Countable sums and products of Loeb and selective metric spaces

Horst Herrlich, Kyriakos Keremedis, Eleftherios Tachtsis

Dedicated with affection to Věra Trnková on the occasion of her 70-th birthday.

Abstract. We investigate the role that weak forms of the axiom of choice play in countable Tychonoff products, as well as countable disjoint unions, of Loeb and selective metric spaces.

Keywords: axiom of choice, weak axioms of choice, Loeb metric spaces, selective metric spaces, countable Tychonoff products of metric spaces, countable sums of metric spaces

Classification: 03E25, 54A35, 54D45, 54E35, 54E50, 54E99

1. Notation and terminology

In the following, any statement "Form x" has been considered in [2] where all known implications between these forms are given in Table 1, see `http://www.math.purdue.edu/~jer/Papers/conseq.html`.

Definition 1. 1. Let (X, T) be a topological space.

(a) X is said to be a *Loeb space* iff the family of all non-empty, closed subsets of X has a choice function.
(b) X is said to be *selective* iff the family of all non-empty, open subsets of X has a choice function.
(c) X is said to be *locally compact* iff each point in X has a neighborhood base consisting of compact sets.
(d) X is called *second countable* if it has a countable base for T.
(e) X is called *separable* if it has a countable dense subset.

2. Let (X, d) be a metric space.

(a) A sequence $(x_n)_{n\in\omega}$ of points in X is said to be a *Cauchy sequence* if $(\forall\,\epsilon > 0)(\exists N \in \omega)(\forall\, n, m \geq N)d(x_n, x_m) < \epsilon$.
(b) X is called *complete* or *Fréchet complete* if every Cauchy sequence converges.

3. **M(S,T)**: Every metric space (X, d) having the property S also has the property T.

4. **FPM(S,T)**: The Tychonoff product of finitely many metric spaces each having the property S has the property T.
5. **CPM(S,T)**: The Tychonoff product of countably many metric spaces each having the property S has the property T.
6. **FSM(S,T)**: The topological sum (or disjoint topological union, see [17]) of finitely many metric spaces each having the property S has the property T.
7. **CSM(S,T)**: The topological sum of countably many metric spaces each having the property S has the property T.
8. **AC, Form 1**: Every family of non-empty sets has a choice function.
9. **CAC, Form 8**: AC restricted to countable families.
10. **CAC(sel)**: CAC restricted to families of non-empty selective metric spaces.
11. **CAC(Loeb)**: CAC restricted to families of non-empty Loeb metric spaces.
12. **AC(WO), Form 40**: Every well ordered family of non-empty sets has a choice function.
13. **AC($\aleph_u^{\aleph_v}$)**: Every family $\mathcal{A} = \{A_i : i \in I \subseteq \aleph_u^{\aleph_v}\}$ of non-empty subsets of $\aleph_u^{\aleph_v}$ has a choice function.
14. **AC$_{\aleph_u^{\aleph_v}}$**: For all ordinals u and v, AC($\aleph_u^{\aleph_v}$).
15. **CUC, Form 31**: A countable union of countable sets is countable.
16. **AC($\mathbb{R}$), Form 79**: AC restricted to families of non-empty subsets of $\mathbb{R}$.
17. **CAC($\mathbb{R}$), Form 94**: AC($\mathbb{R}$) restricted to countable families.
18. **Form 212**: AC($\mathbb{R}$) restricted to continuum sized families.
19. **Form 421**: A countable union of well orderable sets is well orderable.

We shall use the following abbreviations: **2** for "second countable", **S** for "separable", **hS** for "hereditarily separable", **sel** for "selective", and **hLoeb** for "hereditarily Loeb".

2. Introduction and some known results

It is a famous result of Kelley's, see [5], that Tychonoff's compactness theorem,

(TCT) *Tychonoff products of compact topological spaces are compact*

is equivalent to the full axiom of choice AC. On the other hand, P. Loeb [15] gave a different proof of this theorem in ZFC (Zermelo-Fraenkel set theory (ZF) plus AC) using choice functions on the closed subsets of the coordinate spaces, thus using implicitly the notion of the Loeb space. Furthermore, it is known, see [9], that the weakening of TCT, *every product of non-empty compact T_2 spaces is non-empty*, is equivalent to the proposition *every compact T_2 space is a Loeb space*. Now, if in TCT we require the component spaces to be Loeb pseudometric or selective pseudometric instead of compact, then the statement:

(*) *every product of Loeb* (*selective*) *pseudometric spaces is a Loeb* (*selective*) *space* is equivalent to TCT, thus equivalent to AC. Indeed, let $\mathcal{A} = \{A_i : i \in I\}$ be a pairwise disjoint family of non-empty sets. Let $\{\infty_i : i \in I\}$ be a set of pairwise distinct points which are not elements of $\bigcup \mathcal{A}$. For each $i \in I$, define a pseudometric d_i on $X_i = A_i \cup \{\infty_i\}$ by requiring: $d_i(\infty_i, \infty_i) = 0$, $d_i(x,y) = 0$ if $x, y \in A_i$, and $d_i(x,y) = d_i(y,x) = 1$ if $x = \infty_i$ and $y \in A_i$. Clearly, each A_i is a clopen (closed and open) subset of (X_i, d_i). Let X be the Tychonoff product of the X_i's. If X is a Loeb (selective) space, then the family $\mathcal{B} = \{B_i = \pi_i^{-1}(A_i) : i \in I\}$ which consists of clopen non-empty subsets of the space X admits a choice function, say f. Then $g = \{(i, \pi_i(f(i))) : i \in I\}$ is a choice function for the original family $\mathcal{A}$. Thus, AC holds. Similarly, one shows that the countable version of (*) implies the axiom of countable choice CAC.

Furthermore, if in (*) we replace "pseudometric" by "T_2", then the statement:

(**) *every product of non-empty Loeb T_2 spaces is a Loeb space*

implies van Douwen's choice principle, Form 118 in [2]: *every family $A = \{(A_i, \leq_i) : i \in k\}$, where for each $i \in k$, $A_i \neq \emptyset$ and $\leq_i$ is a conditionally complete linear order on A_i (i.e., every non-empty subset of A_i with an upper bound has a least upper bound), has a choice function*, see [9].

However, the class of pseudometric spaces and the class of T_2 spaces are properly larger than the class of metric spaces, so the deductive strength of the metric version of (*) or its countable metric version or even its finite version comes into consideration. From the forthcoming Theorems 3(i) and 5(i) we will immediately deduce that the statements: *finite products of selective metric spaces are selective* and *finite sums of Loeb (selective) metric spaces are Loeb (selective)* are theorems of ZF. Our aim in this paper is to elucidate on the set theoretic strength of the countable metric versions of (*) and finite versions for Loeb metric spaces as well as on the strength of the corresponding statements for countable sums and the interrelation between them.

Before we begin with the main results let us first recall some theorems we shall be needing.

Theorem 1. *Equivalent are:* (i) *CAC*($\mathbb{R}$).

(ii) ([1]) *Every subspace of $\mathbb{R}$ is separable.*
(iii) ([2], [3]) *M(S,hS).*

Theorem 2. (i) ([15]) (*ZF*) $\mathbb{R}$ *is a Loeb space.*

(ii) ([6]) (*ZF*) $\mathbb{R}$ *is a selective space.*
(iii) ([6]) $\mathbb{R}$ *is hereditarily Loeb implies CAC*($\mathbb{R}$).
(iv) ([6], [13]) *CAC*($\mathbb{R}$) *iff* $\mathbb{R}$ *is hereditarily selective iff the Baire space ω^ω is hereditarily selective.*
(v) ([13]) 212 *iff* $\mathbb{R}$ *is hereditarily Loeb iff ω^ω is hereditarily Loeb iff M(S,Loeb) iff M(2,Loeb).*

	CAC	CAC(Loeb)	CAC(sel)	CPM(Loeb,Loeb)	CSM(Loeb,Loeb)	CPM(sel,sel)	31	94	212	421
CAC	1	1	1	0	1	1	1	1	3	1
CAC(Loeb)	2	1	0	0	0	0	0	0	3	0
CAC(sel)	2	1	1	0	0	0	0	0	3	0
CPM(Loeb,Loeb)	2	1	0	1	1	0	0	0	0	1
CSM(Loeb,Loeb)	2	1	0	0	1	0	0	0	3	1
CPM(sel,sel)	2	1	1	0	0	1	0	0	3	1
31	3	2	2	2	2	2	1	3	3	2
94	3	3	3	3	3	3	2	1	3	2
212	3	3	3	3	3	3	2	1	1	2
421	3	0	0	0	0	0	3	3	3	1

Table 1: A "0" at position (P,Q) means that it is unknown whether the row statement P implies the column statement Q. A "1" at position (P, Q) means that $(P \to Q)$ in ZF, a "2" at (P, Q) means that $(P \not\to Q)$ in ZF^0 (= ZF minus the axiom of foundation), and a "3" at (P, Q) means that $(P \not\to Q)$ in ZF. Because of the equivalence (CPM(sel,sel) $\leftrightarrow$ CSM(sel,sel)) which is proved in Theorem 7(ii) we only use CPM(sel,sel) in the table. Regarding the positive or independence results which appear in the table and concern the forms CAC, 31, 94, 212 and 421 of [2], but are not proved here, the reader is referred to [2].

Theorem 3. (i) ([12]) (*ZF*) *A metric space is selective if and only if it has a well ordered dense subset.*
(ii) ([13]) (*ZF*) *Every Loeb* (*selective*) *metric space has a Loeb* (*selective*) *completion* (*i.e., it embeds as a dense subspace of a complete Loeb* (*selective*) *metric space*)*.*

Theorem 4 ([13])**.** *If* (A, ρ) *is a discrete metric space with well ordered underlying set, then the Tychonoff product* A^ω *of* ω *copies of* A *taken with the discrete metric is a Loeb metric space. In particular, the Baire space* ω^ω *is Loeb.*

Theorem 5 ([13])**.** (i) *M(Loeb,sel) is a theorem of ZF.*
(ii) *A complete metric space is Loeb iff it is selective.*
(iii) *M(Loeb,hLoeb) iff dense subspaces of complete Loeb metric spaces are Loeb.*
(iv) *M(sel,Loeb)* $\rightarrow AC_{\aleph^\omega} \rightarrow$ *212.*
(v) *M(Loeb,hLoeb)* $\leftrightarrow$ *M(sel,Loeb).*

3. Main results

It is well-known that if d is a metric on a set X then d can be replaced by an equivalent metric ρ which is bounded by 1 (e.g. $\rho(x, y) = \min\{1, d(x, y)\}$). If $\{(X_i, d_i) : i \in \mathbb{N}\}$ is a disjoint family of metric spaces, where each d_i is bounded by 1, then $d(x, y) = \sum_{i\in\mathbb{N}} \frac{d_i(x(i),y(i))}{2^i}$ is a metric on $X = \prod_{i\in\mathbb{N}} X_i$ producing the product topology on X, see [17], and $\sigma(x, y) = d_i(x, y)$ if $x, y \in X_i$, $\sigma(x, y) = 1$ otherwise, is a metric on $Y = \bigcup\{X_i : i \in \mathbb{N}\}$ producing the disjoint union topology on Y, see [17]. In the sequel we shall always assume that countable Tychonoff products of metric spaces carry the metric d and countable sums of metric spaces carry the metric σ.

Theorem 6. (i) *CAC* $\rightarrow$ *CSM(Loeb,Loeb)* $\rightarrow$ (421+ *CAC*(*Loeb*)).
(ii) (*CAC*(*Loeb*) +212) $\rightarrow$ *CUC.*
(iii) *CPM*(*Loeb,Loeb*) $\rightarrow$ *CSM*(*Loeb,Loeb*).
(iv) $AC_{\aleph^\aleph} \rightarrow$ (X^ω *is hereditarily Loeb for every selective metric space* (X, d)).
(v) ($AC_{\aleph^\aleph}$+ *CSM*(*Loeb,Loeb*)) $\rightarrow$ *CPM*(*Loeb,Loeb*).

Proof: (i) (CAC $\rightarrow$ CSM(Loeb,Loeb)). Fix $\mathcal{F} = \{(X_i, d_i) : i \in \mathbb{N}\}$ a disjoint family of Loeb metric spaces. Let X be the sum of $\mathcal{F}$ and $\mathbb{G}$ be the family of all non-empty closed subsets of X. Put $\mathcal{G} = \{\mathcal{G}_i : i \in \omega\}$, where $\mathcal{G}_i$ is the family of all choice functions of the set $\mathbf{G}_i$ of all non-empty closed subsets of X_i. Let, by CAC, f be a choice function of the family $\mathcal{G}$. On the basis of f we describe a choice function h of $\mathbb{G}$ as follows. For $G \in \mathbb{G}$ first let $i_G = \min\{i \in \omega : G \cap X_i \neq \emptyset\}$ and then define $h(G) = f(i_G)(G \cap X_{i_G})$. It can be readily verified that h is a choice function of $\mathbb{G}$ as required.

(CSM(Loeb,Loeb) $\to$ 421). Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of non-empty well orderable sets and let d_i be the discrete metric on A_i. Clearly, (A_i, d_i) is a Loeb metric space. Since $\bigcup \mathcal{A}$ carries the discrete topology, it follows by CSM(Loeb,Loeb) that the family of all non-empty subsets of $\bigcup \mathcal{A}$ has a choice function. This means that $\bigcup \mathcal{A}$ is well orderable.

(CSM(Loeb,Loeb) $\to$ CAC(Loeb)). This is straightforward.

(ii) Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of countably infinite sets. For every $i \in \omega$, let d_i be the discrete metric on A_i and consider the Tychonoff product $X_i = A_i^\omega$. Clearly, X_i is a metric space which is homeomorphic to the Baire space ω^ω, hence by Theorem 2(v), X_i is hereditarily Loeb. Thus, the subspace $Y_i = \{x \in X_i : x \text{ is } 1:1 \text{ and onto}\}$ of X_i is a Loeb metric space for all $i \in \omega$. By CAC(Loeb), let $f = \{(i, f_i) : i \in \omega\}$ be a choice function of the family $Y = \{Y_i : i \in \omega\}$. Then $\bigcup \mathcal{A} = \{f_i(n) : i, n \in \omega\}$ is countable finishing the proof of (ii).

(iii) Fix $\mathcal{F} = \{(X_i, d_i) : i \in \mathbb{N}\}$ a disjoint family of Loeb metric spaces. For each $i \in \mathbb{N}$, let $Y_i = X_i \cup \{*_i\}$, $*_i \notin X_i$, and ϕ_i the metric on Y_i given by:

$$\phi_i(x,y) = \phi_i(y,x) = \begin{cases} \rho_i(x,y), & \text{if } x, y \in X_i \\ 1, & \text{if } x \in X_i \text{ and } y = *_i \\ 0, & \text{if } x = y = *_i, \end{cases}$$

where ρ_i is an equivalent metric to d_i bounded by 1. Let X be the sum of the family $\mathcal{F}$ and Y the product of the Y_i's. Clearly, $\mathcal{G} = \{\pi_i^{-1}(G) : G \subset X_i, \emptyset \neq G = \overline{G}, i \in \omega\}$ is a family of non-empty closed subsets of Y. Let, by CPM(Loeb,Loeb), f be a choice function on the family $\mathcal{G}$. On the basis of f we define a choice function h on the family $\mathcal{Q}$ of all non-empty closed subsets of X as follows: For $G \in \mathcal{Q}$, let $i_G = \min\{i \in \omega : (G \cap X_i) \neq \emptyset\}$ and define $h(G) = f(\pi_i^{-1}(G \cap X_{i_G}))(i)$.

(iv) Let $D = \{d_i : i \in \aleph\}$ be a well ordered dense subset of a selective metric space (X, d). Since $\mathbf{D} = \bigcup\{\mathbf{D}_n = D^n \times \prod_{i \geq n}\{d_0\} : n \in \omega\}$ is dense in $Y = X^\omega$ and each $\mathbf{D}_n$ is a well ordered (under the lexicographic ordering) set of cardinality $\aleph$, it follows that $\mathbf{D}$ has size $\aleph$. Therefore, Y, as well as, every subspace of Y have a well ordered base of size $\aleph$. Let Z be a subspace of Y. It follows that $|Z| \leq |Y| \leq |\aleph^\aleph|$ (the function which maps every element x to the set of all basic open sets which contain x is obviously injective) and $|\mathcal{C}_Z| = |\mathcal{C}| \leq |2^\aleph| = |\aleph^\aleph|$ where $\mathcal{C}$ and $\mathcal{C}_Z$ are the families of all closed, non-empty subsets of Y and Z respectively (the function which maps every open set U to the set of all basic open sets contained in U is obviously injective). By AC($\aleph^\aleph$), it follows that $\mathcal{C}_Z$ has a choice function, hence Z is Loeb and Y is hereditarily Loeb as required.

(v) Let $\mathcal{F} = \{(X_i, d_i) : i \in \mathbb{N}\}$ be a disjoint family of Loeb metric spaces and X, Y be respectively the sum and product of the family $\mathcal{F}$. By CSM(Loeb,Loeb) it follows that X is Loeb and by $\mathrm{AC}_{\aleph^\aleph}$ that $Z = X^\omega$ is hereditarily Loeb. Since Y is a subspace of Z it follows that Y is Loeb as required. $\square$

Theorem 7. (i) *CAC* $\rightarrow$ *CSM(sel,sel)* $\rightarrow$ (421+ *CAC*(*sel*)).
(ii) *CPM(sel,sel)* $\leftrightarrow$ *CSM(sel,sel)*.
(iii) (*CAC*(*sel*) + *CAC*($\mathbb{R}$)) $\rightarrow$ *CUC.*

Proof: (i) and (CPM(sel,sel) $\rightarrow$ CSM(sel,sel)) can be proved as in (i) and (iii) of Theorem 6.

(CSM(sel,sel) $\rightarrow$ CPM(sel,sel)). Fix $\mathcal{F} = \{(X_i, d_i) : i \in \mathbb{N}\}$ a disjoint family of selective metric spaces. Let X and Y be respectively the sum and the product of the family $\mathcal{F}$. Fix, by CSM(sel,sel) and Theorem 3(i), a well ordered dense subset $G = \{g_i : i \in \aleph\}$ of X. Then for every $i \in \omega$, $G_i = G \cap X_i$ is a well ordered dense subset of X_i and as in the proof of (iv) of Theorem 6 we may construct a well ordered dense subset of Y. The conclusion now follows from Theorem 3(i).

(iii) This can be proved as in (ii) of Theorem 6, but now taking into account Theorem 2(iv). □

Corollary 1. *In every permutation model, each one of CAC*(*Loeb*) *and CAC*(*sel*) *implies CUC.*

Theorem 8. (i) *FSM*(*Loeb,Loeb*), *FSM*(*sel,sel*), *FPM*(*sel,sel*) *are theorems of ZF.*
(ii) *The product of a well ordered metric space* (X, d) *with a Loeb metric space* (Y, σ) *is Loeb.*
(iii) *Let* $(X, d), (Y, \sigma)$ *be two Loeb metric spaces. If the canonical projection* π_X *of* $X \times Y$ *on* X *is a closed map, then* $X \times Y$ *is Loeb. In particular, the product of two Loeb metric spaces one of which is compact is a Loeb metric space.*
(iv) *The product of two Loeb metric spaces one of which is locally compact is a Loeb space.*
(v) *The finite product of compact, Loeb metric spaces is compact and Loeb.*
(vi) *If* (X, d), (Y, ρ) *are two Loeb metric spaces, then the family of all non-empty, compact subsets of* $X \times Y$ *has a choice function.*
(vii) *The product of two complete, Loeb metric spaces is a Loeb space.*
(viii) *M*(*Loeb,hLoeb*) *implies FPM*(*Loeb,Loeb*).
(ix) (*M*(*Loeb,hLoeb*) + *CSM*(*Loeb,Loeb*)) $\rightarrow$ *CPM*(*Loeb,Loeb*).

Proof: (i) This is evident.

(ii) Let $\{y_i : i \in \aleph\}, \aleph$ a well ordered cardinal, be an enumeration of X and f a choice function of the family of all non-empty closed subsets of Y. Let A be a closed subset of $X \times Y$. Clearly, $(y_{i_A}, f(\pi_Y[\pi_X^{-1}(\{y_{i_A}\}) \cap A])) \in A$, $i_A = \min\{i \in \aleph : \pi_X^{-1}(\{y_i\}) \cap A \neq \emptyset\}$ and $X \times Y$ is Loeb.

(iii) Let f and g be choice functions on the families of all non-empty closed subsets of X and Y respectively. Fix A a closed subset of $X \times Y$. Clearly, $(f(\pi_X(A)), g(\pi_Y[(\{f(\pi_X(A))\} \times Y) \cap A])) \in A$ (for every $x \in X$, $\pi_Y : \{x\} \times Y \longrightarrow$

Y is a homeomorphism) and $X \times Y$ is Loeb as required. The second assertion follows from the fact that if (X,T), (Y,P) are two topological spaces and X (or Y) is compact, then in ZF, π_Y (resp. π_X) is a closed map, see [17].

(iv) Let (X,d) and (Y,ρ) be two Loeb metric spaces. Suppose that X is locally compact and let f and g be respectively choice functions on the families of all non-empty, closed subsets of X and Y. Since X and Y are Loeb spaces, they have well ordered bases, say $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Moreover, as X is locally compact, it follows that $\mathcal{C}_X = \{\overline{B} : (B \in \mathcal{B}_X) \wedge (B \subset U,\ U$ a compact neighborhood of some $x \in X)\}$ is a well ordered base for X consisting of compact, hence closed, sets. Then $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, where $\mathcal{C}_Y = \{\overline{c} : c \in \mathcal{B}_Y\}$, is a base for $X \times Y$. Now let A be any non-empty closed subset of $X \times Y$ and $U \times V$ the first element of $\mathcal{C}$ which meets A non trivially. Then $A' = (U \times V) \cap A$ is a non-empty, closed subset of the subspace $U \times V$ of $X \times Y$. Since U is compact and both U and V, being respectively closed subsets of the Loeb spaces X and Y, are Loeb spaces, we may choose an element of A', hence of A, exactly as in (iii) using the choice functions $f|_U$ and $g|_V$.

(v) It is well known that in ZF a finite product of compact topological spaces is compact, see [17]. The second assertion follows from (iv).

(vi) If A is a non-empty compact subset of $X \times Y$, then $\pi_X(A)$ is a compact, hence closed, subset of X and we may continue as in (iii) in order to choose an element from A.

(vii) The product of two complete, Loeb metric spaces is obviously a complete metric space which, by Theorems 5(i) and 3, has a well ordered dense set, hence it is selective. The conclusion now follows from Theorem 5(ii).

(viii) Fix (X,d) and (Y,ρ) two Loeb metric spaces. By Theorem 3(ii), let (X_1,d_1) and (Y_1,ρ_1) be respectively the Loeb completions of X and Y. By (vii), $X_1 \times Y_1$ is a complete, Loeb metric space and since $\overline{X \times Y} = X_1 \times Y_1$, it follows by Theorem 5(iii) that $X \times Y$ is a Loeb metric space as required.

(ix) Let $\mathcal{F} = \{(X_i,d_i) : i \in \omega\}$ be a disjoint family of Loeb metric spaces and X, Y be respectively the sum and the product of the family $\mathcal{F}$. By CSM(Loeb,Loeb), X is Loeb, thus it has a well ordered dense subset which in turn implies that $Y \neq \emptyset$ and that Y also has a well ordered dense subset, see the proof of (iv) of Theorem 6. Let (Z,ρ) be the completion of Y. Since Z obviously has a well ordered dense subset, it is selective and consequently, by Theorem 5(ii), Z is Loeb. By M(Loeb,hLoeb), the subspace Y of Z is a Loeb space as required. □

Corollary 2. *In ZF, the Euclidean metric space $\mathbb{R}^n$ is a Loeb space for all $n \in \mathbb{N}$.*

4. Independence results

Theorem 9. (i) *CPM(S,Loeb) implies* 212 *and the implication is not reversible in ZF^0.*

(ii) *In ZF, CPM(S,sel) does not imply CPM(S,Loeb).*

Proof: (i) Clearly, CPM(S,Loeb) implies M(S,Loeb) which, by Theorem 2(v), is equivalent to 212. On the other hand, in [8] a permutation model was constructed (model $\mathcal{N}55$ in [2]) in which there is a countable family of compact separable metric spaces with empty product. Hence, CPM(S,Loeb) fails in that model and since 212 is true in every permutation model, the independence result follows.

(ii) In Feferman's model ($\mathcal{M}2$ in [2]), CPM(S,sel) holds since CAC clearly implies CPM(S,sel) and CAC holds in $\mathcal{M}2$, see [2]. However, 212 fails in $\mathcal{M}2$, hence by (i), CPM(S,Loeb) also fails in this model. □

Theorem 10. (i) *In every permutation model, CAC(Loeb) iff CAC(sel) iff 421 iff CSM(sel,sel) iff CPM(Loeb,Loeb) iff CSM(Loeb,Loeb).*

(ii) *In ZF^0, none of CAC(Loeb), CAC(sel), CSM(sel,sel), CPM(Loeb,Loeb), CSM(Loeb,Loeb) implies CAC.*

(iii) *In ZF^0, CUC does not imply any of the statements appearing in part (i).*

(iv) *In every permutation model, (CAC $\rightarrow$ CPM(Loeb,Loeb)) is true.*

(v) *In ZF, CAC does not imply CPM(sel,Loeb).*

Proof: (i) Let $\mathcal{N}$ be a permutation model. The implications (421$\rightarrow$CSM(sel,sel)) and (421 $\rightarrow$ CSM(Loeb,Loeb)) follow at once from the observation that every selective or Loeb metric space is well orderable in a permutation model. Indeed, from Theorems 3(i) and 5(i) of the introduction, it follows that every selective, hence Loeb, metric space (X,d) has cardinality at most $|\aleph^\omega|$ for some infinite well ordered cardinal number $\aleph$ (if G is a well ordered dense subset of X of size $\aleph$, then since X is first countable, one readily constructs for each $x \in X$ a sequence $\mathbf{x}$ of points in G which converges to x. Then the function which maps each $x \in X$ to $\mathbf{x}$ is clearly 1:1). Since the statement *the powerset of a well orderable set is well orderable* (= Form 91 in [2]) holds true in every permutation model, see [2], it follows that X is well orderable.

(CAC(Loeb) $\leftrightarrow$ CAC(sel)) and (CPM(sel,sel) $\leftrightarrow$ CPM(Loeb,Loeb)) follow from the fact that in permutation models the notions selective and Loeb coincide (since in such models these spaces are well orderable).

(CAC(sel) $\rightarrow$ 421). Let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be a pairwise disjoint family of well orderable sets. For each $i \in \mathbb{N}$, let κ_i be the unique aleph such that $|A_i| = \kappa_i$. For every $i \in \mathbb{N}$, define $B_i = \{f \in A_i^{\kappa_i} : f$ is a bijection$\}$, and let d_i be the discrete metric on B_i. Since $B_i \subset \wp(\kappa_i \times A_i)$ and the powerset of a well orderable set is well orderable in every permutation model, it follows that B_i is well orderable, hence (B_i, d_i) is a selective metric space. By CAC(sel) let f be a choice function on the family $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$. On the basis of the functions $f(i)$, $i \in \mathbb{N}$, it is straightforward to construct a well-ordering on $\bigcup \mathcal{A}$.

(CSM(Loeb,Loeb) $\rightarrow$ CPM(Loeb,Loeb)). Fix $\{(X_i, d_i) : i \in \mathbb{N}\}$ a family of pairwise disjoint Loeb metric spaces. Since each X_i is well orderable and

CSM(Loeb,Loeb) implies 421, we have that $X = \bigcup\{X_i : i \in \mathbb{N}\}$ is well orderable. Therefore, $A = \wp(\omega \times X)$ is well orderable and since $Y = \prod_{i\in\mathbb{N}} X_i \subseteq A$ we have that Y is a well orderable metrizable space, hence Y is a Loeb space as required.

(ii) It is well known, see [2], that in the basic Fraenkel permutation model (model $\mathcal{N}1$ in [2]) CAC fails whereas 421 holds. The independence result follows from part (i) of the present theorem.

(iii) In [11] a permutation model $\mathcal{N}$ was constructed in which CUC is true. Let us recall the description of $\mathcal{N}$. The set of atoms $A = \bigcup\{A_n : n \in \omega\}$, where $A_n = \{a_{n,x} : x \in \mathbb{R}\}$ and A_n is ordered like the reals by $\leq_n$. Thus, $(A_n, \leq_n)$ is order isomorphic to $(\mathbb{R}, \leq)$ for all $n \in \omega$. $\mathcal{G}$ is the group of all permutations π on A such that $\pi|_{A_n} \in \mathrm{Aut}(A_n, \leq_n)$ for all $n \in \omega$, where $\mathrm{Aut}(A_n, \leq_n)$ is the group of all order automorphisms on A_n. The normal ideal I of supports is the ideal generated by the set of all finite unions $\bigcup_{i\leq n} A_i$, $n \in \omega$. $\mathcal{N}$ is the resulting permutation model. In [13] it is shown that AC(WO) fails in $\mathcal{N}$ for the family $\mathcal{A} = \{A_n : n \in \omega\}$. Since for each $n \in \omega$, A_n is well orderable in $\mathcal{N}$ (A_n is a support of A_n), it follows that 421 is false in $\mathcal{N}$ and the independence result follows from part (i) of the present theorem.

(iv) Clearly CAC implies CAC(Loeb), hence by (i) it follows that if $\mathcal{N}$ is a permutation model which satisfies CAC, then CPM(Loeb,Loeb) is valid in $\mathcal{N}$.

(v) Clearly CPM(sel,Loeb) implies M(sel,Loeb) which, by Theorem 5(iv), implies 212. In Feferman's forcing model, $\mathcal{M}2$ model in [2], CAC holds whereas 212 fails in that model, see [hr]. Thus, CPM(sel,Loeb) also fails in $\mathcal{M}2$. $\square$

Theorem 11. (i) *In ZF, none of CAC(Loeb), CAC(sel) implies* 212.

(ii) *In ZF, AC($\mathbb{R}$) does not imply any of the statements CAC(Loeb), CAC(sel), CPM(Loeb,Loeb), CSM(Loeb,Loeb), CPM(sel,sel).*

(iii) *In ZF, the statements M(Loeb,hLoeb) and P, $P \in \{$CAC(Loeb), CAC(sel), CPM(sel,sel), CSM(Loeb,Loeb)$\}$, are mutually independent.*

(iv) *In ZF, $AC_{\aleph^\omega}$ does not imply CPM(Loeb,Loeb).*

(v) *In ZF, FPM(Loeb,Loeb) does not imply CPM(Loeb,Loeb).*

Proof: (i) In Feferman's model $\mathcal{M}2$ in [2], CAC(sel), hence CAC(Loeb) (see Theorem 5(i)), holds whereas 212 fails in that model.

(ii)–(v) In [16, Theorem 2.1] a symmetric extension $(\mathcal{N}, \in)$ of a countable, transitive model $(\mathcal{M}, \in)$ of ZF + $V = L$ was constructed by using an ω_1-closed partially ordered set $\mathbb{P} \in \mathcal{M}$ (see [14, Definition 6.12, p. 214]) of forcing conditions. It follows that all cardinals of the model $\mathcal{M}$ are preserved in $\mathcal{N}$ and for every $\aleph \in \mathcal{M}$, no new functions $f \in \aleph^\omega$ are added in $\mathcal{N}$, see [14, pp. 214–215, Theorems 6.14 and 6.16]. Consequently, $(\aleph^\omega)^{\mathcal{M}} = (\aleph^\omega)^{\mathcal{N}}$, hence $2^{\aleph_0} = \aleph_1$ in $\mathcal{N}$ which means that AC($\mathbb{R}$) is true of $\mathcal{N}$. Furthermore, since AC is valid in the ground model $\mathcal{M}$, $\aleph^\omega$ is well orderable in $\mathcal{N}$ for every well ordered cardinal $\aleph$.

Now, in [13] it is shown that the statement $\mathrm{WO}_{\aleph^\omega}$: *For every* $\aleph$, $\aleph^\omega$ *is well orderable* implies M(Loeb,hLoeb), thus, in view of the above, the latter proposition, as well as $\mathrm{AC}_{\aleph^\omega}$ and FPM(Loeb,Loeb) (see Theorems 5(iv), (v) and 8(viii)) hold true in the forcing model $\mathcal{N}$.

Furthermore, in Theorem 2.1 of [16], it is shown that in the model $\mathcal{N}$ there exists a countable family $A = \{(A_n, d_n) : n \in \mathbb{N}\}$ of metric spaces such that for all $n \in \mathbb{N}$, (A_n, d_n) is homeomorphic to the unit circle $B(0, 1/n)$, hence A_n is compact, and $\prod_{n\in\mathbb{N}} A_n = \emptyset$ in $\mathcal{N}$. Now, $B(0, 1/n)$ being a closed subset of the Loeb metric space $\mathbb{R}^2$, see Corollary 2, is also Loeb, therefore for all $n \in \mathbb{N}$, (A_n, d_n) is a Loeb, hence selective, metric space in $\mathcal{N}$. As $\prod_{n\in\mathbb{N}} A_n = \emptyset$, we deduce that all the statements CAC(Loeb), CAC(sel), CPM(Loeb,Loeb), CSM(Loeb,Loeb) and CPM(sel,sel) fail in the model $\mathcal{N}$.

Finally, to see that none of $P \in$ {CAC(Loeb), CAC(sel), CPM(sel,sel), CSM(Loeb,Loeb)} implies M(Loeb,hLoeb), notice that P holds in $\mathcal{M}2$ (due to CAC) but M(Loeb,hLoeb) fails (since the latter implies 212, see Theorem 5(iv), (v)). □

References

[1] Herrlich H., Strecker G.E., *When is* $\mathbb{N}$ *Lindelöf?*, Comment. Math. Univ. Carolinae **38** (1998), 553–556.

[2] Howard P., Rubin J.E., *Consequences of the Axiom of Choice*, Amer. Math. Soc., Math. Surveys and Monographs, **Vol. 59**, Providence (RI), 1998.

[3] Jech T., *The Axiom of Choice*, North-Holland, Amsterdam, 1973.

[4] Keremedis K., *The failure of the axiom of choice implies unrest in the theory of Lindelöf metric spaces*, Math. Log. Quart. **49** (2003), no. 2, 179–186.

[5] Kelley J., *The Tychonoff product theorem implies the axiom of choice*, Fund. Math. **37** (1950), 75–76.

[6] Keremedis K., *Consequences of the failure of the axiom of choice in the theory of Lindelöf metric spaces*, Math. Log. Quart. **50** (2004), no. 2, 1–11.

[7] Keremedis K., Tachtsis E., *On Lindelöf metric spaces and weak forms of the axiom of choice*, Math. Log. Quart. **46** (2000), 35–44.

[8] Keremedis K., Tachtsis E., *Compact metric spaces and weak forms of the axiom of choice*, Math. Log. Quart. **47** (2001), 117–128.

[9] Keremedis K., Tachtsis E., *On Loeb and weakly Loeb Hausdorff spaces*, Sci. Math. Jpn. **53** (2001), 247–251.

[10] Keremedis K., Tachtsis E., *Some weak forms of the axiom of choice restricted to the real line*, Math. Log. Quart. **47** (2001), 413–422.

[11] Keremedis K., Tachtsis E., *Countable sums and products of metrizable spaces in ZF*, Math. Log. Quart. **51** (2005), no. 1, 95–103.

[12] Keremedis K., Tachtsis E., *Topology in the absence of the axiom of choice*, Sci. Math. Jpn. **59** (2004), no. 2, 357–406.

[13] Keremedis K., Tachtsis E., *Weak axioms of choice for metric spaces*, to appear in Proc. Amer. Math. Soc.

[14] Kunen K., *Set Theory. An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1983.

[15] Loeb P.A., *A new proof of the Tychonoff theorem*, Amer. Math. Monthly **72** (1965), 711–717.

[16] Tachtsis E., *Disasters in metric topology without choice*, Comment. Math. Univ. Carolinae **43** (2002), no. 1, 165-174.
[17] Willard S., *General Topology*, Addison-Wesley Publ. Co., Reading, 1970.

Department of Mathematics, University of Bremen, P.O. Box 33 04 40, 28334 Bremen, Germany

E-mail: herrlich@math.uni-bremen.de

Department of Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece

E-mail: kker@aegean.gr

Department of Statistics and Actuarial Science, University of the Aegean, Karlovassi 83200, Samos, Greece

E-mail: ltah@aegean.gr

(*Received* May 19, 2004)