Berezin-Weyl quantization for Cartan motion groups

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Abstract. We construct adapted Weyl correspondences for the unitary irreducible representations of the Cartan motion group of a noncompact semisimple Lie group by using the method introduced in [B. Cahen, *Weyl quantization for semidirect products*, Differential Geom. Appl. **25** (2007), 177–190].

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1. Introduction

In [3] and [4], we introduced the notion of adapted Weyl correspondence as a generalization of the usual quantization rules [1], [15]. The present paper is part of a larger program to study adapted Weyl correspondences for semisimple Lie groups and for semidirect products.

Let G be a connected Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let π be a unitary irreducible representation of G on a Hilbert space $\mathcal{H}$. Suppose that π is associated with an orbit $\mathcal{O}$ for the coadjoint action of G on $\mathfrak{g}^*$ by the Kostant-Kirillov method of orbits [18], [20]. In [6], we gave the following definition for the notion of adapted Weyl correspondence.

Definition 1.1. An adapted Weyl correspondence is an isomorphism W from a vector space $\mathcal{A}$ of complex-valued smooth functions on the orbit $\mathcal{O}$ (called symbols) onto a vector space $\mathcal{B}$ of (not necessarily bounded) linear operators on $\mathcal{H}$ satisfying the following properties:

(1) the elements of $\mathcal{B}$ preserve a fixed dense domain $\mathcal{D}$ of $\mathcal{H}$;
(2) the constant function 1 belongs to $\mathcal{A}$, the identity operator I belongs to $\mathcal{B}$ and $W(1)=I$;
(3) $A\in\mathcal{B}$ and $B\in\mathcal{B}$ implies $AB\in\mathcal{B}$;
(4) for each f in $\mathcal{A}$ the complex conjugate $\bar{f}$ of f belongs to $\mathcal{A}$ and the adjoint of $W(f)$ is an extension of $W(\bar{f})$;
(5) the elements of $\mathcal{D}$ are C^∞-vectors for the representation π, the functions $\tilde{X}$ $(X\in\mathfrak{g})$ defined on $\mathcal{O}$ by $\tilde{X}(\xi)=\langle\xi,X\rangle$ are in $\mathcal{A}$ and $W(i\tilde{X})\,v=d\pi(X)v$ for each $X\in\mathfrak{g}$ and each $v\in\mathcal{D}$.

Adapted Weyl correspondences were obtained in various situations, see the introduction of [6]. In particular, we constructed adapted Weyl correspondences for the principal series representations of a noncompact semisimple Lie group in [3] and [7]. We also obtained adapted Weyl correspondences for the unitary irreducible representations of the semidirect product $V \rtimes K$ of the real vector space V by a Lie group K acting linearly on V in the following situations:

(1) K is a connected noncompact semisimple Lie group and the little group associated with the representation of $V \times K$ is a maximal compact subgroup of K [6];

(2) K is a connected compact semisimple Lie group and the little group is the centralizer of a torus of K [10].

Let us mention that adapted Weyl correspondences have various applications in Harmonic Analysis and Deformation Theory as the construction of covariant star-products on coadjoint orbits [3] and the study of contractions of Lie group unitary representations [13], [5], [8], [9]. Recently, in [11], we have studied a contraction of the principal series of a semisimple Lie group to the unitary irreducible representations of its Cartan motion group by using the deformed Weyl calculus introduced in [3].

The present paper can be considered as a sequel of [6] and [10]. Let G_0 be a connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g}_0$ and let K be a maximal compact subgroup of G_0. Then we have the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus V$ where $\mathfrak{k}$ is the Lie algebra of K and V is an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{g}_0$. The Cartan motion group associated with the pair (G_0, K) is the semidirect product $V \rtimes K$ formed with respect to the adjoint action of K on V.

It is known for a long time that the unitary irreducible representations of $V \rtimes K$ are similar to the principal series representations of G_0 [21]. This has been illustrated by means of contractions of representations in [14] (see also [11]). Here, we exploit this similarity to construct adapted Weyl correspondences for unitary irreducible representations of $V \rtimes K$ as it was done for principal series representations of G_0 in [7]. The method is essentially the same as in [6] and the explicit computations are partially based on those of [7].

More precisely, let $\mathcal{O}$ be a coadjoint orbit of $V \rtimes K$ which is associated with a generic unitary irreducible representation π of $V \rtimes K$. We realize π on a Hilbert space of functions on $\mathbb{R}^n$ where $n = (1/2)\dim \mathcal{O}$ and we compute the corresponding derived representation $d\pi$ (Section 3). We dequantize $d\pi$ by using a combination of the usual Weyl calculus on $\mathbb{R}^{2n}$ and of the Berezin calculus on the little group orbit $\mathcal{O}'$ (Section 4). Then we obtain an explicit symplectomorphism from $\mathbb{R}^{2n} \times \mathcal{O}'$ onto a dense open subset of $\mathcal{O}$ and an adapted Weyl correspondence on $\mathcal{O}$ (Section 5). In the case when G_0 is a complex Lie group, we verify that the adapted Weyl correspondence coincide with that of [10].

2. Preliminaries

In this section, we introduce some general facts on noncompact semisimple Lie groups and Cartan motion groups. Our main references are [16, Chapter VI], [19, Chapter V] and [22].

Let G_0 be a connected noncompact semisimple real Lie group with finite center. Let $\mathfrak{g}_0$ be the Lie algebra of G_0. We denote by β the Killing form of $\mathfrak{g}_0$ defined by $\beta(X,Y) = \mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y)$ for X and Y in $\mathfrak{g}_0$. Let θ be a Cartan involution of $\mathfrak{g}_0$ and let $\mathfrak{g}_0 = \mathfrak{k} \oplus V$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$. Let K be the connected compact (maximal) subgroup of G_0 with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of V and let M be the centralizer of $\mathfrak{a}$ in K. Let $\mathfrak{m}$ denote the Lie algebra of M. Let $\Delta := \Delta(\mathfrak{a}, \mathfrak{g}_0)$ be the set of restricted roots and let

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

be the root space decomposition of $\mathfrak{g}_0$. We fix a Weyl chamber in $\mathfrak{a}$ and we denote by Δ^+ the corresponding set of positive roots. We set $\mathfrak{n} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$ and $\bar{\mathfrak{n}} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_{-\lambda}$. Then $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$. Let A, N and $\bar{N}$ denote the analytic subgroups of G with algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively.

Recall that $\bar{N}MAN$ is a open dense subset of G. We denote by $g = \bar{n}(g)m(g)a(g)n(g)$ the decomposition of $g \in \bar{N}MAN$. Also, recall that we have the Iwasawa decomposition $G = KAN$. We denote by $g = \tilde{k}(g)\tilde{a}(g)\tilde{n}(g)$ the decomposition of $g \in G$.

The Cartan motion group associated with the pair (G_0, K) is the semidirect product $G := V \rtimes K$. The group law of G is given by

$$(v,k).(v',k') = (v + \mathrm{Ad}(k)v', kk')$$

for v, v' in V and $k, k' \in K$. The Lie algebra $\mathfrak{g}$ of G is the space $V \times \mathfrak{k}$ endowed with the Lie bracket

$$[(w,U),(w',U')] = ([U,w']_0 - [U',w]_0, [U,U']_0)$$

where $[\cdot,\cdot]_0$ denotes the Lie bracket of $\mathfrak{g}_0$.

Recall that β is positive defined on V and negative defined on $\mathfrak{k}$ [16, p. 184]. Then, by using β, we can identify V^* with V and $\mathfrak{k}^*$ with $\mathfrak{k}$, hence $\mathfrak{g}^* \simeq V^* \times \mathfrak{k}^*$ with $V \times \mathfrak{k}$. Under this identification, the coadjoint action of G on $\mathfrak{g}^* \simeq V \times \mathfrak{k}$ is given by

$$(v,k) \cdot (w,U) = (\mathrm{Ad}(k)w, \mathrm{Ad}(k)U + [v, \mathrm{Ad}(k)w]_0)$$

for v, w in V, k in K and U in $\mathfrak{k}$. This is a particular case of the general formula for the coadjoint action of a semidirect product, see for instance [22].

Let $p^c_{\mathfrak{k}}$ and p^c_V be the projections of $\mathfrak{g}_0$ on $\mathfrak{k}$ and V associated with the decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus V$. Recall that an element ξ_1 of $\mathfrak{a}$ is said to be regular if $\lambda(\xi_1) \neq 0$ for each $\lambda \in \Delta$. We shall need the following lemma.

Lemma 2.1. *For each regular element ξ_1 of $\mathfrak{a}$, the space* $\operatorname{ad}\xi_1\,(V)$ *is the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$.*

Proof: For each $\lambda \in \Delta^+$, let $E_\lambda \neq 0$ be in $\mathfrak{g}_\lambda$. Note that the space $p^c_{\mathfrak{k}}(\mathfrak{n}) = p^c_{\mathfrak{k}}(\bar{\mathfrak{n}})$ is generated by the elements $E_\lambda + \theta(E_\lambda)$ and hence orthogonal to $\mathfrak{m}$. Now, by applying successively $p^c_{\mathfrak{k}}$ and p^c_V to the decomposition $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} + \bar{\mathfrak{n}}$ we get the decompositions $\mathfrak{k} = \mathfrak{m} + p^c_{\mathfrak{k}}(\mathfrak{n})$ and $V = \mathfrak{a} + p^c_V(\mathfrak{n})$. This shows that $p^c_{\mathfrak{k}}(\mathfrak{n})$ is the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$. On the other hand, since $p^c_V(\mathfrak{n})$ is generated by the elements $E_\lambda - \theta(E_\lambda)$, we see that the space $\operatorname{ad}\xi_1\,(V)$ is generated by the elements

$$\operatorname{ad}\xi_1\,(E_\lambda - \theta(E_\lambda)) = \lambda(\xi_1)(E_\lambda + \theta(E_\lambda))$$

where $\lambda(\xi_1) \neq 0$ for $\lambda \in \Delta$. Hence $\operatorname{ad}\xi_1\,(V) = p^c_{\mathfrak{k}}(\mathfrak{n})$ is the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$. □

The coadjoint orbits of the semidirect product of a Lie group by a vector space were described in [22]. For each $(w,U) \in \mathfrak{g}^* \simeq \mathfrak{g}$, we denote by $O(w,U)$ the orbit of (w,U) under the coadjoint action of G. The following lemma shows that, for almost all (w,U), the orbit $O(w,U)$ is of the form $O(\xi_1,\xi_2)$ with $\xi_1 \in \mathfrak{a}$ and $\xi_2 \in \mathfrak{m}$.

Lemma 2.2. (1) *Let $\mathcal{O}$ be a coadjoint orbit for the coadjoint action of G on $\mathfrak{g}^* \simeq \mathfrak{g}$. Then there exists an element of $\mathcal{O}$ of the form (ξ_1, U) with $\xi_1 \in \mathfrak{a}$. Moreover, if ξ_1 is regular then there exists $\xi_2 \in \mathfrak{m}$ such that $(\xi_1,\xi_2) \in \mathcal{O}$.*
(2) *Let ξ_1 be a regular element of $\mathfrak{a}$. Then M is the stabilizer of ξ_1 in K.*

Proof: (1) Let $(w,U) \in \mathcal{O}$. For each $k \in K$ we have

$$(0,k)\cdot(w,U) = (\operatorname{Ad}(k)w, \operatorname{Ad}(k)U).$$

By [19, p. 120], we have $\operatorname{Ad}(K)\mathfrak{a} = V$ and then one can choose $k \in K$ so that $\operatorname{Ad}(k)w \in \mathfrak{a}$. We set $\xi_1 := \operatorname{Ad}(k)w$. If we assume that ξ_1 is regular then, by Lemma 2.1, we can write $U = \xi_2 + [\xi_1, v]$ where $\xi_2 \in \mathfrak{m}$ and $v \in V$. Then $(\xi_1, U) = (v,e)\cdot(\xi_1,\xi_2)$. Hence $\mathcal{O} = O(\xi_1,\xi_2)$.

(2) By [7, Lemma 4.2], the stabilizer of ξ_1 in $\mathfrak{g}_0$ is MA. Then, the stabilizer of ξ_1 in K is $MA \cap K = M$. □

Let $\xi_1 \in \mathfrak{a}$ be a regular element. Denote by $O_V(\xi_1)$ the orbit of ξ_1 in V under the action of K. In the next section, we shall need the chart on $O_V(\xi_1) \simeq K/M$ which is given by the following lemma.

Lemma 2.3 ([26, Lemma 7.6.8])**.** *The map $\tau : y \to \operatorname{Ad}(\tilde{k}(y))\xi_1$ is a diffeomorphism from $\bar{N}$ onto a dense open subset of $O_V(\xi_1)$. Let us consider the action of $k \in K$ on $y \in \bar{N}$ defined by $\tau(k\cdot y) = \operatorname{Ad}(k)\tau(y)$ or, equivalently, by $k\cdot y = \bar{n}(ky)$. Then the K-invariant measure on $\bar{N}$ is given by $e^{-2\rho(\log \tilde{a}(y))}dy$ where dy is a Haar measure on $\bar{N}$.*

In the rest of the paper, we fix the normalization of dy as follows. Let $(E_i)_{1\leq i\leq n}$ be an orthonormal basis of $\bar{\mathfrak{n}}$ with respect to the scalar product defined by

$(Y,Z) := -\beta(Y,\theta(Z))$. Denote by $(y_1, y_2, \ldots, y_n)$ the coordinates of $Y \in \bar{\mathfrak{n}}$ in this basis and let $dY = dy_1 dy_2 \ldots dy_n$ be the Euclidean measure on $\bar{\mathfrak{n}}$. The exponential map exp is a diffeomorphism from $\bar{\mathfrak{n}}$ onto $\bar{N}$ and we set $dy := (\exp^{-1})^*(dY)$.

We shall also denote by $k \cdot Y$ the action of $k \in K$ on $Y \in \bar{\mathfrak{n}}$ defined by $\exp(k \cdot Y) = k \cdot \exp(Y)$.

3. Representations

We retain the notation of Section 2. Let $\xi_1 \in \mathfrak{a}$ be a regular element and let $\xi_2 \in \mathfrak{m}$. We denote by $o(\xi_2)$ the orbit of ξ_2 under the adjoint action of M on $\mathfrak{m}$. Let σ be a unitary irreducible representation of M on a complex (finite-dimensional) vector space E. In the rest of the paper, we assume that σ is associated with the orbit $o(\xi_2)$ in the following sense, see [27, Section 4]. Given a maximal torus T of M with Lie algebra $\mathfrak{t}$ and a set of positive roots in $\Delta(\mathfrak{t}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}})$, the element $i\beta(\xi_2, \cdot)$ of $i\mathfrak{t}^*$ is the highest weight of σ. Under these assumptions, the orbit $O(\xi_1, \xi_2)$ is associated with the unitarily induced representation

$$\hat{\pi} = \mathrm{Ind}_{V\times M}^{G} \left(e^{i\beta(\xi_1,\cdot)} \otimes \sigma\right)$$

(see [20], [22] and [23]). By a result of Mackey, $\hat{\pi}$ is irreducible since σ is irreducible [24, p. 149]. Moreover, in the terminology of [22] and [23], the group M is called the little group and the orbit $o(\xi_2)$ the little orbit.

The representation $\hat{\pi}$ is usually realized on the space of square-integrable sections of a Hermitian vector bundle over $O_V(\xi_1)$, see [10], [22]. Following [23] and using Lemma 2.3 and the section $y \to \tilde{k}(y)$ from $\bar{N}$ to K, we immediately obtain the realization π_0 of $\hat{\pi}$ defined by

$$(\pi_0(v,k)\psi)(y) = e^{i\beta(\mathrm{Ad}(\tilde{k}(y))\xi_1, v)}\, \sigma\big(\tilde{k}(y)^{-1}k\tilde{k}(k^{-1}\cdot y)\big)\psi(k^{-1}\cdot y)$$

on the Hilbert space $\mathcal{H}_0$ which is the completion of the space of compactly supported smooth functions $\psi : \bar{N} \to E$ with respect to the norm

$$\|\psi\|_0^2 = \int_{\bar{N}} \langle \psi(y),\, \psi(y)\rangle_E\, e^{-2\rho(\log \tilde{a}(y))}\, dy.$$

For the Weyl calculus, it is more convenient to realize $\hat{\pi}$ on the Hilbert space $\mathcal{H} := L^2(\bar{\mathfrak{n}}, E)$ which is the completion of the space $C_0(\bar{\mathfrak{n}}, E)$ of compactly supported smooth functions $\phi : \bar{\mathfrak{n}} \to E$ with respect to the norm

$$\|\phi\|^2 = \int_{\bar{\mathfrak{n}}} \langle \phi(Y),\, \phi(Y)\rangle_E\, dY.$$

To this end, we use the unitary isomorphism B from $\mathcal{H}$ onto $\mathcal{H}_0$ defined by $B(\phi)(\exp Y) = e^{\rho(\log \tilde{a}(y))}\phi(Y)$ and we set $\pi(g) := B^{-1}\pi_0(g)B$ for $g \in G$. We immediately obtain, for $(v,k) \in G$,

$$(\pi(v,k)\phi)(Y) = e^{i\beta(\mathrm{Ad}(\tilde{k}(y))\xi_1, v) + \rho(\log \tilde{a}(k^{-1}\cdot y) - \log \tilde{a}(y))}\, \sigma(m(k,y))\phi(k^{-1}\cdot Y)$$

where we have set $y = \exp Y$ and $m(k,y) := \tilde{k}(y)^{-1}k\tilde{k}(k^{-1}\cdot y) \in M$. This formula can be simplified as follows. Let $k \in K$ and $y \in \bar{N}$. Write $k^{-1}y = \bar{n}(k^{-1}y)m(k^{-1}y)a(k^{-1}y)n(k^{-1}y)$. Then $k^{-1}\tilde{k}(y) = \tilde{k}(\bar{n}(k^{-1}y))m(k^{-1}y)$. Thus $m(k,y) = m(k^{-1}y)^{-1}$. Also, we have that

$$\tilde{a}(y) = \tilde{a}(k^{-1}y) = \tilde{a}(\bar{n}(k^{-1}y))a(k^{-1}y) = \tilde{a}(k^{-1}\cdot y)a(k^{-1}y).$$

This gives

$$\begin{aligned}(\pi(v,k)\phi)(Y) = e^{-\rho(\log a(k^{-1}\exp Y))+i\beta(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1,v)}\,\sigma(m(k^{-1}\exp Y))^{-1}\\ \times\,\phi(\log\bar{n}(k^{-1}\exp Y)).\end{aligned}$$

Now, we compute the derived representation $d\pi$. Let $p_{\mathfrak{a}}$, $p_{\mathfrak{m}}$ and $p_{\bar{\mathfrak{n}}}$ be the projections of $\mathfrak{g}_0$ onto $\mathfrak{a}$, $\mathfrak{m}$ and $\bar{\mathfrak{n}}$ associated with the direct decomposition $\mathfrak{g}_0 = \mathfrak{a}+\mathfrak{m}+\mathfrak{n}+\bar{\mathfrak{n}}$. For $X \in \bar{\mathfrak{n}}$ we denote by X^+ the right invariant vector field on $\bar{N}$ generated by X, that is, $X^+(y) = \frac{d}{dt}(\exp tX)y\big|_{t=0}$ for $y \in \bar{N}$.

Lemma 3.1 ([7]). (1) *For each $X \in \bar{\mathfrak{n}}$ and each $Y \in \bar{\mathfrak{n}}$, we have*

$$d\log(\exp Y)\left(X^+(\exp Y)\right) = \frac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}(X).$$

(2) *For each $X \in \mathfrak{g}_0$ and each $y \in \bar{N}$, we have*

$$\begin{aligned}\frac{d}{dt}a(\exp(tX)y)\Big|_{t=0} &= p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})X)\\ \frac{d}{dt}m(\exp(tX)y)\Big|_{t=0} &= p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})X)\\ \frac{d}{dt}\bar{n}(\exp(tX)y)\Big|_{t=0} &= \left(\mathrm{Ad}(y)\,p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\,X)\right)^+(y).\end{aligned}$$

From this lemma, we immediately deduce the following proposition.

Proposition 3.2. *For each $(w,U) \in \mathfrak{g}$, $\phi \in C_0(\bar{\mathfrak{n}},E)$ and $Y \in \bar{\mathfrak{n}}$, we have*

$$\begin{aligned}(d\pi(w,U)\phi)(Y) &= i\beta\big(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1,w\big)\phi(Y)\\ &+\rho\big(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))U)\big)\phi(Y)+d\sigma\big(p_{\mathfrak{m}}(\mathrm{Ad}(\exp(-Y))U)\big)\phi(Y)\\ &-d\phi(Y)\left(\frac{\mathrm{ad}\,Y}{1-e^{-\mathrm{ad}\,Y}}p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(\exp(-Y))\,U)\right).\end{aligned}$$

4. Dequantization

In this section,we first introduce the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2\times o(\xi_2)$. Recall that the Berezin calculus on $o(\xi_2)$ associates with each operator B on the finite-dimensional complex vector space E a complex-valued function $s(B)$ on the orbit $o(\xi_2)$ called the symbol of the operator B (see [2]). The following properties of the Berezin calculus are well-known, see [12], [3], [10].

Proposition 4.1. (1) *The map $B \to s(B)$ is injective.*

(2) *For each operator B on E, we have $s(B^*) = \overline{s(B)}$.*

(3) *For each operator B on E, each $m \in M$ and each $\varphi \in o(\xi_2)$, we have*

$$s(B)(\mathrm{Ad}(m)\varphi) = s(\sigma(m)^{-1} B \sigma(m))(\varphi).$$

(4) *For $X \in \mathfrak{m}$ and $\varphi \in o(\xi_2)$, we have $s(d\sigma(X))(\varphi) = i\beta(\varphi, X)$.*

In particular, we see that the map s^{-1} is an adapted Weyl transform on $o(\xi_2)$ in the sense of Definition 1.1.

We say that a complex-valued smooth function $f : (Y, Z, \varphi) \to f(Y, Z, \varphi)$ is a symbol on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ if for each $(Y, Z) \in \bar{\mathfrak{n}}^2$ the function $\varphi \to f(Y, Z, \varphi)$ is the symbol in the Berezin calculus on $o(\xi_2)$ of an operator on E denoted by $\hat{f}(Y, Z)$. A symbol f on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ is called an S-symbol if the function $\hat{f}$ belongs to the Schwartz space of rapidly decreasing smooth functions on $\bar{\mathfrak{n}}^2$ with values in $\mathrm{End}(E)$. The Weyl calculus for $\mathrm{End}(E)$-valued functions is a slight refinement of the usual Weyl calculus for complex-valued functions [17], [15]. For each S-symbol on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$, we define the operator $\mathcal{W}(f)$ on the Hilbert space $L^2(\bar{\mathfrak{n}}, E)$ by

$$(\mathcal{W}(f)\phi)(Y) = (2\pi)^{-n} \int_{\bar{\mathfrak{n}}^2} e^{i\langle T, Z\rangle} \hat{f}\left(Y + \frac{1}{2}T, Z\right) \phi(Y+T)\, dT\, dZ \tag{4.1}$$

for $\phi \in C_0(\bar{\mathfrak{n}}, E)$.

It is well-known that the Weyl calculus can be extended to much larger classes of symbols (see for instance [17]). Here we only consider a class of polynomial symbols. We say that a symbol f on $\bar{\mathfrak{n}}^2 \times o(\xi_2)$ is a P-symbol if the function $\hat{f}(Y, Z)$ is polynomial in the variable Z. Let f be the P-symbol defined by $f(Y, Z, \varphi) = u(Y) Z^\alpha$ where $u \in C^\infty(\bar{\mathfrak{n}})$ and $Z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$ for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then, by [25], we have

$$(\mathcal{W}(f)\phi)(Y) = \left(i\frac{\partial}{\partial Z}\right)^\alpha \left(u(Y + \frac{1}{2}Z)\, \phi(Y+Z)\right)\Big|_{Z=0}. \tag{4.2}$$

In particular, if $f(Y, Z, \varphi) = u(Y)$ then $(\mathcal{W}(f)\phi)(Y) = u(Y)\,\phi(Y)$ and if $f(Y, Z, \varphi) = u(Y) z_k$ then

$$(\mathcal{W}(f)\phi)(Y) = i\left(\frac{1}{2}\partial_k u(Y)\, \phi(Y) + u(Y)\partial_k \phi(Y)\right) \tag{4.3}$$

where ∂_k denotes partial derivative with respect to the variable y_k.

We need the following lemma. The trace of an endomorphism u of $\bar{\mathfrak{n}}$ is denoted by $\mathrm{Tr}_{\bar{\mathfrak{n}}}\, u$.

Lemma 4.2. *For $U \in \mathfrak{k}$ let $c_U : \bar{\mathfrak{n}} \to \bar{\mathfrak{n}}$ be the map defined by*

$$c_U(Y) = s(\operatorname{ad} Y) p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(\exp(-Y))\, U)$$

where s is the function defined by $s(z)=\frac{e^z}{1-e^{-z}}$ for $z\neq 0$ and $s(0)=1$. Then we have

$$\mathrm{Tr}_{\bar{\mathfrak{n}}}\, dc_U(Y) = -2\rho\left(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))\,X)\right).$$

PROOF: This is a particular case of [7, Lemma 3.3]. □

Then we get the following proposition.

Proposition 4.3. *For each $(w,U)\in\mathfrak{g}$, the Berezin-Weyl symbol of the operator $-id\pi(w,U)$ is the P-symbol $f_{(w,U)}$ on $\bar{\mathfrak{n}}^2\times o(\xi_2)$ defined by*

$$\begin{aligned} f_{(w,U)}(Y,Z,\varphi) = {} & \beta\big(\mathrm{Ad}(\tilde{k}(\exp Y))\,\xi_1, w\big) + \beta\big(p_{\mathfrak{m}}(\mathrm{Ad}(\exp(-Y))U),\varphi\big) \\ & + (c_U(Y),Z). \end{aligned}$$

PROOF: Set $c_U^k(Y)=(c_U(Y),E_k)$ for each $k=1,2,\ldots,n$. By using (4) of Proposition 4.1 and Formula (4.3), we immediately see that the symbol of $-id\pi(w,U)$ is

$$\begin{aligned} f_{(w,U)}(Y,Z,\varphi) = {} & -i\rho\left(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))U)\right) + \beta\left(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1, w\right) \\ & + \beta\left(p_{\mathfrak{m}}(\mathrm{Ad}(\exp(-Y))U),\varphi\right) + \sum_{k=1}^{n} c_U^k(Y) z_k - \frac{i}{2}\sum_{k=1}^{n}\partial_k c_U^k(Y). \end{aligned}$$

But by Lemma 4.2, we have

$$-\frac{i}{2}\sum_{k=1}^{n}\partial_k c_U^k(Y) = -\frac{i}{2}\,\mathrm{Tr}_{\bar{\mathfrak{n}}}\left(dc_U(Y)\right) = i\rho\left(p_{\mathfrak{a}}(\mathrm{Ad}(\exp(-Y))U)\right).$$

The result follows. □

5. Adapted Weyl correspondence

In this section, we use the dequantization procedure of Section 4 in order to obtain an explicit diffeomorphism from $\bar{\mathfrak{n}}^2\times o(\xi_2)$ onto the dense open subset $\tilde{O}(\xi_1,\xi_2)$ of $O(\xi_1,\xi_2)$ defined by

$$\tilde{O}(\xi_1,\xi_2) = \{(v,k)\cdot(\xi_1,\xi_2) : v\in V,\, k\in K\cap\bar{N}MAN\}$$

and then to construct an adapted Weyl correspondence on $O(\xi_1,\xi_2)$.

Proposition 5.1. *Let Ψ be the map from $\bar{\mathfrak{n}}^2\times o(\xi_2)$ to $\mathfrak{g}$ defined by*

$$\Psi(Y,Z,\varphi) = \Big(\mathrm{Ad}(\tilde{k}(\exp Y))\xi_1\,,\; p_{\mathfrak{k}}^c\Big(\mathrm{Ad}(\exp Y)\big(\varphi + p_{\mathfrak{n}}\big(\tfrac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\theta(Z)\big)\big)\Big)\Big).$$

Then, for each $(w,U)\in\mathfrak{g}$, we have

$$f_{(w,U)}(Y,Z,\phi) = \langle\Psi(Y,Z,\varphi),\,(w,U)\rangle.$$

PROOF: We use Proposition 4.3. Note that we have $\beta(\mathfrak{a}+\mathfrak{m},\mathfrak{n}+\bar{\mathfrak{n}})=(0)$, $\beta(\mathfrak{n},\mathfrak{n})=(0)$ and $\beta(\bar{\mathfrak{n}},\bar{\mathfrak{n}})=(0)$. Then for $(Y,Z,\varphi)\in\bar{\mathfrak{n}}^2\times o(\xi_2)$ and $(w,U)\in\mathfrak{g}$, we can write

$$\begin{aligned}(c_U(Y),Z)&=-\beta(c_U(Y),Z)\\&=-\beta\left(\frac{\operatorname{ad}Y}{1-e^{-\operatorname{ad}Y}}\,p_{\bar{\mathfrak{n}}}(\operatorname{Ad}(\exp(-Y))\,U),\,\theta(Z)\right)\\&=\beta\left(p_{\bar{\mathfrak{n}}}(\operatorname{Ad}(\exp(-Y))\,U),\,\frac{\operatorname{ad}Y}{e^{\operatorname{ad}Y}-1}\,\theta(Z)\right)\\&=\beta\left(\operatorname{Ad}(\exp(-Y))\,U,\,p_{\mathfrak{n}}\Big(\frac{\operatorname{ad}Y}{e^{\operatorname{ad}Y}-1}\,\theta(Z)\Big)\right)\\&=\beta\left(U,\,p_{\mathfrak{k}}^c\Big(\operatorname{Ad}(\exp Y)\,p_{\mathfrak{n}}\Big(\frac{\operatorname{ad}Y}{e^{\operatorname{ad}Y}-1}\,\theta(Z)\Big)\Big)\right).\end{aligned}$$

Similarly, we have

$$\beta(\varphi,p_{\mathfrak{m}}(\operatorname{Ad}(\exp(-Y))U))=\beta(\varphi,\operatorname{Ad}(\exp(-Y))U)=\beta(\operatorname{Ad}(\exp Y)\varphi,U).$$

The result then follows from Proposition 4.3. □

Let ω and ω_0 be the Kirillov 2-forms on $O(\xi_1,\xi_2)$ and $o(\xi_2)$, respectively. We endow $\bar{\mathfrak{n}}^2$ with the symplectic form $dY\wedge dZ:=\sum_{k=1}^n dy_k\wedge dz_k$.

Proposition 5.2. *The map Ψ is a symplectomorphism from the symplectic product $(\bar{\mathfrak{n}}^2\times o(\xi_2),(dY\wedge dZ)\otimes\omega_0)$ onto $(\tilde{O}(\xi_1,\xi_2),\omega|_{\tilde{O}(\xi_1,\xi_2)})$.*

PROOF: The proof is similar to that of Proposition 6.2 in [10]. □

Now, we obtain an adapted Weyl transform on $O(\xi_1,\xi_2)$ by transferring to $O(\xi_1,\xi_2)$ the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2\times o(\xi_2)$. We say that a smooth function f on $O(\xi_1,\xi_2)$ is a symbol on $O(\xi_1,\xi_2)$ (respectively a P-symbol, an S-symbol) if $f\circ\Psi$ is a symbol (respectively a P-symbol, an S-symbol) for the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2\times o(\xi_2)$.

Proposition 5.3. *Let $\mathcal{A}$ be the space of P-symbols on $O(\xi_1,\xi_2)$ and let $\mathcal{B}$ be the space of differential operators on $\bar{\mathfrak{n}}$ with coefficients in $C^\infty(\bar{\mathfrak{n}},E)$. Then the map $W:\mathcal{A}\to\mathcal{B}$ that assigns to each $f\in\mathcal{A}$ the operator $\mathcal{W}(f\circ\Psi)$ on $L^2(\bar{\mathfrak{n}},E)$ is an adapted Weyl correspondence in the sense of Definition* 1.1.

PROOF: Properties (1), (2) and (3) of the definition of an adapted Weyl correspondence are clearly satisfied with $\mathcal{D}=C_0(\bar{\mathfrak{n}},E)$. Property (4) follows from (2) of Proposition 4.1 and from the similar result for the usual Weyl calculus, see [17]. Finally, Property (5) is an immediate consequence of Proposition 4.1. □

Finally, let us consider the case when G_0 is a complex Lie group. In this case, we have $V=i\mathfrak{k}$ and M is the maximal torus $\exp(i\mathfrak{a})$ of K [19, p. 143 and p. 468].

Moreover, $o(\xi_2)$ reduces to the point ξ_2, σ is a character of M and $E = \mathbb{C}$. So, the map $\mathcal{W}$ is just the usual Weyl calculus.

Note that the construction of [10] can also be applied in this case. In [10], we have defined a symplectomorphism Ψ_0 from $\mathfrak{n}^2$ onto $\tilde{O}(\xi_1, \xi_2)$ and an adapted Weyl correspondence W_0 on $O(\xi_1, \xi_2)$. We can easily verify that $\Psi(Y, Z) = \Psi_0(\theta(Y), \theta(Z))$ for each $(Y, Z) \in \bar{\mathfrak{n}} \times \bar{\mathfrak{n}}$ and that the spaces of symbols for W and for W_0 are the same. Moreover, choosing the orthonormal basis for $\bar{\mathfrak{n}}$ in Section 2 and for $\mathfrak{n}$ in [10] in compatible ways, we have that $W_0(f)(\phi \circ \theta) = (W(f)\phi) \circ \theta$ for each S-symbol f on $O(\xi_1, \xi_2)$ and for each $\phi \in C_0(\bar{\mathfrak{n}})$.

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