha$, by Theorem 1.2 we have

$$\left\| \overline{I_\gamma^+} F_j \right\|_{\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)} \leq C_{\gamma,\alpha} \left\| F_j \right\|_{\mathcal{H}_{q,\alpha}^{p,+}(\omega)}. \tag{4.13}$$

Using that the operator $\overline{I_\gamma^+}$ is linear and (4.13) we have that for each $j,k \in \mathbb{N}$

$$\begin{aligned}
\left\| \overline{I_\gamma^+} F_j - \overline{I_\gamma^+} F_k \right\|_{\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)} &= \left\| \overline{I_\gamma^+} (F_j - F_k) \right\|_{\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)} \\
&\leq C_{\gamma,\alpha} \left\| F_j - F_k \right\|_{\mathcal{H}_{q,\alpha}^{p,+}(\omega)}.
\end{aligned} \tag{4.14}$$

Since F_j is a Cauchy sequence in $\mathcal{H}_{q,\alpha}^{p,+}(\omega)$, by (4.14) we have that $\overline{I_\gamma^+} F_j$ is a Cauchy sequence in $\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)$. By Lemma 2.3 $\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)$ is complete, thus $\overline{I_\gamma^+} F_j$ has a limit in $\mathcal{H}_{q,\alpha+\gamma}^{p,+}(\omega)$ that we define by $\overline{I_\gamma^+} F$ and so we have that $\overline{I_\gamma^+} F_j \to \overline{I_\gamma^+} F$

in $\mathcal{H}^{p,+}_{q,\alpha+\gamma}(\omega)$. Then by this last conclusion and (4.13) we have (4.12) as follows

$$\begin{aligned}\|\overline{I^+_\gamma}F\|_{\mathcal{H}^{p,+}_{q,\alpha+\gamma}(\omega)} &= \lim_{j\to\infty}\left\|\overline{I^+_\gamma}F_j\right\|_{\mathcal{H}^{p,+}_{q,\alpha+\gamma}(\omega)}\\ &\le C_{\gamma,\alpha}\lim_{j\to\infty}\|F_j\|_{\mathcal{H}^{p,+}_{q,\alpha}(\omega)} = C_{\gamma,\alpha}\|F\|_{\mathcal{H}^{p,+}_{q,\alpha}(\omega)}.\end{aligned}$$

□

5. Final remarks

Remark 5.1. By the characterization given in Lemma 2.1(5) we can observe that Theorem 1.2 gives another proof of the classical result which ensures that $\overline{I^+_\gamma}$ map Λ_α into $\Lambda_{\alpha+\gamma}$ for $\alpha = \mathbb{N} + \beta$ with $0 < \beta + \gamma < 1$.

Remark 5.2. It is not hard to see that as a consequence of Ombrosi's results, see Theorems 4.1.5 and 4.2.2 in [8], we can say that the previous result is also true for the case $\alpha \in \mathbb{N}$.

Remark 5.3. Nevertheless, Theorem 1.1 is false for $\beta + \gamma = 1$, $0 < \beta < 1$. We will see that by an example. We suppose $\omega \equiv 1$. Let ϕ be a in C_0^∞, $0 \le \phi(y) \le 1$, with support contained in $[-8, 8]$, and with $\phi(y) \equiv 1$ in $[-4, 4]$. For $0 < \alpha < 1$, we define

$$a(x) = \phi(x)\left(\sum_{n=1}^{\infty}\frac{1}{2^{\alpha n}}\cos 2^n x\right). \tag{5.1}$$

The previous series defines a function Lipschitz-α (see [14]), and since $\phi(x) \in C_0^\infty$, $a(x)$ also belongs to Lipschitz-α. Then if we denote the class of $a(x)$ in E_0^q by A, we have that $N^+_{q,\alpha}(A, x)$ is bounded, and therefore since $a(x)$ has compact support contained in a bounded interval, A is a multiple of a p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(1)$. Then the class of $a(x)$ in E_0^q belongs to $\mathcal{H}^{p,+}_{q,\alpha}(1)$. If we consider, in particular, the case $\alpha = N + \beta$, $N = 0$ and $\beta + \gamma = 1$. If we suppose that $\overline{I^+_{1-\alpha}}$ is a bounded extension from $\mathcal{H}^{p,+}_{q,\alpha}(1)$ into $\mathcal{H}^{p,+}_{q,1}(1)$, then we have that the class in E_0^q of the function $I^+_{1-\alpha}a(x) \in \mathcal{H}^{p,+}_{q,1}(1)$ and this is false. If we suppose that it is true, by Theorem 4.2.2 in [8] we have that $DI^+_{1-\alpha}a(x) \in H^p$, where H^p is the classical Hardy space, and this is false. A proof of this fact is given in [8].

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