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A bifurcation theorem for noncoercive integral functionals

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Abstract. In this paper we study the existence of critical points for noncoercive functionals, whose principal part has a degenerate coerciveness. A bifurcation result at zero for the associated differential operator is established.

Keywords: critical points, noncoercive and nondifferentiable functionals, bifurcation points

Classification: 35B32, 35B38

1. Introduction and statement of the results

This paper is motivated by a recent study of Arcoya, Boccardo and Orsina (see [1]) on the existence of critical points of noncoercive functionals whose principal part has a degenerate coerciveness of the kind

$$\int_\Omega a(x,v)|\nabla v|^2, \ v \in H_0^1(\Omega)$$

where $a : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumption

$$\frac{c_1}{(1+|s|)^{2\alpha}} \leq a(x,s) \leq c_2$$

for almost every $x \in \Omega$ and all $s \in \mathbb{R}$.

They deal with the existence of critical points of functionals whose model is

$$J(v) = \frac{1}{2}\int_\Omega a(x,v)|\nabla v|^2 - \frac{1}{m}\int_\Omega |v|^m.$$

This functional, which is well defined thanks to the Sobolev embeddings if $m \leq 2^*$ (where $2^* = \frac{2N}{N-2}$), and weakly lower semicontinuous as it follows from the De Giorgi Theorem, is however non coercive on $H_0^1(\Omega)$ (see Example 3.3. of [2]). The lack of coerciveness implies that J may not attain its infimum in $H_0^1(\Omega)$.

Another difficulty arising in this problem is due to the differentiability of the functional in a proper subspace of $H_0^1(\Omega)$, that is $H_0^1(\Omega) \cap L^\infty(\Omega)$.

In order to prove existence and boundedness of minima the authors of [1] need a suitable relationship involving m and α, since the behaviour of J may be different depending on the assumption on m.

In the present paper we deal with the functional

$$J_\lambda(v) = \frac{1}{2}\int_\Omega a(x,v)|\nabla v|^2 - \lambda \int_\Omega F(x,v)$$

depending on a positive parameter λ. We will prove the existence of critical points of J_λ for small λ, just assuming a suitable behaviour of the nonlinearity F at zero without any growth assumption at infinity. In particular, it is possible to show that $\lambda = 0$ is a bifurcation point of J'_λ in $H^1_0(\Omega)$, that is $(0,0)$ belongs to the closure in $\mathbb{R} \times H^1_0(\Omega)$ of the set

$$\{(\lambda, u) \in \mathbb{R} \times (H^1_0(\Omega) \cap L^\infty(\Omega)) : u \ \text{ is a nontrivial critical point of } \ J_\lambda\}.$$

Let us state the precise assumptions on the functional J_λ that we will study below.

Here and in the sequel Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$. Let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, differentiable in $\mathbb{R}$ for almost every $x \in \Omega$, satisfying the following assumption

$$\frac{c_1}{(1+|s|)^{2\alpha}} \le a(x,s) \le c_2 \tag{1}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, where $c_1,\ c_2$ are positive constants and

$$0 \le \alpha < \frac{N}{2N-2} \tag{2}$$

(note that $\frac{N}{2N-2} \in \left]\frac{1}{2}, 1\right[$ for every $N > 2$).

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, such that $f(x,0) = 0$ for almost every x in Ω and F is defined by $F(x,v) = \int_0^v f(x,s)\,ds$. We introduce, for $v \in H^1_0(\Omega)$, the functional

$$J_\lambda(v) = \frac{1}{2}\int_\Omega a(x,v)|\nabla v|^2 - \lambda \int_\Omega F(x,v).$$

If f satisfies the growth condition

$$|f(x,s)| \le c_3(1+|s|^{m-1}) \tag{3}$$

with $1 < m < 2^*$, then it is well known that J_λ is well defined in $H^1_0(\Omega)$ and Gateaux differentiable in $H^1_0(\Omega) \cap L^\infty(\Omega)$ (but not in $H^1_0(\Omega)$!), with derivative given by

$$\langle J'_\lambda(v), w\rangle = \int_\Omega a(x,v)\nabla v \nabla w + \int_\Omega a_s(x,v)|\nabla v|^2 w - \lambda \int_\Omega f(x,v)w$$

for every v and w in $H^1_0(\Omega) \cap L^\infty(\Omega)$.

Our results are

Theorem 1. *Let us assume that there exists* $\delta > 1$ *such that*

$$\sup_{0\le s\le\delta}\ \sup_{x\in\Omega}|f(x,s)| < +\infty.$$

Moreover, suppose that there are a non-empty open set $D \subseteq \Omega$ *and a set* $B \subseteq D$ *of positive measure such that*

$$\limsup_{\xi\to 0^+}\frac{\inf_{x\in B}\int_0^\xi f(x,s)\,ds}{|\xi|^2} = +\infty, \quad \liminf_{\xi\to 0^+}\frac{\inf_{x\in D}\int_0^\xi f(x,s)\,ds}{|\xi|^2} > -\infty. \tag{4}$$

Then, there exists a $\lambda^* > 0$ *such that for all* $0 < \lambda < \lambda^*$, J_λ *has at least a nonnegative critical point* u_λ *in* $H_0^1(\Omega)\cap L^\infty(\Omega)\setminus\{0\}$. *Moreover one has*

$$\lim_{\lambda\to 0^+}\|u_\lambda\|_{H_0^1(\Omega)} = 0$$

and the function $\lambda \to J_\lambda(u_\lambda)$ *is negative and decreasing in* $]0,\lambda^*[$.

Theorem 2. *Let us assume that there exists* $\delta > 1$ *such that*

$$\sup_{-\delta\le s\le 0}\ \sup_{x\in\Omega}|f(x,s)| < +\infty$$

and

$$\limsup_{\xi\to 0^-}\frac{\inf_{x\in B}\int_0^\xi f(x,s)\,ds}{|\xi|^2} = +\infty, \quad \liminf_{\xi\to 0^-}\frac{\inf_{x\in D}\int_0^\xi f(x,s)\,ds}{|\xi|^2} > -\infty \tag{5}$$

with B *and* D *as in Theorem* 1.

Then, there exists a $\lambda^* > 0$ *such that for all* $0 < \lambda < \lambda^*$, J_λ *has at least a non positive critical point* v_λ *in* $H_0^1(\Omega)\cap L^\infty(\Omega)\setminus\{0\}$. *Moreover one has*

$$\lim_{\lambda\to 0^+}\|v_\lambda\|_{H_0^1(\Omega)} = 0$$

and the function $\lambda \to J_\lambda(v_\lambda)$ *is negative and decreasing in* $]0,\lambda^*[$.

Theorem 3. *Let us assume that there exists* $\delta > 1$ *such that*

$$\sup_{|s|\le\delta}\ \sup_{x\in\Omega}|f(x,s)| < +\infty$$

and

$$\limsup_{\xi\to 0^+}\frac{\inf_{x\in B}\int_0^\xi f(x,s)\,ds}{|\xi|^2} = \limsup_{\xi\to 0^-}\frac{\inf_{x\in B}\int_0^\xi f(x,s)\,ds}{|\xi|^2} = +\infty,$$

$$\liminf_{\xi\to 0} \frac{\inf_{x\in D}\int_0^\xi f(x,s)\,ds}{|\xi|^2} > -\infty \tag{6}$$

with B and D as in Theorem 1.

Then, there exists a $\lambda^* > 0$ such that for all $0 < \lambda < \lambda^*$, J_λ has at least two nontrivial critical points u_λ and v_λ in $H_0^1(\Omega) \cap L^\infty(\Omega) \setminus \{0\}$ where u_λ is nonnegative and v_λ is nonpositive. Moreover, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{H_0^1(\Omega)} = \lim_{\lambda\to 0^+} \|v_\lambda\|_{H_0^1(\Omega)} = 0$$

and the functions $\lambda \to J_\lambda(u_\lambda)$ and $\lambda \to J_\lambda(v_\lambda)$ are negative and decreasing in $]0, \lambda^*[$.

We notice that in our results no growth assumption on f is required.
At first we assumed (3) with $1 < m < 2^*(1-\alpha)$. Under this assumption and (4) or (5) we were able to prove (even in the case $m = 2(1-\alpha)$) the existence of a nontrivial minimum for J_λ in $H_0^1(\Omega) \cap L^\infty(\Omega)$. The boundedness of the solution induced us to check whether the growth assumption could be removed, by means of a suitable truncation of the nonlinearity.

We would like to mention briefly the assumptions made in [1].

In Theorem 1.1 of [1] the authors assume (3) with $1 < m < 2(1-\alpha)$ and a stricter assumption at zero on the nonlinearity, that is

$$\lim_{s\to 0} \frac{F(x,s)}{s^2} = +\infty \tag{7}$$

uniformly with respect to $x \in \Omega$. If $2 < m < 2^*(1-\alpha)$ the authors need further assumptions on F (see Theorem 1.2 of [1]) in order to apply a suitable Mountain Pass Theorem for nondifferentiable functionals, namely

$$\lim_{s\to 0} \frac{F(x,s)}{s^2} = 0, \qquad \lim_{s\to +\infty} \frac{F(x,s)}{s^2} = +\infty \tag{8}$$

uniformly with respect to $x \in \Omega$, and $f(x,s)s \geq rF(x,s)$ for some $r > 2$ and all $|s| \geq s_0$. We notice that our existence theorems hold under rather different hypotheses on f if in these results we replace the nonlinearity f with λf.
Let us recall finally Theorem 1.3 of [1] where a positive parameter appears:

Theorem ([1, Theorem 1.3]). *Let f satisfy* (3) *with* $2(1-\alpha) < m < \min\{2, 2^*(1-\alpha)\}$ *and* (7).
Then, there exists a positive λ_0 such that the functional J_λ has at least a non trivial critical point in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for every $0 < \lambda < \lambda_0$.

It is easily seen that our Theorem 3 improves the above result: we are able to find under weakened assumptions two solutions of opposite sign and to give some more information on the energy functional.

We conclude this section with some examples.

Example 1 (of a function satisfying Theorem 3 but not condition (7) appearing in Theorems 1.1 and 1.3 of [1]).

Let $m \in]\frac{3}{2}, 2[$ and

$$f(x,s) = f(s) = \begin{cases} m|s|^{m-2}s\left(\sin\frac{1}{\sqrt{|s|}}+1\right) - \frac{1}{2}|s|^{m-\frac{5}{2}}s\cos\frac{1}{\sqrt{|s|}} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

We notice that, if $s \neq 0$,

$$\frac{F(x,s)}{s^2} = |s|^{m-2}\left(\sin\frac{1}{\sqrt{|s|}}+1\right)$$

and so our condition (6) holds while the limit of the quotient as $s \to 0$ does not exist.

Example 2 (of a function satisfying Theorem 3, but not Theorem 1.2 of [1]).

Let $r \in]1, 2[$, $m > 1$, and $f(x,s) = f(s) = |s|^{r-2}s + |s|^{m-2}s$. It is immediately seen that f satisfies condition (6) but it does not verify assumptions (8).

Notations. In the following we will use the following functions

$$T_k(s) = \max\{-k, \min\{k, s\}\},$$

and the following sets

$$A_k = \{x \in \Omega : |u(x)| \geq k\}$$

with $k > 0$. By $C, C_1, C_2, \ldots$ we will denote various positive constants whose values may vary from line to line.

2. Proofs

Our main tools are a recent variational principle by B. Ricceri that allows us to prove the existence of a minimum without requiring the coerciveness of the energy functional on the space and some regularity results that allow us to prove the boundedness of the minimum. For the convenience of the reader, we recall their statements:

Theorem A ([6, Theorem 2.5]). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals. Assume also that Ψ is (strongly) continuous and satisfies $\lim_{\|x\|\to+\infty} \Psi(x) = +\infty$. For each $\rho > \inf_X \Psi$, put*

$$\varphi(\rho) = \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi}{\rho - \Psi(x)},$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ is the closure in the weak topology.
Then, for each $\rho > \inf_X \Psi$ and each $\lambda > \varphi(\rho)$, the restriction of the functional $\Phi + \lambda\Psi$ to $\Psi^{-1}(]-\infty,\rho[)$ has a global minimum.

Lemma B ([3, Lemma 2.1]). *Let w be a function in $W_0^{1,\sigma}(\Omega)$ with $\sigma < N$ such that for $k \geq k_0 > 0$,*

$$\int_{A_k} |\nabla w|^\sigma \leq c k^{\theta\sigma} |A_k|^{\frac{\sigma}{\sigma^*}+\varepsilon},$$

where $\varepsilon, c > 0$, $0 \leq \theta < 1$, $\sigma^* = \frac{\sigma N}{N-\sigma}$. Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant depending on c, θ, σ, N, ε, k_0, $|\Omega|$.

Lemma C ([5, Lemma 5.2]). *Let w be a function in $W_0^{1,\sigma}(\Omega)$ with $\sigma < N$ such that for $k \geq k_0 > 0$,*

$$\int_{A_k} |\nabla w|^\sigma \leq c k^{\sigma} |A_k|^{1-\frac{\sigma}{N}+\varepsilon},$$

where $\varepsilon, c > 0$. Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant depending on c, σ, N, ε, k_0, $\|w\|_{L^1(A_{k_0})}$.

The following lemma will be useful in the sequel.

Lemma 4. *If $\alpha \in]0, \frac{N}{2N-2}[$ and*

$$q = \frac{2N(1-\alpha)}{N-2\alpha} \tag{9}$$

then, for every $u \in W_0^{1,q}(\Omega)$ such that $a(x,u(x))|\nabla u(x)|^2$ belongs to $L^1(\Omega)$, one has

$$\int_\Omega |\nabla u|^q \leq C \left(\int_\Omega a(x,u)|\nabla u|^2\right)^{\frac{q}{2}} \left(\int_\Omega (1+|u|)^{q^*}\right)^{1-\frac{q}{2}}.$$

Proof: See [1]. □

The technique used in the proof of our results is analogous to the one used in [1]: we extend our functional to a larger space where assumptions of Theorem A are satisfied.

Proof of Theorem 1: Let us introduce the following functions

$$\tilde f(x,s) = \begin{cases} 0 & \text{if } s<0, \\ f(x,s) & \text{if } 0 \leq s \leq \delta, \\ f(x,\delta) & \text{if } s > \delta, \end{cases} \qquad \tilde a(x,s) = \begin{cases} a(x,0) & \text{if } s<0, \\ a(x,s) & \text{if } s \geq 0. \end{cases}$$

Clearly $\tilde f$ is a Carathéodory function, satisfying condition (4) and

$$|\tilde f(x,s)| \leq L \equiv \sup_{0\leq s\leq\delta} \sup_{x\in\Omega} |f(x,s)|$$

for every $x \in \Omega$ and $s \in \mathbb{R}$; $\tilde{a}$ is differentiable in $\mathbb{R}$ for almost every x in Ω, satisfying (1) and (2).

Let q be as in Lemma 4. We notice that $1 < q < 2$. In $W_0^{1,q}(\Omega)$ we introduce the following functionals

$$\Psi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \tilde{a}(x,u)|\nabla u|^2 & \text{if } \int_\Omega \tilde{a}(x,u)|\nabla u|^2 < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\Phi(u) = -\int_\Omega \tilde{F}(x,u)$$

where $\tilde{F}(x,u) = \int_0^u \tilde{f}(x,s)ds$. It is easily seen that Ψ is continuous in $W_0^{1,q}(\Omega)$, while the weak lower semicontinuity of the same functional on $W_0^{1,q}(\Omega)$ is a consequence of the De Giorgi Theorem ([4]). Moreover, Ψ is coercive on $W_0^{1,q}(\Omega)$: reasoning as in [1], let $u \in W_0^{1,q}(\Omega)$, such that $\Psi(u)$ is finite. Hence, using Lemma 4 and Poincaré inequality, one has

$$\begin{aligned} \int_\Omega |\nabla u|^q \le\ & C\Psi(u)^{\frac{q}{2}} \left(\int_\Omega (1+|u|)^{q^*}\right)^{1-\frac{q}{2}} \\ \le\ & C\Psi(u)^{\frac{q}{2}} \left(1 + \left(\int_\Omega |\nabla u|^q\right)^{\frac{q^*}{q}}\right)^{(1-\frac{q}{2})} \\ \le\ & C\Psi(u)^{\frac{q}{2}} \left(1 + \left(\int_\Omega |\nabla u|^q\right)^{\frac{q^*}{q}(1-\frac{q}{2})}\right). \end{aligned}$$

The coerciveness of Ψ in $W_0^{1,q}(\Omega)$ follows from $\frac{q^*}{q}(1-\frac{q}{2}) = \alpha < 1$.

In a standard way it is shown that Φ is weakly lower semicontinuous on $W_0^{1,q}(\Omega)$. Thus, we can apply Theorem A with $X = W_0^{1,q}(\Omega)$: let $\tilde{\rho} > 0 = \inf_{W_0^{1,q}(\Omega)} \Psi$ such that $\varphi(\tilde{\rho}) > 0$. Put $\lambda_0 = \min\{\frac{1}{\varphi(\tilde{\rho})}, 1\}$. For every $\lambda \in]0, \lambda_0[$, the restriction of $\tilde{J}_\lambda = \Psi + \lambda\Phi$ to $\Psi^{-1}]-\infty, \tilde{\rho}[$, has a global minimum, say u_λ.

We are going to prove now that u_λ is different from zero.
Let us prove that

$$\liminf_{\|u\|\to 0} \frac{\Phi(u)}{\Psi(u)} = -\infty\,. \tag{10}$$

Thanks to our assumptions we can fix a sequence $\{\xi_k\}$ of positive numbers converging to zero and two constants σ, and Γ with $\sigma > 0$ such that

$$\lim_{k\to+\infty} \frac{\inf_{x\in B} \int_0^{\xi_k} \tilde{f}(x,s)ds}{|\xi_k|^2} = +\infty$$

and

$$\inf_{x\in D}\int_0^{\xi}\tilde f(x,s)ds \geq \Gamma|\xi|^2$$

for all $\xi \in [0,\sigma]$. Next, fix a set $C \subset B$ of positive measure. It is possible to construct a function $v \in H_0^1(\Omega)$ such that $v(x) \in [0,1]$ for all $x \in \Omega$, $v(x) = 1$ for all $x \in C$, and $v(x) = 0$ for all $x \in \Omega \setminus D$.
Let $Q > 0$, and choose T such that

$$Q < \frac{T|C| + \Gamma\int_{D\setminus C}|v|^2}{\frac{c_2}{2}\int_\Omega|\nabla v|^2}.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_k < \sigma$ and

$$\inf_{x\in B}\int_0^{\xi_k}\tilde f(x,s)ds \geq T|\xi_k|^2$$

for all $k > \nu$. For $k > \nu$, one has

$$\begin{aligned}-\frac{\Phi(\xi_k v)}{\Psi(\xi_k v)} &\geq \frac{\int_C\left(\int_0^{\xi_k}\tilde f(x,s)\,ds\right)dx + \int_{D\setminus C}\left(\int_0^{\xi_k v(x)}\tilde f(x,s)\,ds\right)dx}{\frac{c_2}{2}\xi_k^2\int_\Omega|\nabla v|^2} \\ &\geq \frac{T|C| + \Gamma\int_{D\setminus C}|v|^2}{\frac{c_2}{2}\int_\Omega|\nabla v|^2} > Q.\end{aligned} \tag{11}$$

From (11), clearly (10) follows. Hence, there is a sequence $\{w_k\}$ in $W_0^{1,q}(\Omega)$ converging to zero, such that for k large enough we have $w_k \in \Psi^{-1}(]-\infty,\tilde\rho[)$, and

$$\Psi(w_k) + \lambda\Phi(w_k) < 0.$$

Since u_λ is a global minimum of the restriction of $\Psi + \lambda\Phi$ to $\Psi^{-1}(]-\infty,\tilde\rho[)$, it is proved that $\tilde J_\lambda(u_\lambda) < 0$.

Our next step consists in proving that u_λ belongs to $L^\infty(\Omega)$.
Let $k \geq 1$. It is easily seen that $T_k(u_\lambda)$ belongs to $W_0^{1,q}(\Omega)$ and that $\Psi(T_k(u_\lambda)) < \tilde\rho$. Hence, $\tilde J_\lambda(u_\lambda) \leq \tilde J_\lambda(T_k(u_\lambda))$. This means that, if $A_k^\lambda = \{x \in \Omega : |u_\lambda(x)| \geq k\}$,

$$\begin{aligned}\int_{A_k^\lambda}\tilde a(x,u_\lambda)|\nabla u_\lambda|^2 &\leq 2\lambda\int_{A_k^\lambda}[\tilde F(x,u_\lambda) - \tilde F(x,T_k(u_\lambda))] \\ &\leq 2\lambda\int_{A_k^\lambda}[|\tilde F(x,u_\lambda)| + |\tilde F(x,T_k(u_\lambda))|] \\ &\leq C\lambda\left(\int_{A_k^\lambda}|u_\lambda| + k|A_k^\lambda|\right)\end{aligned}$$

$$\leq C\lambda|A_k^\lambda|^{1-\frac{1}{q^*}}\left(\int_{A_k^\lambda}|u_\lambda|^{q^*}\right)^{\frac{1}{q^*}}+C\lambda k|A_k^\lambda|$$
$$\leq C\lambda|A_k^\lambda|^{1-\frac{1}{q^*}}\left(\int_{A_k^\lambda}|\nabla u_\lambda|^{q}\right)^{\frac{1}{q}}+C\lambda k|A_k^\lambda|.$$

Now, since $\frac{q^*}{q}(1-\frac{q}{2})=\alpha$,

$$\begin{aligned}\int_{A_k^\lambda}|\nabla u_\lambda|^q &= \int_{A_k^\lambda}\frac{|\nabla u_\lambda|^q}{(1+|u_\lambda|)^{\alpha q}}(1+|u_\lambda|)^{\alpha q}\\ &\leq \left(\int_{A_k^\lambda}\frac{|\nabla u_\lambda|^2}{(1+|u_\lambda|)^{2\alpha}}\right)^{\frac{q}{2}}\left(\int_{A_k^\lambda}(1+|u_\lambda|)^{q^*}\right)^{1-\frac{q}{2}}\\ &\leq C\left(\int_{A_k^\lambda}\tilde{a}(x,u_\lambda)|\nabla u_\lambda|^2\right)^{\frac{q}{2}}\left(\int_{A_k^\lambda}|u_\lambda|^{q^*}\right)^{1-\frac{q}{2}}\\ &\leq C\left(\int_{A_k^\lambda}\tilde{a}(x,u_\lambda)|\nabla u_\lambda|^2\right)^{\frac{q}{2}}\left(\int_{A_k^\lambda}|\nabla u_\lambda|^{q}\right)^{\alpha}\\ &\leq C\lambda^{\frac{q}{2}}|A_k^\lambda|^{\frac{q}{2}(1-\frac{1}{q^*})}\left(\int_{A_k^\lambda}|\nabla u_\lambda|^{q}\right)^{\alpha+\frac{1}{2}}\\ &+ C\lambda^{\frac{q}{2}}k^{\frac{q}{2}}|A_k^\lambda|^{\frac{q}{2}}\left(\int_{A_k^\lambda}|\nabla u_\lambda|^{q}\right)^{\alpha}.\end{aligned}\tag{12}$$

Let us suppose first that $\alpha<\frac{1}{2}$.

Dividing both members by $\left(\int_{A_k^\lambda}|\nabla u_\lambda|^q\right)^\alpha$ and taking the power $\frac{1}{1-\alpha}$, we finally obtain

$$\begin{aligned}\int_{A_k^\lambda}|\nabla u_\lambda|^q \leq\ & C\lambda^{\frac{q}{2(1-\alpha)}}|A_k^\lambda|^{\frac{q}{2(1-\alpha)}(1-\frac{1}{q^*})}\left(\int_{A_k^\lambda}|\nabla u_\lambda|^{q}\right)^{\frac{1}{2(1-\alpha)}}\\ &+ C\lambda^{\frac{q}{2(1-\alpha)}}k^{\frac{q}{2(1-\alpha)}}|A_k^\lambda|^{\frac{q}{2(1-\alpha)}}\\ \leq\ & \frac{1}{2(1-\alpha)}\int_{A_k^\lambda}|\nabla u_\lambda|^q+C\lambda^{\frac{q}{1-2\alpha}}|A_k^\lambda|^{\frac{q}{1-2\alpha}(1-\frac{1}{q^*})}\\ &+C\lambda^{\frac{q}{2(1-\alpha)}}k^{\frac{q}{2(1-\alpha)}}|A_k^\lambda|^{\frac{q}{2(1-\alpha)}}\end{aligned}$$

as it is easily seen applying Young inequality. Hence,

$$\begin{aligned} \int_{A_k^\lambda} |\nabla u_\lambda|^q \leq\ & C\lambda^{\frac{q}{1-2\alpha}} |A_k^\lambda|^{\frac{q}{1-2\alpha}(1-\frac{1}{q^*})} \\ & + C\lambda^{\frac{q}{2(1-\alpha)}} k^{\frac{q}{2(1-\alpha)}} |A_k^\lambda|^{\frac{q}{2(1-\alpha)}}. \end{aligned} \tag{13}$$

Now, it is easily seen that there exists a $\lambda^* < \lambda_0$ such that for every $k \geq 1$ and $0 < \lambda < \lambda^*$ one has $|A_k^\lambda| \leq 1$. We notice that $A_k^\lambda \subseteq A_1^\lambda$ for every $\lambda < \lambda_0$ and so it is enough to prove that $|A_1^\lambda| \leq 1$ for every $\lambda < \lambda^*$, by the monotonicity of Lebesgue's measure.
One has

$$\begin{aligned} \int_\Omega |\nabla u_\lambda|^q \leq\ & C\Psi(u_\lambda)^{\frac{q}{2}} \left(1 + \left(\int_\Omega |\nabla u_\lambda|^q\right)^\alpha\right) \\ \leq\ & C\Psi(u_\lambda)^{\frac{q}{2}} + C\Psi(u_\lambda)^{\frac{q}{2}} \left(\int_\Omega |\nabla u_\lambda|^q\right)^\alpha \\ \leq\ & C\Psi(u_\lambda)^{\frac{q}{2}} + C\Psi(u_\lambda)^{\frac{q}{2(1-\alpha)}} + \alpha \int_\Omega |\nabla u_\lambda|^q \end{aligned}$$

and so

$$\int_\Omega |\nabla u_\lambda|^q \leq C\Psi(u_\lambda)^{\frac{q}{2}} + C\Psi(u_\lambda)^{\frac{q}{2(1-\alpha)}}.$$

From the coerciveness of Ψ and the localization of the solutions given by Theorem A, i.e. $\Psi(u_\lambda) < \tilde{\rho}$, it follows that the set $\{u_\lambda : \lambda \in]0, \lambda_0[\}$ is bounded in $W_0^{1,q}(\Omega)$.

Thus, applying Poincaré inequality we have:

$$\begin{aligned} \Psi(u_\lambda) <\ & \lambda \int_\Omega \tilde{F}(x, u_\lambda) \leq \lambda C \|u_\lambda\|_{L^1(\Omega)} \\ \leq\ & \lambda C \|u_\lambda\|_{L^{q^*}(\Omega)} \leq \lambda C \|u_\lambda\|_{W_0^{1,q}(\Omega)} \leq \lambda C. \end{aligned}$$

Let us suppose that there exists a sequence λ_n of positive numbers converging to zero, such that $|A_1^{\lambda_n}| > 1$. Hence, from the above computations, we find out that u_{λ_n} (eventually passing to a subsequence) converges in measure to zero, i.e. for every $\varepsilon, \eta > 0$ there exists $\nu \in \mathbb{N}$ such that for every $n > \nu$ $|A_\eta^{\lambda_n}| < \varepsilon$.
Choosing $\eta = 1$ and $\varepsilon = 1$ we deduce the existence of ν_1 such that $|A_1^{\lambda_n}| < 1$ for every $n > \nu_1$ and our claim follows.

From (13) we deduce that

$$\int_{A_k^\lambda} |\nabla u_\lambda|^q \leq C\lambda^{\frac{q}{2(1-\alpha)}} k^{\frac{q}{2(1-\alpha)}} |A_k^\lambda|^{\frac{q}{2(1-\alpha)}}.$$

We can apply then Lemma B with $\sigma = q$, $\theta = \frac{1}{2(1-\alpha)}$ (that is less than 1 in according to our assumptions) and $\varepsilon = \frac{q}{N(1-\alpha)}$.
In particular, there exists C_1 that does not depend on λ such that $|A_{k^*}^\lambda| = 0$ with $k^* = 1 + C_1 \lambda^{\frac{1}{2(1-\alpha)}}$, i.e. $\|u_\lambda\|_{L^\infty(\Omega)} \leq k^*$. By definition of k^*, for λ smaller than a suitable positive number that we still call λ^*, we have $k^* \leq \delta$.

Let now $\alpha \geq \frac{1}{2}$.

We estimate now the right hand side of (12) in the following way:

$$\begin{aligned}\int_{A_k^\lambda} |\nabla u_\lambda|^q \leq\ & C\lambda^{\frac{q}{2}} |A_k^\lambda|^{\frac{q}{2}(1-\frac{1}{q^*})} \left(\int_{A_k^\lambda} |\nabla u_\lambda|^q \right)^{\alpha - \frac{1}{2}} \int_{A_k^\lambda} |\nabla u_\lambda|^q \\ & + C\lambda^{\frac{q}{2}} k^{\frac{q}{2}} |A_k^\lambda|^{\frac{q}{2}} \left(\int_{A_k^\lambda} |\nabla u_\lambda|^q \right)^{\alpha}.\end{aligned}$$

Reasoning as above, in both cases $\alpha = \frac{1}{2}$, $\alpha > \frac{1}{2}$ it is possible to find a $\lambda^* < \lambda_0$ such that for every $0 < \lambda < \lambda^*$

$$C\lambda^{\frac{q}{2}} |A_k^\lambda|^{\frac{q}{2}(1-\frac{1}{q^*})} \left(\int_{A_k^\lambda} |\nabla u_\lambda|^q \right)^{\alpha - \frac{1}{2}} < \frac{1}{2}.$$

Then (12) becomes

$$\begin{aligned}\int_{A_k^\lambda} |\nabla u_\lambda|^q \leq\ & C\lambda^{\frac{q}{2}} k^{\frac{q}{2}} |A_k^\lambda|^{\frac{q}{2}} \left(\int_{A_k^\lambda} |\nabla u_\lambda|^q \right)^{\alpha} \\ \leq\ & C\lambda^{\frac{q}{2(1-\alpha)}} k^{\frac{q}{2(1-\alpha)}} |A_k^\lambda|^{\frac{q}{2(1-\alpha)}} + \alpha \int_{A_k^\lambda} |\nabla u_\lambda|^q\end{aligned}$$

applying again Young inequality. Finally we have

$$\int_{A_k^\lambda} |\nabla u_\lambda|^q \leq C\lambda^{\frac{q}{2(1-\alpha)}} k^{\frac{q}{2(1-\alpha)}} |A_k^\lambda|^{\frac{q}{2(1-\alpha)}}. \tag{14}$$

If $\alpha = \frac{1}{2}$, (14) reads as follows

$$\int_{A_k^\lambda} |\nabla u_\lambda|^q \leq C\lambda^q k^q |A_k^\lambda|^q. \tag{15}$$

If $\alpha > \frac{1}{2}$, one has

$$k^{q^*} \leq \frac{1}{|A_k^\lambda|} \int_{A_k^\lambda} k^{q^*} \leq \frac{1}{|A_k^\lambda|} \int_{A_k^\lambda} |u_\lambda|^{q^*} \leq \frac{C_1}{|A_k^\lambda|}.$$

Hence, from (14)

$$\begin{aligned}\int_{A_k^\lambda}|\nabla u_\lambda|^q \leq\ & C\lambda^{\frac{q}{2(1-\alpha)}}k^{q(\frac{1}{2(1-\alpha)}-1)}k^q|A_k^\lambda|^{\frac{q}{2(1-\alpha)}}\\ \leq\ & C\lambda^{\frac{q}{2(1-\alpha)}}k^q|A_k^\lambda|^{\frac{q}{2(1-\alpha)}-\frac{q}{q^*}(\frac{1}{2(1-\alpha)}-1)}.\end{aligned} \tag{16}$$

We are ready to apply Lemma C to both (15), (16) with $\sigma = q$, $\varepsilon = q + \frac{q}{N} - 1$ and $\varepsilon = \frac{q}{2(1-\alpha)}(1 - \frac{1}{q^*})$ respectively. (We notice that the constant given by Lemma C depends on the norm of u_λ in $L^1(A_k^\lambda)$, but it is possible to estimate it removing the dependance on u_λ).

In particular, there exist constants C_2 and C_3 not depending on λ such that $\|u_\lambda\|_{L^\infty(\Omega)} \leq 1 + C_2\lambda^{\frac{N}{2}}$ and $\|u_\lambda\|_{L^\infty(\Omega)} \leq 1 + C_3\lambda^{\frac{q}{2\varepsilon}}$ respectively. Hence, in both cases it is possible to find a positive value of λ, still denoted by λ^* such that $\|u_\lambda\|_{L^\infty(\Omega)} \leq \delta$. It is proved that u_λ belongs to $L^\infty(\Omega)$.

This implies that u_λ also belongs to $H_0^1(\Omega)$ since

$$\begin{aligned}\int_\Omega|\nabla u_\lambda|^2 \leq\ & (1+\|u_\lambda\|_{L^\infty(\Omega)})^{2\alpha}\int_\Omega\frac{|\nabla u_\lambda|^2}{(1+|u_\lambda|)^{2\alpha}}\\ \leq\ & C(1+\|u_\lambda\|_{L^\infty(\Omega)})^{2\alpha}\Psi(u_\lambda)\\ \leq\ & \lambda C(1+\|u_\lambda\|_{L^\infty(\Omega)})^{2\alpha}\|u_\lambda\|_{L^\infty(\Omega)}.\end{aligned}$$

We claim that $u_\lambda \geq 0$. Indeed if the set $B = \{x \in \Omega : u_\lambda(x) < 0\}$ has positive measure, then the restriction of u_λ to B belongs to $H_0^1(B)$. Moreover one has that

$$\begin{aligned}\tilde{J}'_\lambda(u_\lambda)(w) =\ & \int_B\tilde{a}(x,u_\lambda)\nabla u_\lambda\nabla w + \int_B\tilde{a}_u(x,u_\lambda)|\nabla u_\lambda|^2 w\\ & -\lambda\int_B\tilde{f}(x,u_\lambda)w = 0\end{aligned}$$

for every $w \in H_0^1(B) \cap L^\infty(\Omega)$. Choosing $w = u_\lambda$, we have

$$\int_B\tilde{a}(x,u_\lambda)|\nabla u_\lambda|^2 = 0$$

that implies $u_\lambda = 0$ in $H_0^1(B)$, that is an absurd.

We have proved that $0 \leq u_\lambda(x) \leq \delta$ for almost every $x \in \Omega$ and every $0 < \lambda < \lambda^*$ and it is clear that $J_\lambda(u_\lambda) < 0$.

Hence, it is immediately seen that u_λ is a critical point of J_λ:

$$\begin{aligned} 0 = \tilde{J}'_\lambda(u_\lambda)(w) = & \int_\Omega \tilde{a}(x,u_\lambda)\nabla u_\lambda \nabla w + \int_\Omega \tilde{a}_u(x,u_\lambda)|\nabla u_\lambda|^2 w \\ & - \lambda \int_\Omega \tilde{f}(x,u_\lambda)w \\ = & \int_\Omega a(x,u_\lambda)\nabla u_\lambda \nabla w + \int_\Omega a_u(x,u_\lambda)|\nabla u_\lambda|^2 w \\ & - \lambda \int_\Omega f(x,u_\lambda)w = J'_\lambda(u_\lambda)(w) \end{aligned}$$

for every $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We point out that from

$$\int_\Omega |\nabla u_\lambda|^2 \leq (1 + \|u_\lambda\|_{L^\infty(\Omega)})^{2\alpha}\Psi(u_\lambda),$$

it follows that $\lim_{\lambda\to 0}\|u_\lambda\|_{H_0^1(\Omega)} = 0$. Finally, if $0 < \lambda_1 < \lambda_2 < \lambda^*$, let

$$m_{\lambda_1} = \Phi(u_{\lambda_1}) + \frac{1}{\lambda_1}\Psi(u_{\lambda_1}) = \inf_{v\in\Psi^{-1}]-\infty,\tilde{\rho}[}\left(\Phi(v) + \frac{1}{\lambda_1}\Psi(v)\right)$$

and

$$m_{\lambda_2} = \Phi(u_{\lambda_2}) + \frac{1}{\lambda_2}\Psi(u_{\lambda_2}) = \inf_{v\in\Psi^{-1}]-\infty,\tilde{\rho}[}\left(\Phi(v) + \frac{1}{\lambda_2}\Psi(v)\right);$$

one has

$$J_{\lambda_1}(u_{\lambda_1}) = \lambda_1 m_{\lambda_1} > \lambda_2 m_{\lambda_1} \geq \lambda_2 m_{\lambda_2} = J_{\lambda_2}(u_{\lambda_2}).$$

Hence, the function $\lambda \to J_\lambda(u_\lambda)$ is decreasing in $]0,\lambda^*[$ and the proof is complete. □

Proof of Theorem 2: It is analogous to the proof of Theorem 1 with functions

$$\bar{f}(x,s) = \begin{cases} f(x,-\delta) & \text{if } s < -\delta, \\ f(x,s) & \text{if } -\delta \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases} \qquad \bar{a}(x,s) = \begin{cases} a(x,s) & \text{if } s < 0, \\ a(x,0) & \text{if } s \geq 0. \end{cases}$$

□

Proof of Theorem 3: It is a combination of the proofs of the previous theorems. □

Remark. It is worth noticing that in general $\tilde{J}_\lambda$ is not coercive.
For instance, let $\frac{1}{2} < \alpha < \frac{N}{2N-2}$, $r < 2$, $f(x,u) = u^{r-1}$ if $0 \le u \le \delta$, $a(x,u) = \frac{1}{(1+|u|)^{2\alpha}}$.

Let us choose $v \in H_0^1(\Omega)$ such that $\operatorname{ess\,inf} v > 0$. If $\tau > \frac{\delta}{\operatorname{ess\,inf} v}$, then

$$\begin{aligned}
\tilde{J}_\lambda(\tau v) &= \frac{\tau^2}{2}\int_\Omega \frac{|\nabla v|^2}{(1+\tau v)^{2\alpha}} - \lambda \int_\Omega \int_0^{\tau v} \tilde{f}(x,s)\,ds \\
&= \frac{\tau^{2(1-\alpha)}}{2}\int_\Omega \frac{|\nabla v|^2}{(\frac{1}{\tau}+v)^{2\alpha}} - \lambda \int_\Omega \left[\int_0^{\delta} \tilde{f}(x,s)\,ds + \int_\delta^{\tau v} \tilde{f}(x,s)ds\right] \\
&= \frac{\tau^{2(1-\alpha)}}{2}\int_\Omega \frac{|\nabla v|^2}{(\frac{1}{\tau}+v)^{2\alpha}} - \lambda \int_\Omega \left[\frac{\delta^r}{r} + \delta^{r-1}(\tau v - \delta)\right] \\
&= \frac{\tau^{2(1-\alpha)}}{2}\int_\Omega \frac{|\nabla v|^2}{(\frac{1}{\tau}+v)^{2\alpha}} + \lambda\left(1-\frac{1}{r}\right)\delta^r|\Omega| - \lambda\tau\delta^{r-1}\int_\Omega v.
\end{aligned}$$

Since $\alpha > \frac{1}{2}$, it is immediately seen that $2(1-\alpha) < 1$ and so the left hand side of the previous inequality goes to -∞ as τ tends to $+\infty$.

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