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Add(U) of a uniserial module

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Abstract. A module is called uniserial if it has totally ordered submodules in inclusion. We describe direct summands of $U^{(I)}$ for a uniserial module U. It appears that any such a summand is isomorphic to a direct sum of copies of at most two uniserial modules.

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1. Introduction

The aim of this paper is to give a classification of objects in Add(U), where U is a uniserial module over an arbitrary associative ring. Recall that a module U is said to be *uniserial* if the lattice of its submodules is a chain. Direct sums of uniserial modules are called *serial*. If a uniserial module U has local endomorphism ring, then any object in Add(U) is isomorphic to $U^{(I)}$ for a suitable set I because any uniserial module is $\sigma-$small and we can use [3, Theorem 2.52]. In general, the situation is a bit worse but still easy enough to understand. Recall that a module K is *quasi-small* if for any family $\{M_i \mid i \in I\}$ of modules such that K is isomorphic to a direct summand of $\oplus_{i\in I} M_i$ there exists a finite set $I' \subseteq I$ such that K is isomorphic to a direct summand of $\oplus_{i\in I'} M_i$. It is possible to prove that a uniserial module U is not quasi-small if and only if it is isomorphic to a non-zero direct summand of $V^{(\omega)}$, where V is a uniserial module not isomorphic to U.

Before we formulate the main result of the paper, we summarize several results of [4, Section 2] we shall use in the sequel. If U and V are uniserial modules, we say that U, V are of the same *monogeny* (*epigeny*) class if there are monomorphisms (epimorphisms) $f{:}\, U \to V$ and $g{:}\, V \to U$. In this case we write $[U]_m = [V]_m$ ($[U]_e = [V]_e$). We can get some information about monogeny and epigeny classes of U from the lattice of submodules of U. Let S be the set of all monomorphisms in $\mathrm{End}_R(U)$ and let T be the set of all epimorphisms in $\mathrm{End}_R(U)$. We define $U_m = \bigcap_{f\in S} \operatorname{Im} f$ and $U_e = \sum_{f\in T} \operatorname{Ker} f$. Then U_m, U_e are fully invariant submodules of U, $[V]_m = [U]_m$ if and only if V is isomorphic to a submodule of U properly containing U_m or $U \simeq V$, $[V]_e = [U]_e$ if and only if V is isomorphic to U/U',

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where $U' = 0$ or $U' \subsetneq U_e$. If U does not have local endomorphism ring, then $0 \neq U_e$ and $U_m \subsetneq U$. Further, a uniserial module U is not quasi-small if and only if $U_m \subsetneq U_e = U$ and U is countably generated. If $U_e \subseteq U_m$, then any module of the same monogeny class as U is quasi-small. On the other hand if $U_m \subsetneq U_e$, there is unique module V up to isomorphism such that $[V]_m = [U]_m$ and V is not quasi-small. Moreover, for any $u \in U_e$ there exists a submodule $U' \subseteq U_e$ such that $U' \simeq V$ and $u \in U'$.

Now we can formulate the main result of the paper:

Theorem 1.1. *Let U be a non-zero uniserial right module over a ring R. Then*

(i) *if for any monomorphism $f: U \to U$ and any epimorphism $g: U \to U$, the homomorphism gf is not zero, then any object in $\operatorname{Add}(U)$ is isomorphic to $U^{(I)}$ for a suitable set I;*

(ii) *if U is quasi-small and there is a monomorphism $f: U \to U$ and an epimorphism $g: U \to U$ such that $gf = 0$, then any object of $\operatorname{Add}(U)$ is isomorphic to $U^{(I)} \oplus V^{(J)}$, where I, J are suitable sets and V is the unique uniserial module of the same monogeny class as U that is not quasi-small;*

(iii) *if U is not quasi-small, then any object of $\operatorname{Add}(U)$ is isomorphic to $U^{(I)}$ for a suitable set U.*

2. The result

Throughout this paper we suppose that R is an associative ring with unit and U is a uniserial right module over R such that U is a quasi-small module of *type* 2. This means that there is a monomorphism $f: U \to U$ and an epimorphism $g: U \to U$ such that neither of them is an isomorphism. If a uniserial module is not of type 2, then it has local endomorphism ring by [3, Theorem 9.1] and our main theorem holds for such uniserial modules as remarked above.

Before we start let us fix the following notation: Let $M = A \oplus B = \oplus_{i \in I} N_i$ be two direct sum decompositions of M. We denote $\pi_A: M \to A$, $\pi_B: M \to B$, $\pi_i: M \to N_i$, $i \in I$ the canonical projections and we denote $\iota_A: A \to M$, $\iota_B: B \to M$, $\iota_i: N_i \to M$ the canonical injections.

We start with an auxiliary lemma whose modifications are quite used in the literature. Recall that a nonzero module is called *uniform* if any pair of its nonzero submodules has a nonzero intersection. Obviously, any nonzero uniserial module is uniform.

Lemma 2.1. *Let I be a nonempty set and let $\{M_i\}_{i \in I}$ be a family of R-modules. Suppose that N is a uniform submodule of $\oplus_{i \in I} M_i$. Then there exists a nonempty finite set $I' \subseteq I$ such that $\pi_i|_N: N \to M_i$ is injective if and only if $i \in I'$. Moreover, for any $i \in I'$, $N \cap (\oplus_{j \neq i} M_j) = 0$.*

Proof: Since N is nonzero, there exists $0 \neq n \in N$. Let I'' be a finite set such that for any $i \in I$, $\pi_i(n) = 0$ if and only if $i \notin I''$. Now $0 = \bigcap_{i \in I}(N \cap \operatorname{Ker} \pi_i) \supseteq$

$nR \cap \bigcap_{i\in I''}(\operatorname{Ker}\pi_i \cap N)$. Since N is uniform and I'' finite, $\operatorname{Ker}\pi_i \cap N = 0$ for some $i \in I''$. So the set $I' = \{i \in I'' \mid \pi_i|_N \text{ is mono}\}$ is nonempty. $\square$

The following lemma gives a criterion when a uniserial submodule of $U^{(\mathbb{N})}$ has a complement. Recall that a family f_i, $i \in I$ of homomorphisms from M to N is called *summable*, if for any $m \in M$ there is a finite set $I' \subseteq I$ such that $f_i(m) = 0$ for any $i \in I \setminus I'$. In this case the sum of this family gives a homomorphism $\sum_{i\in I} f_i\colon M \to N$.

Lemma 2.2. *Let V be a submodule of $M = \oplus_{i\in\mathbb{N}}U_i$, where V is uniserial and $U_i = U$ for any $i \in \mathbb{N}$. If there is $j \in \mathbb{N}$ such that $\pi_j(V) = U_j$, then V is a direct summand of M isomorphic to U. Conversely, if V is a direct summand of M and $V \simeq U$, then there is $j \in \mathbb{N}$ such that $\pi_j(V) = U_j$.*

Proof: Suppose that $\pi_j(V) = U_j$ for some $j \in \mathbb{N}$. Since V is uniform, we can use Lemma 2.1 to find $i \in \mathbb{N}$ such that $f = \pi_i|_V$ is a monomorphism. If we put $V_i = f(V)$ and if $g_k\colon V_i \to U_k$ is a homomorphism given by $\pi_k \circ f^{-1}$ for any $k \in \mathbb{N}$, we see that $\{g_k\}_{k\in\mathbb{N}}$ can be considered as a summable family of homomorphism from V_i to M and $V = \operatorname{Im}\sum_{k\in\mathbb{N}} g_k$. We know that g_j is an epimorphism. If g_j is an isomorphism, then $\pi_j|_V$ is an isomorphism and thus $M = V \oplus (\oplus_{k\neq j}U_k)$. If $V_i = U_i$, then $\pi_i|_V$ is an isomorphism and $M = V \oplus (\oplus_{k\neq i}U_k)$. Thus we can suppose $V_i \neq U_i$, $i \neq j$ and $g_j : V_i \to U_j$ is a non-monic epimorphism. Now, let $V' = \operatorname{Im}\iota_i + \iota_j$, where $\iota_i, \iota_j : U \to M$ are the canonical injections. Then it is easy to see $V \oplus V' \oplus (\oplus_{k\neq i,j}U_k) = M$. Since $\pi_i|_V\colon V \to U$ is a monomorphism and $\pi_j|_V\colon V \to U$ is an epimorphism, $V \simeq U$ by [3, Lemma 9.2(i)].

Now suppose V is a direct summand of M isomorphic to U. For any $n \in \mathbb{N}$ consider decomposition $M = V \oplus X = \oplus_{i=1}^n U_i \oplus Y_n$, where $Y_n = \oplus_{i>n}U_i$. One of the homomorphisms $\pi_V\iota_1\pi_1\iota_V, \dots, \pi_V\iota_n\pi_n\iota_V, \pi_V\iota_{Y_n}\pi_{Y_n}\iota_V$ has to be an epimorphism because otherwise their sum cannot be an epimorphism. If it is one of the $\pi_V\iota_i\pi_i\iota_V$ we are done because $\pi_i(V) = U_i$, otherwise for any $n \in \mathbb{N}$, $\pi_V\iota_{Y_n}\pi_{Y_n}\iota_V$ is an epimorphism. But then V is a union of kernels of these epimorphisms, therefore $V_e = V$. This also gives that V is a countable union of proper submodules and hence countably generated. As $V \simeq U$, $V_m \subsetneq V$ and V is not quasi-small. This contradicts our assumption that U is quasi-small. $\square$

If we want to prove that a uniserial module V is isomorphic to a direct summand of a module A, it is enough to find $f, f'\colon V \to A$ and $g, g'\colon A \to V$ such that gf is a monomorphism and $g'f'$ is an epimorphism according to [2, Proposition 2.4] and [3, Theorem 9.1]. The following lemma says that if A is a non-zero direct summand of $U^{(\omega)}$, it is enough to find the epimorphisms.

Lemma 2.3. *Let U_i, $i \in I$, be a family of uniform modules. If $A \oplus B = \oplus_{i\in I}U_i$ and $A \neq 0$, then there are $i, j \in I$ such that gf is a monomorphism, where $f = \pi_A\iota_i$ and $g = \pi_j|A$.*

Proof: Consider the homomorphisms $\pi_A\iota_i$, $i \in I$. If none of them is a monomorphism, then $B \cap U_i$ is non-zero for all $i \in I$. Since in this case $\oplus_{i\in I} B \cap U_i$ is essential in $\oplus_{i\in I} U_i$, we have a contradiction to $A \neq 0$. Let $i \in I$ be any index for which $\pi_A\iota_i$ is a monomorphism. Then $V = \pi_A(U_i)$ is uniform and hence there is $j \in I$ such that $\pi_j|_V$ is a monomorphism by Lemma 2.1. Therefore for $f = \pi_A\iota_i$ and $g = \pi_j|_A$ the composition gf is a monomorphism. □

Lemma 2.4. *Let $A \oplus B = \oplus_{i\in\mathbb{N}} U_i$, where $U_i = U$ for any $i \in \mathbb{N}$. If for any $i, j \in \mathbb{N}$ $\pi_j\pi_A(U_i) \neq U_j$, then $B \simeq \oplus_{i\in\mathbb{N}} U_i$.*

Proof: From our assumption, for any $i \in \mathbb{N}$ we have $\pi_i\pi_B(U_i) = U_i$ and $\pi_j\pi_B(U_i) \neq U_j$ whenever $i \neq j$.

Set $U_1' = \pi_B(U_1)$ and observe that $U_1' \oplus B_1 = B$ for suitable module B_1 by Lemma 2.2. Note that, for any $j > 1$, $\pi_j(U_1') \neq U_j$.

Suppose that we have constructed $U_1', \dots, U_k'$ such that $B = U_1' \oplus \dots \oplus U_k' \oplus B_k$ for some $B_k \subseteq B$, $\pi_j(U_1' \oplus \dots \oplus U_k') \neq U_j$ for any $j > k$ and $\pi_B(U_1 \oplus \dots \oplus U_k) = U_1' \oplus \dots \oplus U_k'$. Put $U_{k+1}' = \pi_{B_k}(U_{k+1})$ (projection is with respect to decomposition $\oplus_{i\in\mathbb{N}} U_i = A \oplus U_1' \oplus \dots \oplus U_k' \oplus B_k$). Now we have $\pi_{k+1}(U_{k+1}') = U_{k+1}$, therefore U_{k+1}' is a direct summand of B_k and we have $U_1' \oplus \dots \oplus U_k' \oplus U_{k+1}' \oplus B_{k+1}$ for some $B_{k+1} \subseteq B_k$. From the induction argument we have that $U_1' \oplus \dots \oplus U_{k+1}' = \pi_B(U_1 \oplus \dots \oplus U_{k+1})$ and thus $\pi_j(U_1' \oplus \dots \oplus U_{k+1}') \neq U_j$ for any $j > k+1$. After all $B = \oplus_{i\in\mathbb{N}} U_i'$, where $\pi_i(U_i') = U_i$. Since $\pi_i(U_i') = U_i$, $U \simeq U_i'$ according to Lemma 2.2. □

Corollary 2.5. *Let U be a uniserial module. Let $A \oplus B = U^{(\omega)}$. Then either A contains a direct summand isomorphic to U or $B \simeq U^{(\omega)}$.*

Proof: If $A = 0$ we are done. Suppose $A \neq 0$. From Lemma 2.3 we have existence of homomorphisms $f : U \to A$ and $g: A \to U$ such that gf is a monomorphism. If there are no homomorphisms $f': U \to A$ and $g': A \to U$ such that $g'f'$ is an epimorphism, we have $B \simeq U^{(\omega)}$ according to Lemma 2.4 and Lemma 2.2. Otherwise we have U isomorphic to a direct summand of A. □

Observation 2.6. *Let V, V' be uniserial modules of type 2 having the same epigeny class. Then $f(V_e) \subseteq V_e'$ for any homomorphism $f: V \to V'$.*

Proof: Let $v \in V_e$ be such that $f(v) \notin V_e'$. This is impossible if f is an epimorphism by [4, Lemma 2.3(iv)]. But there is an epimorphism $g: V \to V'$ such that $g(v) = 0$ since $[V]_e = [V']_e$. Then $h = f + g$ is an epimorphism such that $h(v) \notin V_e'$, a contradiction to [4, Lemma 2.3(iv)]. □

The next proposition gives an answer to [3, Problem 13] for the remaining case (i.e. there is no superdecomposable direct summand of $X^{(I)}$ if X is a quasi-small uniserial module of type 2).

Proposition 2.7. *Let A be a non-zero direct summand of $U^{(\omega)}$. Then A contains a non-zero uniserial direct summand. Moreover, if A does not contain a direct summand isomorphic to U, then there exists a non-quasi-small module V of the same monogeny class as U and A is a direct sum of modules isomorphic to V.*

PROOF: Let $A \oplus B = \oplus_{i\in\mathbb{N}} U_i$, where $A \neq 0$ and $U_i = U$ for any $i \in \mathbb{N}$. We can suppose $\pi_j \pi_A(U_i) \neq U_j$ for any $i, j \in \mathbb{N}$, otherwise A contains a direct summand isomorphic to U by Lemma 2.2.

Let us analyze the proof of Lemma 2.4 a bit. We keep the notation from the proof of Lemma 2.4. For any $u \in \oplus_{i\in\mathbb{N}}(U_i)_e$, $\pi_B(u) \subseteq \oplus_{i\in\mathbb{N}}(U_i')_e$ according to Observation 2.6. From the construction $\pi_B(U_i) \subseteq U_1' \oplus \cdots \oplus U_i'$ and $\pi_{U_i'}\pi_B(U_i) = U_i'$. Thus since $[U_i]_e = [U_i']_e$ and $\pi_{U_i'}\pi_B|_{U_i}$ is an epimorphism, we have $\pi_{U_i'}\pi_B(u) \notin (U_i')_e$ for any $u \in U_i \setminus U_e$. Now let $a = a_1 + \cdots + a_k \in A$, and $a_i \in U_i$. Suppose that $a \notin \oplus_{i\in\mathbb{N}}(U_i)_e$. Let l be the greatest index $1 \leq l \leq k$ such that $a_l \notin (U_l)_e$. Then $\pi_B(a_1 + \cdots + a_{l-1}) \in U_1' \oplus \cdots \oplus U_{l-1}'$, $\pi_{U_l'}\pi_B(a_{l+1} + \cdots + a_k) \in (U_l')_e$, and $\pi_{U_l'}\pi_B(a_l) \notin (U_l')_e$. Thus $\pi_B(a) \neq 0$, a contradiction. From this fact we see that $A \subseteq \oplus_{i\in\mathbb{N}}(U_i)_e$. But since $A \neq 0$, Lemma 2.3 gives $i, j \in \mathbb{N}$ such that $\pi_j \iota_A \pi_A \iota_i$ is a monomorphism. Therefore $\pi_j(A) \subseteq (U_j)_e$ contains an isomorphic copy of U and $U_m \subsetneq U_e$ follows.

If $U_m \subsetneq U_e$ and there are no homomorphisms $f : U \to A$ and $g : A \to U$ such that $g \circ f$ is an epimorphism, then for any $i \in \mathbb{N}$ we have $\pi_j(A) \neq \pi_j \pi_A(U_i)$ whenever $(U_j)_m \subsetneq \pi_j(A)$ because $\pi_j \pi_A(U_i) \simeq U$ in this case. Therefore $\pi_j(A)$ is countably generated whenever $(U_j)_m \subsetneq \pi_j(A)$. Since $U_m \subsetneq U_e$, any countably generated submodule of U_e is contained in a submodule of U_e that is not quasi-small and that properly contains U_m (if U_e is countably generated it is not quasi-small, otherwise we can adapt the proof of [4, Lemma 2.9]). Any such module is isomorphic to V (the unique module of the same monogeny class as U that is not quasi-small). It follows that for any $i \in \mathbb{N}$ there exists $W_i \simeq V$ such that $\pi_i(A) \subseteq W_i \subseteq (U_i)_e$. Therefore A can be considered as a direct summand of $V^{(\omega)}$. By [4, Theorem 3.12], A is isomorphic to a direct sum of copies of V. $\square$

The next proposition can be seen as an analogy to the result "uniformly big projective modules are free" which was proved by Bass in [1]. In fact, we just adapted his proof to our setting. Let us recall the notions we shall need in the proof of the proposition. A module M is called *small* if for any family of modules M_i, $i \in I$ and any homomorphism $f\colon M \to \oplus_{i\in I} M_i$, there is a finite set $I' \subseteq I$ such that $f(M) \subseteq \oplus_{i\in I'} M_i$. A module is called *$\sigma$-small*, if it is a union of a countable chain of its small submodules. As noted above, any uniserial module is σ-small.

Proposition 2.8. *Let $M = A \oplus B = \oplus_{i\in\mathbb{N}} U_i$, where for any $i \in \mathbb{N}$ $U_i = U$. Suppose for any $n \in \mathbb{N}$ there exists a direct summand of A isomorphic to U^n. Then A is isomorphic to $U^{(\omega)}$.*

Proof: Let V be a proper submodule of U such that there is a non-monic epimorphism $g: V \to U$. By induction we construct submodules $U'_1, U'_2, \dots, A_1, A_2, \dots$ of A and we find $j_1, j_2, \dots \in \mathbb{N}$ such that for any $i \in \mathbb{N}$ the following are satisfied:

(i) for any $k > i$, $U'_k \subseteq A_i$,
(ii) $A_i \oplus (\oplus_{j \le i} U'_j) = A$,
(iii) $\pi_{j_i}(U'_i) = U_{j_i}$,
(iv) for every $k \ge i$ is $\pi_{j_i}(A_k) \neq U_{j_i}$,
(v) $U'_i \simeq U$ for any $i \in \mathbb{N}$.

According to the assumption A contains a uniserial direct summand U'_1 isomorphic to U. By Lemma 2.2 there exists j_1 such that $\pi_{j_1}(U'_1) = U_{j_1}$. If $\pi_{j_1}|_{U_1}$ is an isomorphism, we set $A_1 = A \cap \oplus_{k \neq j_1} U_k$. Otherwise there is $i_1 \neq j_1$ such that $\pi_{i_1}|U'_1$ is a monomorphism. Let $h: V \to U_{i_1} \oplus U_{j_1}$ be given by the sum of $\iota_{i_1} g$ and an inclusion of V into U_{j_1}. Then $A \oplus B = U'_1 \oplus \operatorname{Im} h \oplus (\oplus_{i \neq i_1, j_1} U_i)$. Set $A_1 = A \cap (\operatorname{Im} h \oplus (\oplus_{i \neq i_1, j_1} U_i))$. In both cases (ii), (iii), (iv) and (v) are satisfied for $i = 1$.

Now suppose that $j_1, \dots, j_k$, $U'_1, \dots, U'_k$, A_k have been defined such that conditions (i)–(v) are satisfied when restricted to constructed objects. From (ii), $A = \oplus_{i=1}^k U'_i \oplus A_k$. According to our assumption A_k contains a direct summand U'_{k+1} isomorphic to U (recall that any uniserial module cancels from direct sums by [3, Corollary 4.6]). Therefore there is j_{k+1} such that $\pi_{j_{k+1}}(U'_{k+1}) = U_{j_{k+1}}$. In the same way as above we find X such that $A \oplus B = X \oplus U'_{k+1}$ and $\pi_{j_{k+1}}(X) \neq U_{j_{k+1}}$. Then we put $A_{k+1} = A_k \cap X$. Then conditions (i)–(v) are satisfied by the objects we have defined.

For the modules U'_k defined in the construction we have indices $i_k, j_k \in \mathbb{N}$ such that $\pi_{i_k}|_{U'_k}$ is a monomorphism and $\pi_{j_k}|_{U'_k}$ is an epimorphism. We know that j_k are pairwise different. We can suppose that, for any $k < l \in \mathbb{N}$, $i_k, j_k < i_l, j_l$ if we remove some of U'_ns since indices i_k can be chosen such that the set $\{i_k \mid k \in \mathbb{N}\}$ is infinite as it follows from considerations about Goldie dimension.

For any $k \in \mathbb{N}$ such that $i_k \neq j_k$ let V_{i_k} be a projection of U'_k to $U_{i_k} \oplus U_{j_k}$ and V_{j_k} be a complement of V_{i_k} in $U_{i_k} \oplus U_{j_k}$. For any $i \in \mathbb{N} \setminus \{i_k, j_k \mid i_k \neq j_k, k \in \mathbb{N}\}$ set $V_i = U_i$. Then $A \oplus B = \oplus_{i \in \mathbb{N}} V_i$. Let $\pi'_i: M \to V_i, \iota'_i: V_i \to M$ be canonical projections and injections with respect to this decomposition. Observe that for any $k \in \mathbb{N}$ $\pi'_{i_k}|_{U'_k}$ is an isomorphism. Therefore there are $f_{k,l}: V_{i_k} \to V_l$ such that for any $k \in \mathbb{N}$ homomorphism f_{k,i_k} is an isomorphism, $\{f_{k,l}\}_{l \in \mathbb{N}}$ is a summable family of homomorphisms from V_{i_k} to M and $U'_k = \operatorname{Im} \sum_{l \in \mathbb{N}} f_{k,l}$.

We are going to define a sequence $k_1 < k_2 < \dots \in \mathbb{N}$ such that $\oplus_{l \in \mathbb{N}} U'_{k_l}$ is a direct summand of M. Since any uniserial module is σ-small, there are modules $V_{k,l} \subseteq V_{i_k}$ such that $V_{k,l}$ is small for any $k, l \in \mathbb{N}$ and $V_{i_k} = \bigcup_{l \in \mathbb{N}} V_{k,l}$ for any $k \in \mathbb{N}$.

First put $k_1 = 1$. Observe that $M = U'_{k_1} \oplus (\oplus_{i \neq i_1} V_i)$. Suppose $k_1, \dots, k_n$ have been defined such that if $I' = \mathbb{N} \setminus \{i_{k_1}, \dots, i_{k_n}\}$, then $M = \oplus_{j=1}^n U'_{k_j} \oplus (\oplus_{i\in I'} V_i)$. Clearly, module $N = \oplus_{j=1}^n V_{k_j,n}$ is small. Therefore there is k' such that $N \subseteq \oplus_{j=1}^n U'_{k_j} \oplus (\oplus_{i<k', i\in I'} V_i)$. Now, let k_{n+1} be an integer greater that k' such that $M = \oplus_{j=1}^{n+1} U'_{k_j} \oplus (\oplus_{i\in I''} V_i)$, where $I'' = I' \setminus \{i_{k_{n+1}}\}$. (Any $j > \max(k', k_n)$ such that f_{k_l,i_j} is neither a monomorphism nor an epimorphism for any $1 \le l \le n$ can be chosen for k_{n+1}.)

By our construction, $M = \oplus_{j\in\mathbb{N}} U'_{k_j} \oplus (\oplus_{i\in I} V_i)$, where $I = \mathbb{N} \setminus \{i_{k_1}, i_{k_2}, \dots\}$. Of course, $\oplus_{j\in\mathbb{N}} U'_{k_j} \subseteq A$ and thus A contains a direct summand isomorphic to $U^{(\omega)}$. To finish the proof we use the Eilenberg's trick as usually: Recall that $X \oplus U^{(\omega)} \simeq U^{(\omega)}$ whenever X is a direct summand of $U^{(\omega)}$. Therefore if $A \oplus A' \simeq U^{(\omega)}$ and $U^{(\omega)} \oplus A'' \simeq A$, then $A \simeq U^{(\omega)}$. $\square$

Corollary 2.9. *Let W be a uniserial module. If $A \oplus B \simeq W^{(\omega)}$, then either $A \simeq W^{(\omega)}$ or $B \simeq W^{(\omega)}$.*

PROOF: If W is not of type 2, we use [3, Corollary 2.54]. If W is not quasi-small, we use [4, Theorem 3.12]. If W is quasi-small and of type 2, we use Proposition 2.8, Corollary 2.5 and the fact that uniserial modules cancel from direct sums (see [3, Corollary 4.6]). $\square$

PROOF OF THEOREM 1.1: Any direct summand of $U^{(I)}$ can be decomposed as a direct sum of direct summands of $U^{(\omega)}$ by [3, Corollary 2.49]. Therefore it is possible to suppose I countable. Uniserial modules with local endomorphism ring satisfy the hypothesis of (i) and the theorem holds for such modules as explained above. Also the case (iii) was already proved in [4, Theorem 3.12] So it remains to prove the theorem for quasi-small uniserial modules of type 2.

Let $A \oplus B = \oplus_{i\in\mathbb{N}} U_i$, where $U_i = U$ for any $i \in \mathbb{N}$. It is enough to see that A is a direct sum of uniserial modules since any non-zero uniserial direct summand has the same monogeny class as U by [3, Proposition 9.6] and thus the uniserial direct summand can be isomorphic only to U or, in case (ii), to V. We can suppose that A does not have finite Goldie dimension otherwise we use Proposition 2.7 to see that A is serial. If A contains a direct summand isomorphic to U^k for arbitrary $k \in \mathbb{N}$, then $A \simeq U^{(\omega)}$ by Proposition 2.8. In the other case there exist $k \in \mathbb{N}$ and $A' \subseteq A$ such that $A \simeq U^k \oplus A'$ and A' contains no direct summand isomorphic to U.

(i) In this case $A' = 0$ by Proposition 2.7. (ii) A' is isomorphic to a direct sum of copies of V by Proposition 2.7. $\square$

Remark 2.10. The reader could observe that we proved that summands of $U^{(\omega)}$ having infinite Goldie dimension in case (ii) can be only modules isomorphic to $U^{(\omega)}$ or $U^k \oplus V^{(\omega)}$, $k \in \mathbb{N}_0$. This reflects the main result of [5] that imply that for cardinals $\kappa, \lambda, \kappa', \lambda'$ the modules $U^{(\kappa)} \oplus V^{(\lambda)}$ and $U^{(\kappa')} \oplus V^{(\lambda')}$ are isomorphic if and only if $\kappa = \kappa'$ and $\kappa + \lambda = \kappa' + \lambda'$.

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