Linear hyperbolic problems in the whole scale of Sobolev-type spaces of periodic functions

I. Kmit

Abstract. We study one-dimensional linear hyperbolic systems with L^∞-coefficients subjected to periodic conditions in time and reflection boundary conditions in space. We derive a priori estimates and give an operator representation of solutions in the whole scale of Sobolev-type spaces of periodic functions. These spaces give an optimal regularity trade-off for our problem.

Keywords: hyperbolic systems, periodic-Dirichlet problems, anisotropic Sobolev spaces, a priori estimates

Classification: 35L50

1. Introduction

The traveling wave models in laser dynamics [1], [3], [8], [9], [11], [12], [13] describe the dynamical behavior of distributed feedback multisection semiconductor lasers. The models include a couple of dissipative semilinear first-order hyperbolic equations of a single space variable describing the forward and backward propagating complex amplitudes of the light. We investigate a linearized version of this system in the case of small periodic forcing of stationary states (see the discussion in Section 5). Specifically, in the domain $\{(x,t)\,|\,0 < x < 1,\ -\infty < t < \infty\}$ we consider system

$$\begin{aligned} &\partial_t u + \partial_x u + a(x)u + b(x)v = f(x,t) \\ &\partial_t v - \partial_x v + c(x)u + d(x)v = g(x,t), \end{aligned} \tag{1}$$

subjected to periodic conditions

$$\begin{aligned} u(x,t+T) &= u(x,t) \\ v(x,t+T) &= v(x,t) \end{aligned} \tag{2}$$

and reflection boundary conditions

$$\begin{aligned} u(0,t) &= r_0 v(0,t) \\ v(1,t) &= r_1 u(1,t). \end{aligned} \tag{3}$$

The unknown functions u and v and all the data in (1) are complex functions, r_0 and r_1 are complex constants, and $T > 0$. The functions f and g are assumed to be T-periodic in t and $a, b, c, d \in L^\infty(0, 1)$.

We are motivated by the fact that a linearization is a first step in many techniques for local investigation of nonlinear equations. Here the linearization is done in a neighborhood of a stationary solution (the coefficients in (1) depend only on x). Our analysis covers practically interesting cases of discontinuous coefficients and right-hand sides in (1). Discontinuity here corresponds to the fact that different sections of multisection semiconductor lasers have different electrical and optical properties.

Our goal is to investigate uniqueness and solvability questions in spaces providing us with an optimal regularity trade-off (see a preamble of Section 2 for details). Since (1)–(3) allows separation of variables, it is natural to study our problem within the spaces of periodic in t functions using their t-Fourier representations (see, e.g., [10, Chapter 5.10] and [14, Chapter 2.4]). In Section 2 we introduce Sobolev-type spaces of periodic functions of any real index γ including distributions of any desired order of singularity. They serve as the spaces of solutions to (1)–(3) and the spaces of right hand sides of (1). We show that, for all sufficiently large γ, the spaces of solutions are embedded into the algebra L^∞ with pointwise multiplication. This fact makes possible to apply our main results to the aforementioned nonlinear problems of laser dynamics. In Section 3 we derive a priori estimates, thereby proving a uniqueness result. This result is proved under some not too restrictive and quite applicable to problems of laser dynamics conditions imposed on the coefficients of (1). In Section 4, under some smallness assumption on the coefficients b and c, we give an operator representation of solutions.

For the clarity of presentation, we restrict ourselves to the 2×2 hyperbolic system, which is well enough for applications. However, similar results hold true for the $n \times n$ hyperbolic systems (Remark 14).

In [7] we consider the case of $b, c \in BV(0, 1)$ and $a, d \in L^\infty(0, 1)$ and prove Fredholm Alternative in the Sobolev-type spaces of periodic functions continuously embedded into L^∞.

2. Sobolev-type spaces of periodic functions and their properties

We here construct two scales of Banach spaces V^γ (for solutions) and W^γ (for right-hand sides in (1)) with a scale parameter $\gamma \in \mathbb{R}$, consisting of complex valued functions. We will achieve the following properties:

- elements of V^γ satisfy (2) and have traces in x, while elements of W^γ satisfy (2),
- elements of W^γ allow discontinuities in x,
- for any $\gamma \in \mathbb{R}$, the pair (V^γ, W^γ) gives an optimal regularity for (1)–(3). This means that, from one side, for all $(u, v) \in V^\gamma$ the left-hand side

of (1) belongs to W^γ and, from the other side, for all $(f,g) \in W^\gamma$ solutions to (1)–(3) belong to V^γ.

We first introduce Sobolev spaces of T-periodic functions (see, e.g. [4], [5], [10], [14]). Let $S^T = \mathbb{R}/T\mathbb{Z}$. Define the (Banach) space $C^k(S^T)$ of k-times continuously differentiable functions on S^T by

$$C^k(S^T) = \{f : S^T \to \mathbb{C} \mid f \circ q \in C^k(\mathbb{R})\},$$

where q is the quotient map $q : \mathbb{R} \to \mathbb{R}/T\mathbb{Z}$. For the (Freshet) space of smooth functions we hence have

$$C^\infty(S^T) = \bigcap_k C^k(S^T).$$

As a topological vector space this is a projective limit of $C^k(S^T)$ with projections being the natural inclusions. Now the space of distributions on S^T is defined as the ascending union (colimit) of duals of the spaces $C^k(S^T)$:

$$C^\infty(S^T)^* = \bigcup_k C^k(S^T)^* = \operatorname{colim}_{k\to\infty} C^k(S^T)^*.$$

We are now prepared to define Sobolev spaces of periodic functions: Set $\omega = 2\pi/T$ and $\varphi_k(t) = e^{ik\omega t}$ and define

$$H^\gamma(S^T) = \left\{ u \in C^\infty(S^T)^* \,\middle|\, \|u\|^2_{H^\gamma(S^T)} = T^{-1} \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \left| [u, \varphi_{-k}]_{C^\infty(S^T)} \right|^2 < \infty \right\},$$

where $[\cdot,\cdot]_{C^\infty(S^T)} : C^\infty(S^T)^* \times C^\infty(S^T) \to \mathbb{C}$ is the dual pairing.

Given $l \in \mathbb{N}_0$, denote

$$H^{l,\gamma} = H^l(0,1; H^\gamma(S^T)) = \Bigg\{ u(\cdot,t) : (0,1) \to H^\gamma(S^T) \,\Bigg|$$

$$\|u\|^2_{H^{l,\gamma}} = T^{-1} \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \sum_{m=0}^{l} \int_0^1 \left| \frac{d^m}{dx^m} [u(x,\cdot), \varphi_{-k}]_{C^\infty(S^T)} \right|^2 dx < \infty \Bigg\}.$$

Set

$$u_k(x) = T^{-1} [u(x,\cdot), \varphi_{-k}]_{C^\infty(S^T)}, \tag{4}$$

the t-Fourier coefficients of $u \in H^{l,\gamma}$.

Finally, for each $\gamma \in \mathbb{R}$ we define the spaces W^γ and V^γ by

$$W^\gamma = H^{0,\gamma} \times H^{0,\gamma}$$

and

$$V^\gamma = \Big\{(u,v) \in W^\gamma \,|\, (\partial_t u + \partial_x u, \partial_t v - \partial_x v) \in W^\gamma\Big\}.$$

These spaces will be endowed with norms

$$\|(u,v)\|^2_{W^\gamma} = \|u\|^2_{H^{0,\gamma}} + \|v\|^2_{H^{0,\gamma}}$$

and

$$\|(u,v)\|^2_{V^\gamma} = \|(u,v)\|^2_{W^\gamma} + \|(\partial_t u + \partial_x u, \partial_t v - \partial_x v)\|^2_{W^\gamma}.$$

Now we describe some useful properties of the function spaces introduced above.

Lemma 1. *W^γ is a Hilbert space.*

PROOF: It is known that $H^\gamma(S^T)$ is a Hilbert space (see, e.g., [4]). This implies that $H^{0,\gamma}$ is a Hilbert space as well [2]. The lemma follows. $\square$

Lemma 2. *V^γ is a Banach space.*

PROOF: Let $(u^j, v^j)_{j\in\mathbb{N}}$ be a fundamental sequence in V^γ. Then $(u^j, v^j)_{j\in\mathbb{N}}$ and $(\partial_t u^j + \partial_x u^j, \partial_t v^j - \partial_x v^j)_{j\in\mathbb{N}}$ are fundamental sequences in W^γ. Since W^γ is complete (Lemma 1), there exist $(u,v) \in W^\gamma$ and $(\tilde{u}, \tilde{v}) \in W^\gamma$ such that

$$(u^j, v^j) \to (u,v) \quad \text{and} \quad (\partial_t u^j + \partial_x u^j, \partial_t v^j - \partial_x v^j) \to (\tilde{u}, \tilde{v})$$

in W^γ as $j \to \infty$. It remains to show that $\partial_t u + \partial_x u = \tilde{u}$ and $\partial_t v - \partial_x v = \tilde{v}$ in the sense of generalized derivatives. Indeed, take a smooth function $\psi : (0,1) \times (0,T) \to \mathbb{R}$ with compact support. Then

$$[u, (\partial_t + \partial_x)\psi] = \lim_{j\to\infty} \Big[u^j, (\partial_t + \partial_x)\psi\Big] = -\lim_{j\to\infty} \Big[(\partial_t + \partial_x)u^j, \psi\Big] = -\left[\tilde{u}, \psi\right],$$

and similarly with v and $\tilde{v}$. Here $[\cdot,\cdot]$ is the dual pairing on $C_0^\infty((0,1) \times (0,T))$. The lemma follows. $\square$

Define a Euclidian space

$$E^\gamma = \left\{(u_k(x))_{k\in\mathbb{Z}} \,\middle|\, u_k(x) \in L^2(0,1) \;\; \text{for each } k, \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \|u_k\|^2_{L^2(0,1)} < \infty\right\}$$

with inner product

$$\langle (u_k)_k, (w_k)_k \rangle = \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 u_k(x)\overline{w_k(x)}\, dx.$$

Lemma 3. *E^γ is a Hilbert space.*

Proof: Let $(u^j)_{j\in\mathbb{N}}$, where $u^j = (u_k^j)_{k\in\mathbb{Z}}$ be a fundamental sequence in E^γ. This means that for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int_0^1 |u_k^n - u_k^m|^2\,dx < \varepsilon \tag{5}$$

for all $n, m \geq N$. It follows that for all $k \in \mathbb{Z}$ the sequence $(u_k^j)_{j\in\mathbb{N}}$ is fundamental and hence convergent in $L^2(0,1)$ (by completeness of $L^2(0,1)$). Set $u_k(x) = \lim_{j\to\infty} u_k^j(x)$ and $u = (u_k)_{k\in\mathbb{Z}}$. Our aim is to show that $\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int_0^1 |u_k|^2\,dx < \infty$ and $\lim_{j\to\infty} u^j(x) = u(x)$ in E^γ. Indeed, from (5) we have

$$\sum_{|k|\leq M}(1+k^2)^\gamma \int_0^1 |u_k^n - u_k^m|^2\,dx < \varepsilon,$$

the estimate being uniform in $M \in \mathbb{N}$. Fix n and pass the latter sum to the limit as $m \to \infty$. We get

$$\sum_{|k|\leq M}(1+k^2)^\gamma \int_0^1 |u_k^n - u_k|^2\,dx \leq \varepsilon,$$

which is true for any $M \in \mathbb{N}$. This implies that $\lim_{j\to\infty} u^j(x) = u(x)$. Moreover,

$$\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int_0^1 |u_k^n - u_k|^2\,dx \leq \varepsilon.$$

Since the series $\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int_0^1 |u_k^n|^2\,dx$ is convergent for any $n \in \mathbb{N}$, the last inequality implies the convergence of the series $\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int_0^1 |u_k|^2\,dx$. The proof is therewith complete. □

The following lemma is an analog of a result for the Sobolev spaces $H^\gamma(S^T)$ in [5, Section 2, §6].

Lemma 4. *The map $u \to (u_k(x))_{k\in\mathbb{Z}}$ is a Hilbert space isomorphism from $H^{0,\gamma}$ onto E^γ.*

Proof: By the definition of $H^{0,\gamma}$, for any $u \in H^{0,\gamma}$ the sequence $(u_k(x))_{k\in\mathbb{Z}}$ defined by (4) is in E^γ. Hence the injectivity of the map $H^{0,\gamma} \to E^\gamma$ is a straightforward consequence.

If $\gamma \geq 0$, the surjectivity is a simple, well-known fact. Let us prove the surjectivity if $\gamma < 0$. We can consider $H^{0,\gamma}$ as a space of distributions, namely,

$$H^{0,\gamma} = \left\{ u \in L^2(0,1; C^\infty(S^T))^* \,\middle|\, \|u\|_{H^{0,\gamma}} < \infty \right\}.$$

Given $(u_k)_{k\in\mathbb{Z}}$ in E^γ, let us define a distribution $u \in H^{0,\gamma}$ by

$$[u, f]_{L^2(0,1;C^\infty(S^T))} = \sum_{k\in\mathbb{Z}} \int_0^1 u_{-k}(x) f_k(x)\, dx,$$

where $f \in L^2(0,1; C^\infty(S^T))$ and $f_k(x)$ is the k-th Fourier coefficient of f in t. The following estimate is straightforward:

$$\begin{aligned}
&\left| \sum_{k\in\mathbb{Z}} \int_0^1 u_{-k}(x) f_k(x)\, dx \right| \\
&\leq \sum_{k\in\mathbb{Z}} \int_0^1 (1+k^2)^{\gamma/2} |u_{-k}(x)| (1+k^2)^{-\gamma/2} |f_k(x)|\, dx \\
&\leq \left(\sum_{k\in\mathbb{Z}} \int_0^1 (1+k^2)^{\gamma} |u_k(x)|^2\, dx \right)^{1/2} \left(\sum_{k\in\mathbb{Z}} \int_0^1 (1+k^2)^{-\gamma} |f_k(x)|^2\, dx \right)^{1/2} \\
&= \|(u_k)_{k\in\mathbb{Z}}\|_{E^\gamma} \|f\|_{H^{0,-\gamma}}.
\end{aligned}$$

This means that u is a continuous linear functional on $H^{0,-\gamma}$, namely, that $u \in L^2(0,1; H^{-\gamma}(S^T))^*$. Since $C^\infty(S^T)$ is continuously embedded into $H^{-\gamma}(S^T)$, $u \in L^2(0,1; C^\infty(S^T))^*$. The surjectivity of the t-Fourier coefficient map $H^{0,\gamma} \to E^\gamma$ is therewith proved. Thus, this map is a bijection and it is obviously an isomorphism. □

Corollary 5. *For any $u, v \in H^{l,\gamma}$ there exist sequences $(u_k)_{k\in\mathbb{Z}}$, $(v_k)_{k\in\mathbb{Z}}$ in $H^l(0,1)$ given by (4) such that the series*

$$\sum_{k\in\mathbb{Z}} u_k \varphi_k, \qquad \sum_{k\in\mathbb{Z}} v_k \varphi_k \tag{6}$$

converge, respectively, to u and v in $H^{l,\gamma}$. Vice versa, for any sequences $(u_k)_{k\in\mathbb{Z}}$, $(v_k)_{k\in\mathbb{Z}}$ in $H^l(0,1)$ such that $\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \|u_k\|^2_{H^l(0,1)} < \infty$ and $\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \|v_k\|^2_{H^l(0,1)} < \infty$ there exist unique $u, v \in H^{l,\gamma}$ with u_k and v_k being their t-Fourier coefficients.

In what follows, we will identify distributions $u \in H^{l,\gamma}$ and sequences $(u_k(x))_{k\in\mathbb{Z}}$ in $H^l(0,1)$ corresponding to these distributions by Corollary 5.

Lemma 6. *V^γ is continuously embedded into $[H^{1,\gamma-1}]^2$.*

PROOF: Let $(u,v) \in V^\gamma$. Since $(u,v) \in [H^{0,\gamma}]^2$, we have $(\partial_t u, \partial_t v) \in [H^{0,\gamma-1}]^2$. By the definition of V^γ, $(\partial_x u, \partial_x v) \in [H^{0,\gamma-1}]^2$ and hence $(u,v) \in [H^{1,\gamma-1}]^2$. Moreover, we have

$$\begin{aligned}\|(u,v)\|^2_{[H^{1,\gamma-1}]^2} &= \|u\|^2_{H^{0,\gamma-1}} + \|v\|^2_{H^{0,\gamma-1}} + \|\partial_x u\|^2_{H^{0,\gamma-1}} + \|\partial_x v\|^2_{H^{0,\gamma-1}} \\ &\le \|u\|^2_{H^{0,\gamma-1}} + \|v\|^2_{H^{0,\gamma-1}} + \|\partial_t u + \partial_x u\|^2_{H^{0,\gamma-1}} + \|\partial_t v - \partial_x v\|^2_{H^{0,\gamma-1}} \\ &\quad + \|\partial_t u\|^2_{H^{0,\gamma-1}} + \|\partial_t v\|^2_{H^{0,\gamma-1}} \le C\|(u,v)\|_{V^\gamma},\end{aligned}$$

where the constant C does not depend on (u,v). □

Note that Lemmas 2 and 6 for $\gamma \ge 1$ are proved in [7].

Corollary 7. *If $\gamma > 3/2$, then V^γ is continuously embedded into $\big[C([0,1]\times\mathbb{R})\big]^2$.*

PROOF: By Lemma 6, $V^\gamma \hookrightarrow H^1(0,1;H^{\gamma-1}(S^T))$ continuously. The corollary follows from the embedding (see, e.g., [6])

$$H^\gamma(S^T) \hookrightarrow C(\mathbb{R}), \quad \gamma > \frac{1}{2}.$$

□

Corollary 8. *Let $(u,v) \in V^\gamma$. Then for any $x \in [0,1]$ the traces $u(x,\cdot)$ and $v(x,\cdot)$ are distributions in $H^{\gamma-1}(S^T)$ and satisfy the estimate*

$$\|(u(x,\cdot),v(x,\cdot))\|^2_{[H^{\gamma-1}(S^T)]^2} \le C\|(u,v)\|^2_{V^\gamma},$$

where C does not depend on x, u, and v.

PROOF: The corollary follows from the continuous embedding

$$V^\gamma \hookrightarrow H^1(0,1;H^{\gamma-1}(S^T)) \hookrightarrow C(0,1;H^{\gamma-1}(S^T)).$$

□

3. A priori estimates

We here give conditions ensuring the uniqueness of generalized solutions to (1)–(3). We start from the definition of a generalized solution.

Definition 9. A function $(u,v) \in V^\gamma$ is called a generalized solution to the problem (1)–(3) if it satisfies (1) in $H^{0,\gamma}$ and (3) in $H^{\gamma-1}(S^T)$.

To formulate the main result of this section, we will make the following assumption about the coefficients of the differential equations and the reflection coefficients r_0 and r_1: There exist $p, q \in \mathbb{R}$ such that one of the following conditions is fulfilled:

$$
\begin{aligned}
&\operatorname{ess\,inf}\Big[2\operatorname{Re} a - |b|^{2p} - |c|^{2q}\Big] > 0\\
&\operatorname{ess\,inf}\Big[2\operatorname{Re} d - |b|^{2(1-p)} - |c|^{2(1-q)}\Big] > 0
\end{aligned} \tag{7}
$$

or

$$
\begin{aligned}
&\operatorname{ess\,inf}\left[-2\operatorname{Re} a - \left|\frac{b}{r_0}\right|^{2p} - \left|\frac{c}{r_1}\right|^{2q}\right] + 2(1-m)\ln\frac{1}{|r_0 r_1|} > 0\\
&\operatorname{ess\,inf}\left[-2\operatorname{Re} d - \left|\frac{b}{r_0}\right|^{2(1-p)} - \left|\frac{c}{r_1}\right|^{2(1-q)}\right] + 2m\ln\frac{1}{|r_0 r_1|} > 0,
\end{aligned} \tag{8}
$$

for $m = 0$ or $m = 1$.

Theorem 10. *Let $\gamma \in \mathbb{R}$ be a fixed real. Let $a, b, c, d \in L^\infty(0,1)$ and $(f, g) \in W^\gamma$. Assume that $|r_0| < 1$ and $|r_1| < 1$. If at least one of the conditions (7) and (8) (the latter with $m = 0$ or $m = 1$) is true, then every generalized solution to the problem (1)–(3) satisfies the a priory estimate*

$$
\|(u, v)\|_{V^\gamma} \le C\|(f, g)\|_{W^\gamma} \tag{9}
$$

for some constant $C > 0$ not depending on (f, g).

Note that the assumptions $|r_0| < 1$ and $|r_1| < 1$ are caused by physical reasons. The condition (7) (resp. (8)) is fulfilled if $\operatorname{Re} a$ and $\operatorname{Re} d$ are sufficiently large (resp. sufficiently small). These conditions cover, in particular, a wide range of piecewise smooth coefficients. Note that even piecewise constant coefficients are of physical and practical interest.

Proof: Due to the assumptions imposed on the functions f and g, they allow the following series representations:

$$
\sum_{k\in\mathbb{Z}} f_k \varphi_k, \qquad \sum_{k\in\mathbb{Z}} g_k \varphi_k, \tag{10}
$$

where $f_k(x) = T^{-1}\left[f(x,\cdot), \varphi_{-k}\right]_{C^\infty(S^T)}$ and $g_k(x) = T^{-1}\left[g(x,\cdot), \varphi_{-k}\right]_{C^\infty(S^T)}$. Clearly, $f_k, g_k \in L^2(0,1)$,

$$
\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \|f_k(x)\|^2_{L^2(0,1)} < \infty, \quad \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \|g_k(x)\|^2_{L^2(0,1)} < \infty, \tag{11}
$$

and the series (10) converge to f and g in $H^{0,\gamma}$. Assume that (u,v) is a generalized solution to the problem (1)–(3). Represent u and v as series (6). Hence u_k, v_k for each $k \in \mathbb{Z}$ are in $H^1(0,1)$ and satisfy the boundary value problem

$$
\begin{aligned}
u_k' &= f_k(x) - (a(x) + ik\omega)u_k - b(x)v_k \\
v_k' &= -g_k(x) + (d(x) + ik\omega)v_k + c(x)u_k,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
u_k(0) &= r_0 v_k(0) \\
v_k(1) &= r_1 u_k(1).
\end{aligned}
\tag{13}
$$

Our aim is to show that

$$
\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \Big[\|u_k(x)\|^2_{L^2(0,1)} + \|v_k(x)\|^2_{L^2(0,1)} \Big] < \infty, \tag{14}
$$

$$
\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \Big[\|u_k'(x) + ik\omega u_k(x)\|^2_{L^2(0,1)} + \|v_k'(x) - ik\omega v_k(x)\|^2_{L^2(0,1)} \Big] < \infty. \tag{15}
$$

The estimate (15) follows from (11), (14), and (12). It remains to prove (14). We will distinguish two cases.

Case 1. *Condition* (7) *is fulfilled*. Fix $k \in \mathbb{Z}$. Multiplying the equations of the system (12) by $\overline{u}_k$ and $\overline{v}_k$, respectively, and then summing up the resulting equalities with their complex conjugations, we arrive at the system

$$
\begin{aligned}
&\int_0^1 \Big(\overline{u}_k u_k' + u_k \overline{u}_k' \Big)\, dx + 2\int_0^1 \operatorname{Re} a |u_k|^2\, dx \\
&= \int_0^1 \Big(f_k \overline{u}_k + \overline{f}_k u_k \Big)\, dx - \int_0^1 \Big(b\overline{u}_k v_k + \overline{b} u_k \overline{v}_k \Big)\, dx \\
&\int_0^1 \Big(\overline{v}_k v_k' + v_k \overline{v}_k' \Big)\, dx - 2\int_0^1 \operatorname{Re} d |v_k|^2\, dx \\
&= -\int_0^1 \Big(g_k \overline{v}_k + \overline{g}_k v_k \Big)\, dx + \int_0^1 \Big(c\overline{v}_k u_k + \overline{c} v_k \overline{u}_k \Big)\, dx.
\end{aligned}
\tag{16}
$$

Using integration by parts and boundary conditions (13), we get

$$
\begin{aligned}
\int_0^1 \Big(\overline{u}_k u_k' + u_k \overline{u}_k' \Big)\, dx &= |u_k(1)|^2 - |r_0|^2 |v_k(0)|^2, \\
\int_0^1 \Big(\overline{v}_k v_k' + v_k \overline{v}_k' \Big)\, dx &= |r_1|^2 |u_k(1)|^2 - |v_k(0)|^2.
\end{aligned}
$$

Subtraction of the second equality of (16) from the first one yields

$$(17)\quad \begin{aligned} &(1-|r_1|^2)|u_k(1)|^2+(1-|r_0|^2)|v_k(0)|^2+2\int_0^1\Big(\operatorname{Re}a|u_k|^2+\operatorname{Re}d|v_k|^2\Big)\,dx\\ &=\int_0^1\Big(f_k\overline{u}_k+\overline{f}_ku_k\Big)\,dx+\int_0^1\Big(g_k\overline{v}_k+\overline{g}_kv_k\Big)\,dx\\ &\quad-\int_0^1\Big(b\overline{u}_kv_k+\overline{b}u_k\overline{v}_k\Big)\,dx-\int_0^1\Big(c\overline{v}_ku_k+\overline{c}v_k\overline{u}_k\Big)\,dx. \end{aligned}$$

Since $|r_0|<1$ and $|r_1|<1$, the sum of the boundary terms is positive. We will make use of the following simple inequalities: Given $\varepsilon_0>0$, we have

$$\begin{aligned} \left|\int_0^1 f_k\overline{u}_k\,dx\right| &\le \frac{1}{2\varepsilon_0}\int_0^1|f_k|^2\,dx+\frac{\varepsilon_0}{2}\int_0^1|u_k|^2\,dx,\\ \left|\int_0^1 b\overline{u}_kv_k\,dx\right| &\le \int_0^1|b|^p|u_k||b|^{1-p}|v_k|\,dx\\ &\le \frac{1}{2}\|b\|^{2p}_{L^\infty(0,1)}\int_0^1|u_k|^2\,dx+\frac{1}{2}\|b\|^{2(1-p)}_{L^\infty(0,1)}\int_0^1|v_k|^2\,dx. \end{aligned}$$

We estimate all other integrals in the right-hand side of (17) similarly. Finally, by assumption (7), one can choose $\varepsilon_0>0$ so small that

$$(18)\quad \|u_k(x)\|^2_{L^2(0,1)}+\|v_k(x)\|^2_{L^2(0,1)}\le C\Big[\|f_k(x)\|^2_{L^2(0,1)}+\|g_k(x)\|^2_{L^2(0,1)}\Big],$$

where the constant C depends on a,b,c,d, but not on k. Now, (11) and (18) imply (14).

Case 2. *Condition* (8) *is fulfilled.* We first give the proof under the condition (8) with $m=0$. We start from the observation that (u_k,v_k) is a solution to the problem (12)–(13) iff (w_k,v_k), where $e^{x\alpha+(1-x)\beta}w_k=u_k$ and $\alpha,\beta\in\mathbb{R}$ are fixed reals, is a solution to the problem

$$(19)\quad \begin{aligned} w_k'&=e^{-\alpha x-(1-x)\beta}f_k(x)-(a(x)+ik\omega+\alpha-\beta)w_k-e^{-\alpha x-(1-x)\beta}b(x)v_k\\ v_k'&=-g_k(x)+(d(x)+ik\omega)v_k+c(x)w_ke^{\alpha x+(1-x)\beta}, \end{aligned}$$

$$(20)\quad \begin{aligned} e^\beta w_k(0)&=r_0v_k(0)\\ v_k(1)&=r_1e^\alpha w_k(1). \end{aligned}$$

Let us write down an analog of the equality (17) for the problem (19)–(20):

$$
\begin{aligned}
&-\Big(1-|r_1|^2e^{2\alpha}\Big)|w_k(1)|^2-\Big(1-|r_0|^2e^{-2\beta}\Big)|v_k(0)|^2\\
&\quad+2\int_0^1\Big[(-\operatorname{Re}a-\alpha+\beta)|w_k|^2-\operatorname{Re}d|v_k|^2\Big]\,dx\\
&=-\int_0^1 e^{-x\alpha-(1-x)\beta}\Big(f_k\overline{w}_k+\overline{f}_kw_k\Big)\,dx-\int_0^1\Big(g_k\overline{v}_k+\overline{g}_kv_k\Big)\,dx\\
&\quad+\int_0^1 e^{-x\alpha-(1-x)\beta}\Big(b\overline{w}_kv_k+\overline{b}w_k\overline{v}_k\Big)\,dx+\int_0^1 e^{x\alpha+(1-x)\beta}\Big(c\overline{v}_kw_k+\overline{c}v_k\overline{w}_k\Big)\,dx.
\end{aligned}
$$

Fix $\varepsilon_1>0$ and set $\alpha=\ln|r_1|^{-1}+\varepsilon_1$ and $\beta=-\ln|r_0|^{-1}-\varepsilon_1$. This clearly forces

$$-(1-|r_1|^2e^{2\alpha})|w_k(1)|^2-(1-|r_0|^2e^{-2\beta})|v_k(0)|^2>0.$$

On the account of the assumption (8), similarly to Case 1, $\varepsilon_1>0$ can be chosen so small that estimate (18) with u_k replaced by w_k is true for some C independent of k. Replacing w_k with $e^{-x\alpha-(1-x)\beta}u_k$, we arrive at the estimate (18).

The estimate (18) under the condition (8) with $m=1$ can be obtained in much the same way, the only difference being in considering the problem (12)–(13) with $v_k=e^{x\alpha+(1-x)\beta}w_k$.

The estimate (14) is proved. This finishes the proof of the theorem. □

The following corollary is straightforward.

Corollary 11. *Under the conditions of Theorem* 10 *a generalized solution to the problem* (1)–(3) (*if such exists*) *is unique.*

4. Operator representation of generalized solutions

In this section, under some smallness assumption on the coefficients b and c, we give an explicit formula (an operator representation) of a generalized solution.

Set

$$V^\gamma(r_0,r_1)=\Big\{(u,v)\in V^\gamma\,\Big|\,u(0,\cdot)=r_0v(0,\cdot),\ v(1,\cdot)=r_1u(1,\cdot)\Big\},$$

where the traces $u(0,\cdot),u(1,\cdot),v(0,\cdot)$, and $v(1,\cdot)$ are interpreted as distributions in $H^{\gamma-1}(S^T)$ according to Corollary 8. Given $a,b,c,d\in L^\infty$, let us introduce linear operators $A\in\mathcal{L}(V^\gamma(r_0,r_1);W^\gamma)$ by

$$A\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}\partial_tu+\partial_xu+au\\\partial_tv-\partial_xv+dv\end{bmatrix}$$

and $B\in\mathcal{L}(W^\gamma)$ by

$$B\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}bv\\cu\end{bmatrix}.$$

Lemma 12. *Assume that $a, b, c, d \in L^\infty(0,1)$ and*

$$|r_0 r_1| \neq \exp \int_0^1 (\operatorname{Re} a + \operatorname{Re} d)\, dx. \tag{21}$$

Then the operator A is an isomorphism from $V^\gamma(r_0, r_1)$ onto W^γ.

In the case $\gamma \geq 1$ this lemma is proved in [7].

Proof: Fix an arbitrary $(f, g) \in W^\gamma$. The functions f and g are represented by the series (10) with coefficients $f_k(x) \in L^2(0,1)$ and $g_k(x) \in L^2(0,1)$ satisfying (11). We have to show that there exists exactly one $(u, v) \in V^\gamma(r_0, r_1)$ such that

$$\partial_t u + \partial_x u + au = f, \quad \partial_t v - \partial_x v + du = g.$$

Representing u and v as the series (6), we have to show that there exists exactly one pair of sequences $(u_k)_{k\in\mathbb{Z}}$ and $(v_k)_{k\in\mathbb{Z}}$ with $u_k, v_k \in H^1(0,1)$ satisfying (13), (14), (15), and

$$\begin{aligned} u_k' + (a(x) + ik\omega) u_k &= f_k(x) \\ v_k' - (d(x) + ik\omega) v_k &= -g_k(x). \end{aligned} \tag{22}$$

To simplify the formulae below, let us introduce the following notation:

$$\begin{gathered} \alpha(x) = \int_0^x a(y)\, dy, \quad \delta(x) = \int_0^x d(y)\, dy, \\ \Delta_k = e^{ik\omega + \delta(1)} - r_0 r_1 e^{-ik\omega - \alpha(1)}. \end{gathered}$$

By a straightforward calculation, the boundary value problem (22), (13) has a unique solution $(u_k, v_k) \in [H^1(0,1)]^2$, and this solution is explicitly given by

$$\begin{aligned} u_k(x) &= e^{-ik\omega x - \alpha(x)} \left(\int_0^x e^{ik\omega y + \alpha(y)} f_k(y)\, dy + \frac{r_0}{\Delta_k} w_k(f_k, g_k) \right), \\ v_k(x) &= e^{ik\omega x + \delta(x)} \left(\int_0^x e^{-ik\omega y - \delta(y)} g_k(y)\, dy + \frac{1}{\Delta_k} w_k(f_k, g_k) \right) \end{aligned} \tag{23}$$

with

$$\begin{aligned} &w_k(f_k, g_k) = \\ &\quad = r_1 e^{-ik\omega - \alpha(1)} \int_0^1 e^{ik\omega y + \alpha(y)} f_k(y)\, dy - e^{ik\omega + \delta(1)} \int_0^1 e^{-ik\omega y - \delta(y)} g_k(y)\, dy. \end{aligned}$$

Here we used assumption (21), which implies

$$|\Delta_k| \geq \left| e^{\delta(1)} - |r_0 r_1| e^{-\alpha(1)} \right| > 0 \tag{24}$$

for all $k \in \mathbb{Z}$. From (23) and (24) it follows that

$$|u_k(x)| + |v_k(x)| \leq C\Big(\|f_k\|_{L^2(0,1)} + \|g_k\|_{L^2(0,1)}\Big) \tag{25}$$

for all $x \in [0,1]$, where the constant C does not depend on k, f_k, g_k, and x. Finally, (11) and (25) imply (14). Noting that the estimate (15) follows from (11), (14), and (22), we finish the proof. □

Theorem 13. *Let γ be a real, $a,b,c,d \in L^\infty(0,1)$, and $(f,g) \in W^\gamma$. Assume that the condition* (21) *is true. Suppose also that*

$$\Big[1 + \exp\{3\|a\|_{L^\infty} + 3\|d\|_{L^\infty}\}\Big(1 + (1+|r_0|)(1+|r_1|)\Big|e^{\delta(1)} - |r_0 r_1| e^{-\alpha(1)}\Big|^{-1}\Big) \times (1 + \|a\|_{L^\infty} + \|d\|_{L^\infty})\Big]\Big(\|b\|_{L^\infty} + \|c\|_{L^\infty}\Big) < 1. \tag{26}$$

Then the problem (1)–(3) *has a unique generalized solution given by the formula*

$$(u,v) = \sum_{n=0}^{\infty}(-A^{-1}B)^n A^{-1}(f,g). \tag{27}$$

Note that the conditions (7) and (8) do not cover the condition (26), nor vice versa. Indeed, given a, d, r_0, and r_1, the inequality (26) is true for all sufficiently small b and c, which is not so for (7) and (8). On the other hand, given b and c, the inequality (7) (resp. (8)) is true for all sufficiently large (resp. sufficiently small) a and d, which is not so for (26).

Proof: By Lemma 12, the problem (1)–(3) is equivalent to

$$(u,v) = -A^{-1}B(u,v) + A^{-1}(f,g), \tag{28}$$

where A^{-1} is defined by means of (23). Since $\|A^{-1}B\|_{\mathcal{L}(V^\gamma;V^\gamma)}$ is bounded from above by the left-hand side of (26), $\|A^{-1}B\|_{\mathcal{L}(V^\gamma;V^\gamma)} < 1$. Since V^γ is a Banach space for any $\gamma \in \mathbb{R}$ (Lemma 2), application of the Banach fixed point theorem to the equation (28) gives the unique solvability of the latter. Hence (1)–(3) has a unique generalized solution. Iteration of (28) now gives the desired formula (27). □

Remark 14. The results of Sections 3 and 4 (Theorems 10 and 13) can be easily generalized to $n \times n$ hyperbolic systems, namely, to the problems of the following kind:

$$\begin{aligned}
\partial_t u_j + \lambda_j(x)\partial_x u_j + \sum_{k=1}^n a_{jk}(x)u_k = f_j(x,t), \quad j=1,\dots,n, \quad 0<x<1, \quad t\in\mathbb{R}\\
u_j(x,t+T) = u_j(x,t), \qquad j=1,\dots,n, \quad 0<x<1, \quad t\in\mathbb{R}\\
u_j(0,t) = \sum_{k=m+1}^n r^0_{jk}u_k(0,t), \qquad j=1,\dots,m, \quad t\in\mathbb{R}\\
u_j(1,t) = \sum_{k=1}^m r^1_{jk}u_k(1,t), \qquad j=m+1,\dots,n, \quad t\in\mathbb{R}.
\end{aligned}$$

5. Concluding remarks

Let us turn back to the area of laser dynamics that is a motivation of this paper. Our overall goal here, which remains a subject of future research, is to obtain a local existence and uniqueness result for semilinear hyperbolic systems with small periodic forcing

$$\begin{aligned}
\partial_t u + \partial_x u + g_1(x,u,v,\lambda) = \varepsilon f_1(x,t,u,v,\lambda,\varepsilon)\\
\partial_t v - \partial_x v + g_2(x,u,v,\lambda) = \varepsilon f_2(x,t,u,v,\lambda,\varepsilon).
\end{aligned} \tag{29}$$

Let us choose a solution space to be V^γ with $\gamma > 3/2$, which is advantageous due to Corollary 7. A natural way to achieve our goal can now consist in application of the Implicit Function Theorem to (29), (2), (3). For this purpose we would need to have, first, the isomorphism property of a linearization of (29), second, the C^1-smoothness property of the Nemitsky composition operators defined by the nonlinearities of (29).

Explicit sufficient conditions for the former property are provided by the results of [7] and Theorem 10 of this paper (note that the explicitness here is really important from the point of view of applications). More specifically, assume that $(u,v) = (0,0)$ is a (stationary) solution to (29), (2), (3) and linearize this system in a neighborhood of the stationary solution. Putting $\varepsilon = 0$ and $\lambda = 0$, we arrive at the system (1) with zero right hand side. Denote the operator corresponding to this system by F. In [7] we proved that, if the condition (21) is fulfilled, then F is a Fredholm operator from V^γ onto W^γ. Theorem 10 states constructive conditions ensuring the injectivity of F. Since any Fredholm operator is an isomorphism between two Banach spaces iff it is injective, we therefore have the desired isomorphism property for F under rather wide explicit conditions on the data of (29). Note that Theorem 13 ensures the isomorphism property for a range of data, in which we cannot use the Fredholmness result from [7].

Acknowledgments. I thank Lutz Recke for helpful discussions.

References

[1] Bandelow U., Recke L., Sandstede B., *Frequency regions for forced locking of self-pulsating multi-section DFB lasers*, Optics Comm. **147** (1998), 212–218.

[2] Bourbaki N., *Integration, Chapters 1/4*, Actualités Scientifiques et Industrielles, Hermann, Paris, 1966.

[3] Jochmann F., Recke L., *Well-posedness of an initial boundary value problem from laser dynamics*, Math. Models Methods Appl. Sci. **12** (2002), 593–606.

[4] Garrett P., *Functions on circles*, 2006; Eprint: `www.math.umn.edu/~garrett/m/mfms/notes/09_sobolev.ps`.

[5] Gorbachuk V.I., Gorbachuk M.L., *Boundary value problems for operator differential equations*, Naukova Dumka, Kiev, 1984; English translation: Kluwer Academic Publishers, Dordrecht, 1991.

[6] Herrmann L., *Periodic solutions of abstract differential equations: the Fourier method*, Czechoslovak Math. J. **30** (105) (1980), 177–206.

[7] Kmit I., Recke L., *Fredholm alternative for periodic-Dirichlet problems for linear hyperbolic systems*, J. Math. Anal. Appl. **335** (2007), 355–370.

[8] Recke L., Schneider K.R., Strygin V.V., *Spectral properties of coupled wave equations*, Z. Angew. Math. Phys. **50** (1999), no. 6, 925–933.

[9] Rehberg J., Wünsche H.-J., Bandelow U., Wenzel H., *Spectral properties of a system describing fast pulsating DFB lasers*, Z. Angew. Math. Mech. **77** (1997), 75–77.

[10] Robinson J.C., *Infinite-Dimensional Dynamical Systems*, Cambridge Texts in Appl. Math., Cambridge University Press, Cambridge, 2001.

[11] Sieber J., *Numerical bifurcation analysis for multi-section semiconductor lasers*, SIAM J. Appl. Dyn. Syst. **1** (2002), 248–270.

[12] Sieber J., Recke L., Schneider K., *Dynamics of multisection semiconductor lasers*, J. Math. Sci. (New York) **124** (2004), no. 5, 5298–5309.

[13] Tromborg B., Lassen H.E., Olesen H., *Traveling wave analysis of semiconductor lasers*, IEEE J. of Quant. El. **30** (1994), no. 5, 939–956.

[14] Vejvoda O. et al., *Partial Differential Equations: Time-Periodic Solutions*, Sijthoff Noordhoff, 1981.

Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, Naukova St. 3b, 79060 Lviv, Ukraine

E-mail: kmit@informatik.hu-berlin.de

(*Received* April 1, 2007, *revised* July 29, 2007)