Characterizations of L^1-predual spaces by centerable subsets

Yanzheng Duan, Bor-Luh Lin

Abstract. In this note, we prove that a real or complex Banach space X is an L^1-predual space if and only if every four-point subset of X is centerable. The real case sharpens Rao's result in [*Chebyshev centers and centerable sets*, Proc. Amer. Math. Soc. **130** (2002), no. 9, 2593–2598] and the complex case is closely related to the characterizations of L^1-predual spaces by Lima [*Complex Banach spaces whose duals are L_1-spaces*, Israel J. Math. **24** (1976), no. 1, 59–72].

Keywords: Chebyshev radius, centerable subsets and L^1-predual spaces

Classification: Primary 41A65; Secondary 46B20

1. Introduction

Let X be a Banach space. For $a \in X$, $r > 0$, let $B[a, r] = \{x \in X : \|a-x\| \leq r\}$ and let $S(X) = \{x \in X : \|x\| = 1\}$. If A is a bounded subset of X, let

$$r(A, x) = \sup_{a \in A} \|x - a\|$$

and let

$$r(A) = \inf_{x \in X} r(A, x)$$

denote the Chebyshev radius of A. Let

$$\delta(A) = \sup\{\|a - b\| : a, b \in A\}$$

denote the diameter of A. Then $\delta(A) \leq 2r(A)$ for every bounded subset A of X.

Definition 1.1 ([4]). Let X be a Banach space and A a bounded subset of X. If $\delta(A) = 2r(A)$, then A is said to be centerable.

Research of the second author partially supported by NSF/OISE-03-52523.

Definition 1.2 ([6])**.** A Banach space X whose dual X^* is isometrically isomorphic to $L^1(\mu)$ for some positive measure μ is called an L^1-predual space.

L^1-predual spaces are also called Lindenstrauss spaces. In [9], Lindenstrauss gave several characterizations of real L^1-predual spaces using intersection properties of balls. In [7], Lima gave several characterizations of complex L^1-predual spaces using intersection properties of balls.

A Banach space X is called a $\mathcal{P}_1$ space if X is norm-one complemented in every Banach space Z containing X. Let X be a real or complex Banach space. By Theorem 6.1 of [9] and Theorem 4.1 of [7], X is an L^1-predual space if and only if X^{**} is a $\mathcal{P}_1$ space. By [4, p. 193], every bounded subset of a $\mathcal{P}_1$ space is centerable. In 1977, W.J. Davis [2] proved that the converse is true.

In 2002, Rao [10] proved that a real Banach space X is an L^1-predual space if and only if every finite subset of X is centerable. In this note, we prove that if every four-point subset of a real or complex Banach space X is centerable, then X is an L^1-predual space. The result for the real case sharpens Rao's result in [10] and our proof is different from Rao's. The result for the complex case is a new form of characterizations of L^1-predual spaces. We also point out that it cannot be sharpened anymore, i.e., that every three-point subset of a real or complex Banach space X is centerable does not imply that X is an L^1-predual space.

2. Main results

We first give a characterization of n-point subsets of Banach spaces to be centerable.

Proposition 2.1. *Let X be a real or complex Banach space and $n \geq 3$ be an integer. Then every n-point subset of X is centerable if and only if for every $r > 0$ and every family of pairwise intersecting closed balls $\{B[a_i, r]\}_{i=1}^n$ in X, $\bigcap_{i=1}^n B[a_i, r+\epsilon] \neq \emptyset$ for all $\varepsilon > 0$.*

Proof: $\Rightarrow$. Let $A = \{a_1, a_2, \dots, a_n\}$. Since $\{B[a_i, r]\}_{i=1}^n$ are pairwise intersecting, $\|a_i - a_j\| \leq 2r$ for all i, j. Hence, $2r(A) = \delta(A) \leq 2r$. So $r(A) \leq r$. Therefore, for any $\varepsilon > 0$, there exists $x_0 \in X$ such that $r(A, x_0) \leq r + \varepsilon$, which implies that $x_0 \in \bigcap_{i=1}^n B[a_i, r+\varepsilon]$.

$\Leftarrow$. For any n-point subset $A = \{a_1, a_2, \dots, a_n\}$ of X, to prove that $\delta(A) = 2r(A)$, it suffices to prove that $\delta(A) \geq 2r(A)$. In fact, let $r = \frac{1}{2}\delta(A)$. Then the family $\{B[a_i, r]\}_{i=1}^n$ are pairwise intersecting. Hence, for any $\varepsilon > 0$, $\bigcap_{i=1}^n B[a_i, r+\varepsilon] \neq \emptyset$. Let $x_0 \in \bigcap_{i=1}^n B[a_i, r+\varepsilon]$, then $\|a_i - x_0\| \leq r + \varepsilon$, $i = 1, 2, \dots, n$, i.e., $r(A, x_0) \leq r + \varepsilon$. Hence, $r(A) = \inf_{x \in X} r(A, x) \leq r = \frac{1}{2}\delta(A)$. $\square$

Next theorem is due to Lindenstrauss [9].

Theorem 2.2 ([9])**.** *Let X be a real Banach space and $n \geq 3$ an integer. Then the following statements are equivalent.*

(1) *For every $r > 0$ and every family of pairwise intersecting closed balls*

$\{B[a_i, r]\}_{i=1}^n$ *in* X, $\bigcap_{i=1}^n B[a_i, r+\varepsilon] \neq \emptyset$ *for all* $\varepsilon > 0$.

(2) *For every* $r > 0$ *and every family of pairwise intersecting closed balls* $\{B[a_i, r]\}_{i=1}^n$ *in* X, $\bigcap_{i=1}^n B[a_i, r] \neq \emptyset$.

(3) *For every family of pairwise intersecting closed balls* $\{B[a_i, r_i]\}_{i=1}^n$ *in* X, $\bigcap_{i=1}^n B[a_i, r_i] \neq \emptyset$.

(4) *For every family of pairwise intersecting closed balls* $\{B[a_i, r_i]\}_{i=1}^n$ *in* X, $\bigcap_{i=1}^n B[a_i, r_i+\varepsilon] \neq \emptyset$ *for all* $\varepsilon > 0$.

Combining Proposition 2.1 and Theorem 2.2, we have the following theorem.

Theorem 2.3. *Let* X *be a real Banach space and* $n \geq 3$ *an integer. Then every* n*-point subset of* X *is centerable if and only if one of the four conditions in Theorem 2.2 holds.*

The following theorem is a special case of Lindenstrauss's Theorem 4.1 in [9].

Theorem 2.4 ([9])**.** *Let* X *be a real Banach space and* $n \geq 3$ *an integer. Then the following statements are equivalent.*

(1) *For every family of pairwise intersecting closed balls* $\{B[a_i, r_i]\}_{i=1}^4$ *in* X, $\bigcap_{i=1}^4 B[a_i, r_i] \neq \emptyset$.

(2) *For every family of pairwise intersecting closed balls* $\{B[a_i, r_i]\}_{i=1}^n$ *in* X, $\bigcap_{i=1}^n B[a_i, r_i] \neq \emptyset$.

Following Theorem 2.3 and Theorem 2.4, we have the following.

Theorem 2.5. *Let* X *be a real Banach space. Then every four-point subset of* X *is centerable if and only if every finite subset of* X *is centerable.*

In [1], P. Bandyopadhyay and T.S.S.R.K. Rao proved the following result.

Theorem 2.6 ([1, Theorem 3.9])**.** *Let* X *be a real or complex* L^1*-predual space. Then any compact subset* A *of* X *is centerable.*

Now we are ready to give characterizations of real L^1-predual spaces by centerable subsets.

Theorem 2.7. *Let* X *be a real Banach space. Then the following statements are equivalent.*

(1) X *is an* L^1*-predual space.*
(2) *Every four-point subset of* X *is centerable.*
(3) *Every finite subset of* X *is centerable.*
(4) *Every compact subset of* X *is centerable.*

Proof: (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Trivial. (2) $\Rightarrow$ (1). Following Theorem 2.3, Theorem 2.5 and Theorem 6.1 in [9]. (1) $\Rightarrow$ (4). Following Theorem 2.6. $\square$

In order to give a similar characterization of complex L^1-predual spaces by centerable subsets, we need Lima's results [7], [8] about characterizations of complex L^1-predual spaces.

Definition 2.8 ([5]). A family of closed balls $\{B[a_i, r_i]\}_{i\in I}$ in a complex (real) Banach space X is said to have the weak intersection property if for any $f \in S(X^*)$, $\bigcap_{i\in I} B[f(a_i), r_i] \neq \emptyset$ in $\mathbb{C}(\mathbb{R})$.

Definition 2.9 ([7]). Let $n \geq 3$ be an integer. We say that a real or complex Banach space X is an $E(n)$-space if for every family $\{B[a_i, r_i]\}_{i=1}^n$ of n closed balls in X with the weak intersection property, $\bigcap_{i=1}^n B[a_i, r_i] \neq \emptyset$.

Theorem 2.10 ([8, Corollary 2.5]). *Let X be a real or complex Banach space and let $n \geq 3$ be an integer. Then the following statements are equivalent.*

(1) *For every $r > 0$ and every family of n closed balls $\{B[a_i, r]\}_{i=1}^n$ in X such that any three of them have nonempty intersection, $\bigcap_{i=1}^n B[a_i, r+\varepsilon] \neq \emptyset$ for all $\varepsilon > 0$.*
(2) *For every family of n closed balls $\{B[a_i, r_i]\}_{i=1}^n$ in X such that any three of them have nonempty intersection, $\bigcap_{i=1}^n B[a_i, r_i+\varepsilon] \neq \emptyset$ for all $\varepsilon > 0$.*

Theorem 2.11 ([7, Corollary 4.3]). *Let X be a complex Banach space. If for every family of 4 closed balls $\{B[a_i, r_i]\}_{i=1}^4$ in X such that any three of them have nonempty intersection, $\bigcap_{i=1}^4 B[a_i, r_i+\varepsilon] \neq \emptyset$ for all $\varepsilon > 0$, then X is an $E(n)$-space for all $n \geq 3$.*

Theorem 2.12 ([7, Theorem 4.1]). *Let X be a complex Banach space. Then the following statements are equivalent.*

(1) *X is an L^1-predual space.*
(2) *X is an $E(n)$-space for all $n \geq 3$.*

Theorem 2.13. *Let X be a complex Banach space. Then the following statements are equivalent.*

(1) *X is an L^1-predual space.*
(2) *Every four-point subset of X is centerable.*
(3) *Every finite subset of X is centerable.*
(4) *Every compact subset of X is centerable.*

Proof: (1)$\Rightarrow$(4). Following Theorem 2.6. (4)$\Rightarrow$(3)$\Rightarrow$(2). Trivial.

(2)$\Rightarrow$(1). Let $r > 0$ and let $\{B[a_i, r]\}_{i=1}^4$ be a family of four closed balls such that any three of them intersect. Since $\{B[a_i, r]\}_{i=1}^4$ is pairwise intersecting and $\{a_1, a_2, a_3, a_4\}$ is centerable, by Theorem 2.1, $\bigcap_{i=1}^4 B[a_i, r+\varepsilon] \neq \emptyset$ for all $\varepsilon > 0$. Hence, by Theorem 2.10, for every family of 4 closed balls $\{B[a_i, r_i]\}_{i=1}^4$ in X such that any three of them have nonempty intersection, $\bigcap_{i=1}^4 B[a_i, r_i+\varepsilon] \neq \emptyset$ for all $\varepsilon > 0$. Following Theorem 2.11, X is an $E(n)$-space for all $n \geq 3$. Therefore by Theorem 2.12, X is an L^1-predual space. $\square$

Remark 2.14. Let us show that centerability of all three-point subsets of X does not imply that X is an L^1-predual space. Consider the real or complex space ℓ_1. Since every three pairwise intersecting closed balls in $\mathbb{R}$ or $\mathbb{C}$ intersect (see [3, p. 65]), the same holds also for ℓ_1 by Theorem 4.6(c) in [9]. Hence, by Theorem 2.3, every three-point set in ℓ_1 is centerable. On the other hand, ℓ_1 is not an L^1-predual space since $\ell_1^* = \ell_\infty$.

Acknowledgment. The authors are very grateful to the referee for reading the paper very thoroughly and giving many useful suggestions.

References

[1] Bandyopadhyay P., Rao T.S.S.R.K., *Central subspaces of Banach spaces*, J. Approx. Theory **103** (2000), 206–222.

[2] Davis W.J., *A characterization of $\mathcal{P}_1$ spaces*, J. Approx. Theory **21** (1977), 315–318.

[3] Hanner O., *Intersections of translates of convex bodies*, Math. Scand. **4** (1956), 65–87.

[4] Holmes R.B., *A course on optimization and best approximation*, Lecture Notes in Math., Vol. 257, Springer, Berlin, 1972.

[5] Hustad O., *Intersection properties of balls in complex Banach spaces whose dual are L^1-spaces*, Acta Math. **132** (1974), 282–313.

[6] Lacey H.E., *The isometric theory of classical Banach spaces*, Grundlehren Math. Wiss., Band 208, Springer, New York-Heidelberg, 1974.

[7] Lima A., *Complex Banach spaces whose duals are L_1-spaces*, Israel J. Math. **24** (1976), no. 1, 59–72.

[8] Lima A., *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1977), 1–62.

[9] Lindenstrauss J., *Extension of compact operators*, Mem. Amer. Math. Soc., Vol. 48, Provindence, 1964.

[10] Rao T.S.S.R.K., *Chebyshev centers and centerable sets*, Proc. Amer. Math. Soc. **130** (2002), no. 9, 2593–2598.

Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, U.S.A.

E-mail: yduan@math.uiowa.edu
bllin@math.uiowa.edu

(*Received* July 6, 2006, *revised* October 24, 2006)