Quasiharmonic fields and Beltrami operators

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Abstract. A quasiharmonic field is a pair $\mathcal{F} = [B, E]$ of vector fields satisfying $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$, and coupled by a distorsion inequality. For a given $\mathcal{F}$, we construct a matrix field $\mathcal{A} = \mathcal{A}[B, E]$ such that $\mathcal{A}E = B$. This remark in particular shows that the theory of quasiharmonic fields is equivalent (at least locally) to that of elliptic PDEs.

Here we stress some properties of our operator $\mathcal{A}[B, E]$ and find their applications to the study of regularity of solutions to elliptic PDEs, and to some questions of G-convergence.

Keywords: quasiharmonic fields, Beltrami operator, elliptic partial differential equations, G-convergence

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1. Introduction

In the recent paper [8], the notion of quasiharmonic fields is introduced in order to stress and take advantage of the deep interplay between the theory of quasiconformal mappings and that of partial differential equations.

Given an open subset $\Omega \subset \mathbb{R}^n$ and two vector fields $B \in L^q(\Omega, \mathbb{R}^n)$, $E \in L^p(\Omega, \mathbb{R}^n)$, $1 < p, q < \infty$, $1/p + 1/q = 1$ satisfying $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$ in the sense of distributions, the pair $\mathcal{F} = [B, E]$ will be termed div-curl couple. An important example of a div-curl couple is associated naturally with a variational PDE; if

$$\operatorname{div}\, A(x)\nabla u = 0,$$

then we set $E = \nabla u$ and $B = A\,\nabla u$. The equation is elliptic if the coefficient matrix $A = A(x)$ satisfies

$$\frac{|X|^2}{K} \le \langle A(x)X, X\rangle \le K|X|^2 \quad \text{for all} \quad X \in \mathbb{R}^n \quad \text{and for a.e. } x \in \Omega \tag{1.1}$$

with a measurable function $K = K(x) \ge 1$ on Ω. Hereafter, $\langle\ ,\ \rangle$ denotes the scalar product in $\mathbb{R}^n$. Inequality (1.1) can be rephrased as

$$|A(x)X|^2 + |X|^2 \le \left(K(x) + \frac{1}{K(x)}\right)\langle A(x)X, X\rangle,$$

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for all $X \in \mathbb{R}^n$ and for a.e. $x \in \Omega$. A similar inequality is the so-called distortion inequality for the couple $\mathcal{F} = [B, E]$:

$$|B(x)|^2 + |E(x)|^2 \leq \left(K(x) + \frac{1}{K(x)}\right) \langle B(x), E(x) \rangle. \tag{1.2}$$

For a.e. $x \in \Omega$, it clearly implies that either $B(x) = E(x) = 0$, or $\langle B(x), E(x) \rangle > 0$. A K-quasiharmonic field is a div-curl couple $\mathcal{F} = [B, E]$ satisfying (1.2). The function $K = K(x)$ is called the distortion function of $\mathcal{F}$.

One of the key ingredients in the treatment of [8] is the construction of a symmetric matrix field $\mathcal{A} = \mathcal{A}(x)$ of the form

$$\mathcal{A} = \lambda I + \big(1/\lambda - \lambda\big) e \otimes e \tag{1.3}$$

with $\lambda = \lambda(x) > 0$ and a unit vector $e = e(x)$ of $\mathbb{R}^n$, $|e| = 1$, such that $\mathcal{A}(x)E(x) = B(x)$, for a.e. $x \in \Omega$.

In this paper, our first concern will be to present a simple and abstract construction of that type. It is of course equivalent to deal with linear operators on $\mathbb{R}^n$, or with square real matrices of order n, and we shall use either terminology, depending on the context. The argument we are going to present is point-wise, and thus, for given vectors $B, E \in \mathbb{R}^n$ satisfying $\langle B, E \rangle > 0$, we shall investigate the class $\mathcal{L} = \mathcal{L}(B, E)$ of all linear operators $A : \mathbb{R}^n \to \mathbb{R}^n$ which are symmetric, positive and map E onto B; that is, $A \in \mathcal{L}$ means that for arbitrary $X, Y \in \mathbb{R}^n$ we have

$$\langle AX, Y \rangle = \langle X, AY \rangle \quad , \quad \langle AX, X \rangle \geq 0 \quad , \quad AE = B.$$

We shall show that there exists an operator $\mathcal{A} \in \mathcal{L}$ of the form (1.3), where λ is a positive number and is found from the equality:

$$|B|^2 + |E|^2 = \big(1/\lambda + \lambda\big) \langle B, E \rangle. \tag{1.4}$$

We shall also stress some properties of such an operator $\mathcal{A}$ in certain sense showing its optimality in the class $\mathcal{L}$, regarding to ellipticity bounds.

Our interest for the class $\mathcal{L}$, and in particular for operators in $\mathcal{L}$ of the form (1.3), is motivated by the observation that operators of this type occur repeatedly in some problems in analysis; in addition to the already mentioned [8], we quote e.g. [15], [2], [9].

It is worth mentioning that a systematic method of finding a coefficient matrix from a given *div-curl* couple was first presented in [18].

The paper is organized as follows. In Section 2 we present our construction of the operator $\mathcal{A}$ and investigate some of its properties. In Section 3 we examine the particular case of dimension $n = 2$. Then we present some applications of the preceding study. The subject of Section 4 is a conjecture on the best integrability exponent for derivatives of solutions to an elliptic equation in dimension $n = 2$, see [13], [14], recently proved by Leonetti-Nesi [10]; the proof we present here is based on the arguments of the preceding sections. In Section 5 we see that the class of operators of the form (1.3) is relevant also in some questions on G-convergence.

2. The operator $\mathcal{A}$

Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, let $\{e_1,\dots,e_n\}$ be an orthonormal basis of eigenvectors of A and let $\lambda_1,\dots,\lambda_n$ be the corresponding eigenvalues. Thus we can write

$$A = \lambda_1 e_1 \otimes e_1 + \cdots + \lambda_n e_n \otimes e_n. \tag{2.1}$$

Throughout the paper, given $a = (a_1,\dots,a_n) \in \mathbb{R}^n$, the tensor product $a \otimes a$ denotes the symmetric matrix whose entries are the products $a_i a_j$.

Equality (2.1) is evident identifying, as we did, every matrix of $\mathbb{R}^{n\times n}$ with the operator which it represents. In this case, $a \otimes a$ is the operator defined by the formula

$$(a \otimes a)(x) = \langle a, x\rangle a, \qquad x \in \mathbb{R}^n.$$

Hence (2.1) is immediate on the vectors of the basis $\{e_1,\dots,e_n\}$. Clearly, identity (2.1) is equivalent to diagonalize the matrix A.

In the case $n = 2$, (2.1) reduces to

$$A = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2. \tag{2.2}$$

In particular, the identity matrix can be written as

$$I = e_1 \otimes e_1 + e_2 \otimes e_2 \tag{2.3}$$

and hence

$$A = \lambda_1 I + (\lambda_2 - \lambda_1) e_2 \otimes e_2. \tag{2.4}$$

Moreover if $\det A = 1$, then $\lambda_2 = 1/\lambda_1$. So a symmetric matrix $A \in \mathbb{R}^{2\times 2}$ has $\det A = 1$ if and only if it has the following representation

$$A = \lambda I + \big(1/\lambda - \lambda\big) e \otimes e \tag{2.5}$$

with $\lambda \in \mathbb{R} - \{0\}$ and $|e| = 1$.

In a general dimension n, if the matrix A has the form as in (2.5), e is an eigenvector of A, with eigenvalue $1/\lambda$, while λ is the eigenvalue corresponding to the eigenspace $\langle e\rangle^{\perp}$, hence we have

$$\det A = \lambda^{n-1} 1/\lambda = \lambda^{n-2}.$$

Now, we introduce some notation. Given a symmetric operator $A : \mathbb{R}^n \to \mathbb{R}^n$, we denote by m, M its minimum and the maximum eigenvalue, respectively, that is

$$m = \inf_{|X|=1} \langle AX, X\rangle, \qquad M = \sup_{|X|=1} \langle AX, X\rangle. \tag{2.6}$$

Moreover we denote $K = \max\{\lambda, 1/\lambda\}$, with λ defined in (1.4).

This section is devoted to prove the following

Proposition 2.1. *Let $B, E \in \mathbb{R}^n$ be two vectors satisfying $\langle B, E\rangle > 0$ and define $K \geq 1$ by the equality*

$$|B|^2 + |E|^2 = \big(K + 1/K\big)\langle B, E\rangle.$$

Then there exists an operator $\mathcal{A}$ of the form (1.3) *which satisfies*

$$\frac{|X|^2}{K} \leq \langle \mathcal{A}X, X\rangle \leq K|X|^2, \qquad \forall\, X \in \mathbb{R}^n. \tag{2.7}$$

Moreover, if $|B| = |E|$, then the operator $\mathcal{A}$ minimizes in the class $\mathcal{L}$ the ratio M/m between the maximum and the minimum eigenvalue. Finally, for every symmetric operator A such that $AE = B$, at least one of the following inequalities holds: $m \leq 1/K$ or $M \geq K$.

First, we construct an operator $\mathcal{A} \in \mathcal{L}$ of the form (1.3). Incidentally, we remark that our argument extends to handle, without any additional difficulties, instead of $\mathbb{R}^n$ the more general context of a vector space $\mathcal{H}$ endowed with a scalar product $\langle , \rangle$ and $\dim \mathcal{H} \geq 2$.

As by assumption $\langle B, E\rangle > 0$, we can find $\lambda > 0$ verifying (1.4), that is

$$|B|^2 + |E|^2 = \big(1/\lambda + \lambda\big)\langle B, E\rangle. \tag{2.8}$$

We examine two cases. If $B = \lambda E$, we take any vector $e \in \mathbb{R}^n$, $|e| = 1$, orthogonal to E: $\langle e, E\rangle = 0$. If $B \neq \lambda E$, we define

$$e = \frac{B - \lambda E}{|B - \lambda E|}\,.$$

Let us verify that then $\mathcal{A} \in \mathcal{L}$. The equality $\mathcal{A}E = B$ is trivial in the first case considered. Assuming $B \neq \lambda E$, we find

$$\mathcal{A}E = \lambda E + \left(\frac{1}{\lambda} - \lambda\right)\frac{\langle B - \lambda E, E\rangle}{|B - \lambda E|^2}(B - \lambda E)$$

and the equality follows again by noticing that (1.4) implies

$$\big(1/\lambda - \lambda\big)\frac{\langle B - \lambda E, E\rangle}{|B - \lambda E|^2} = 1.$$

Now we check (2.7). It is enough to consider the case $B \neq \lambda E$. Then we compute

$$\langle \mathcal{A}X, X\rangle = \lambda|X|^2 + \big(1/\lambda - \lambda\big)\langle e, X\rangle^2$$

which immediately implies (2.7) by applying Schwarz inequality.

Next, we minimize the ratio M/m over A in the class $\mathcal{L}$, that is, we consider the problem

$$\min\{M/m\ ;\ A \in \mathcal{L}\}\,. \tag{2.9}$$

Actually, it will be convenient to study the following problem, which is clearly equivalent to (2.9):

$$\min\left\{\sqrt{M/m} + \sqrt{m/M}\ ;\ A \in \mathcal{L}\right\}. \tag{2.10}$$

Lemma 2.1. *The minimum value at* (2.10) *equals* $\frac{2|B||E|}{\langle B,E\rangle}$.

Proof: First, we show that for each $A \in \mathcal{L}$ we have

$$\left\{\sqrt{M/m}+\sqrt{m/M}\right\} \geq 2|B||E|/\langle B,E\rangle. \tag{2.11}$$

To this end, we note that $(A-mI),(MI-A)$ are positive operator which commute and hence the composition $(A-mI)(MI-A)$ is positive as well ([11, p. 155]). Then for every $X \in \mathbb{R}^n$ we compute

$$\begin{aligned}0 \leq \langle (A-mI)(MI-A)X,X\rangle &= \langle MX-AX,AX-mX\rangle \\ &= (m+M)\langle AX,X\rangle - |AX|^2 - mM|X|^2,\end{aligned}$$

thus

$$|AX|^2 + mM|X|^2 \leq (m+M)\langle AX,X\rangle. \tag{2.12}$$

Now we recall that $AE = B$ and, therefore, for $X = E$ (2.12) yields

$$\begin{aligned}(m+M)\langle B,E\rangle \geq |B|^2+mM|E|^2 &= \sqrt{mM}|B||E|\left(\frac{1}{\sqrt{mM}\frac{|E|}{|B|}}+\sqrt{mM}\frac{|E|}{|B|}\right)\\ &\geq 2\sqrt{mM}|B||E|\end{aligned}$$

which immediately implies (2.11).

To conclude the proof of the lemma, we exhibit an operator in $\mathcal{L}$ for which equality at (2.11) occurs. For this, we reduce to the case $|B| = |E|$ introducing $B' = \frac{|E|}{|B|}B$ and considering the couple (B',E) in place of (B,E). It is clear that $A' \in \mathcal{L}(B',E)$ if and only if $A = \frac{|B|}{|E|}A' \in \mathcal{L}(B,E)$, the ratio M/m being the same for both A and A'. Thus we may assume $|B| = |E|$. In this case for the operator $\mathcal{A}$ defined in (1.3) we find

$$\frac{2|B||E|}{\langle B,E\rangle} = \frac{|B|^2+|E|^2}{\langle B,E\rangle} = \lambda + \frac{1}{\lambda} = K + \frac{1}{K}\,.$$

In view of (2.7) we have $\sqrt{M/m} \leq K$ and hence

$$2|B||E|/\langle B,E\rangle \geq \sqrt{M/m}+\sqrt{m/M}\,,$$

which shows that equality at (2.11) holds.
To proceed further, we need the following

Lemma 2.2. *For any symmetric operator A and number $K \geq 1$, the following conditions are equivalent to each other:*

(a) $|X|^2/K \leq \langle AX, X\rangle \leq K|X|^2, \ \forall\, X \in \mathbb{R}^n$;

(b) $|AX|^2 + |X|^2 \leq (K + 1/K)\,\langle AX, X\rangle, \ \forall\, X \in \mathbb{R}^n$.

Proof: Assuming (a), we have inequality (2.12) with $m = 1/K$ and $M = K$, which is exactly (b). For the opposite implication, we note that if (b) holds with $K = 1$, then $AX = X$ and (a) is clear. In case $K > 1$, (b) implies

$$0 \leq |AX - KX|^2 \leq (1 - K^2)\left(\langle AX, X\rangle/K - |X|^2\right)$$

so that $\langle AX, X\rangle \leq K|X|^2$. In the same way

$$0 \leq |AX - X/K|^2 \leq (K - 1/K)\left(\langle AX, X\rangle - |X|^2/K\right)$$

and hence

$$\langle AX, X\rangle \geq |X|^2/K.$$

To conclude the proof of Proposition 2.1, it remains to show that, if A is symmetric and $AE = B$, then $m \leq 1/K$ or $M \geq K$. We argue by contradiction, assuming that $1/K < m \leq M < K$. Set $K' = \max\{M, 1/m\}$, we therefore have $1 \leq K' < K$ and

$$|X|^2/K' \leq \langle AX, X\rangle \leq K'|X|^2, \quad \forall\, X \in \mathbb{R}^n.$$

By Lemma 2.2, for $X = E$, this implies

$$\left(K' + 1/K'\right)\langle B, E\rangle \geq |B|^2 + |E|^2 = (K + 1/K)\,\langle B, E\rangle,$$

that is,

$$K' + 1/K' \geq K + 1/K$$

which is in contrast with the definition of K'. □

Remark. In [9] the optimality of the ratio of eigenvalues has been investigated in connection with Koshelev's number [7].

3. Some special cases

First, we consider the 2-dimensional case. Then we have uniqueness of the operator $\mathcal{A}$ in (1.3) under the assumption $\det \mathcal{A} = 1$, that is, Proposition 2.1 specializes to the following

Proposition 3.1. *For given vectors* $E = (E_1, E_2)$, $B = (B_1, B_2)$ *of* $\mathbb{R}^2$ *satisfying* $\langle B, E\rangle > 0$, *there exists a unique symmetric* 2×2 *matrix* $\mathcal{A} = \mathcal{A}[B, E]$ *such that*

$$\begin{cases} \det \mathcal{A} = 1, \\ \mathcal{A}E = B. \end{cases} \tag{3.1}$$

If we set

$$\mathcal{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

we have

$$a = \frac{B_1^2 + E_2^2}{\langle B, E\rangle}; \qquad b = \frac{B_1B_2 - E_1E_2}{\langle B, E\rangle}; \qquad c = \frac{B_2^2 + E_1^2}{\langle B, E\rangle}. \tag{3.2}$$

We call $\mathcal{A}[B, E]$ the Beltrami operator associated with the couple of vectors $[B, E]$. Formulas (3.2) give the following, in case $B = AE$ for a given symmetric matrix

$$\begin{aligned} A &= \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \\ a &= \frac{\alpha^2E_1^2 + 2\alpha\beta E_1E_2 + (1 + \beta^2)E_2^2}{\langle AE, E\rangle}, \\ b &= \frac{\alpha\beta E_1^2 + (\alpha\gamma + \beta^2 - 1)E_1E_2 + \beta\gamma E_2^2}{\langle AE, E\rangle}, \\ c &= \frac{\gamma^2E_2^2 + 2\beta\gamma E_1E_2 + (1 + \beta^2)E_1^2}{\langle AE, E\rangle} \end{aligned} \tag{3.3}$$

(see [15]).
We note that the ellipticity condition for the matrix A

$$|X|^2/K \le \langle AX, X\rangle \le K|X|^2$$

is preserved precisely for $\mathcal{A}$.

Corollary 3.1. *Given a vector* $E = (E_1, E_2)$ *and a symmetric matrix* $A \in \mathbb{R}^{2\times 2}$ *such that*

$$|X|^2/K \le \langle AX, X\rangle \le K|X|^2,$$

there exists a symmetric matrix $\mathcal{A} \in \mathbb{R}^{2\times 2}$ *such that*

$$\det \mathcal{A} = 1, \qquad \mathcal{A}E = AE$$

and

$$|X|^2/K \le \langle \mathcal{A}X, X\rangle \le K|X|^2.$$

Formulas (3.3) easily generalize to the case of a nonsymmetric matrix A:

$$\begin{aligned} A &= \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, \\ a &= \frac{\alpha^2 E_1^2 + 2\alpha\beta E_1 E_2 + (1+\beta^2)E_2^2}{\langle AE, E\rangle}, \\ b &= \frac{\alpha\delta E_1^2 + (\alpha\gamma + \beta\delta - 1)E_1 E_2 + \beta\gamma E_2^2}{\langle AE, E\rangle}, \\ c &= \frac{\gamma^2 E_2^2 + 2\delta\gamma E_1 E_2 + (1+\delta^2)E_1^2}{\langle AE, E\rangle}. \end{aligned}$$

It is interesting to compare the constant in the ellipticity condition also in this non-symmetric case. So we assume

$$m|X|^2 \le \langle AX, X\rangle, \qquad |AX| \le M|X|, \qquad \forall\, X \in \mathbb{R}^2. \tag{3.4}$$

Clearly, as we require $B = AE$, the best values we can expect for $m,\ M$ are

$$m = \frac{\langle B, E\rangle}{|E|^2}, \qquad M = \frac{|B|}{|E|}. \tag{3.5}$$

Indeed, we have the following

Proposition 3.2. *Given $E, B \in \mathbb{R}^2$ with $\langle B, E\rangle > 0$, there exists a (not necessarily symmetric) matrix A such that $AE = B$ and satisfying (3.4), for m and M defined at (3.5). Moreover, there exists a symmetric matrix $\mathcal{A}$ of the form (1.3) with*

$$K = \max\{\lambda, 1/\lambda\} = \frac{1}{2}\left(\frac{1+M^2}{m} + \sqrt{\left(\frac{1+M^2}{m}\right)^2 - 4}\right).$$

The existence of the matrix A is trivial if B is proportional to E. Otherwise, we may assume $E = \binom{1}{0}$; writing $B = \binom{a}{b}$ we can choose

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

On the other hand, following our construction of $\mathcal{A}$, K is found by solving the equation, see (2.8),

$$K + \frac{1}{K} = \frac{|B|^2 + |E|^2}{\langle B, E\rangle} = \frac{1}{m}(1 + M^2).$$

Remark. Proposition 3.2 extends readily to the case $n \ge 3$. We define A on the 2-dimensional subspace spanned by E and B by the above argument, and on the orthogonal complement as the multiple of identity with factor $\langle B, E\rangle/|E|^2$ to guarantee (3.4).

4. Some applications

Let Ω be an open subset of $\mathbb{R}^2$ and $A = A(x)$ a 2×2 symmetric matrix of measurable functions verifying

$$|X|^2/K \leq \langle AX, X\rangle \leq K|X|^2$$

for all $X \in \mathbb{R}^2$, with $K \geq 1$.
Here we are concerned with weak solutions $u \in W^{1,2}(\Omega)$ to the elliptic equation

$$\operatorname{div}(A\nabla u) = 0. \tag{4.1}$$

We have the following

Theorem 4.1. *Every solution $u \in W^{1,2}(\Omega)$ to equation (4.1) satisfies*

$$|\nabla u| \in weak - L^{\frac{2K}{K-1}}(\Omega') \tag{4.2}$$

for every $\Omega' \subset\subset \Omega$. In particular, $|\nabla u| \in L^s_{loc}(\Omega)$, $\forall\, s \in [1, 2K/(K-1)[$.

Conjectures about the best integrability exponent for the derivatives of solutions to second order elliptic equations in dimension $n = 2$, and its relation to the best Hölder exponent for the solutions, were made by several authors, see [13], [14]. The above result is proved recently by Leonetti-Nesi [10]. Here we present a different proof, which is based on our construction of a matrix $\mathcal{A}$ with the same ellipticity bounds as A

$$|X|^2/K \leq \langle \mathcal{A}X, X\rangle \leq K|X|^2$$

satisfying

$$\det \mathcal{A} = 1 \qquad a.e. \tag{4.3}$$

and such that u solves the equation

$$\operatorname{div}(\mathcal{A}\nabla u) = 0, \tag{4.4}$$

see Section 3. We can assume Ω bounded and simply connected and u non constant. Then (see [1]) by (4.3) the measurable Riemann mapping theorem can be used to factor u as $u = w \circ f$ where $f : \Omega \to B(0,1)$ is K-quasiconformal and $w : B(0,1) \to \mathbb{R}$ is harmonic.
Let now $\Omega' \subset\subset \Omega$. Being f a homeomorphism, $f(\Omega') \subset\subset B(0,1)$ and hence for a.e. $x \in \Omega'$

$$|\nabla u(x)| = |\nabla w(f(x))Df(x)| \leq \left(\sup_{f(\Omega')} |\nabla w|\right) \cdot |Df(x)|.$$

By Astala's result $|Df|^2 \in weak - L^{\frac{K}{K-1}}(\Omega')$ as for any measurable subset $E \subset \Omega'$ we have

$$\int_E |Df|^2\, dx \leq K \int_E \det Df\, dx = K|f(E)| \leq C|E|^{\frac{1}{K}}$$

and the result of Theorem 4.1 follows.

5. Applications to G-convergence

Let Ω be a bounded open subset of $\mathbb{R}^n$, $K \geq 1$ be a constant and $A = (a_{ij})$ be a measurable $n \times n$-matrix valued function defined on Ω satisfying $a_{ij}(x) = a_{ji}(x)$ and

$$|X|^2/K \leq \sum_{ij} a_{ij}(x) X_i X_j \leq K|X|^2 \tag{5.1}$$

for a.e. $x \in \Omega$ and for all $X \in \mathbb{R}^n$. We denote by $\mathbb{M}(K)$ the set of such functions $A = A(x)$. As already noted, inequality (4.1) can be rephrased as

$$|AX|^2 + |X|^2 \leq (K + 1/K)\,\langle AX, X\rangle. \tag{5.2}$$

For each matrix $A \in \mathbb{M}(K)$, we can consider the elliptic differential operator

$$\mathbf{A} = \sum_{ij} D_i(a_{ij} D_j).$$

With abuse of notation, we shall say that $\mathbf{A}$ belongs to $\mathbb{M}(K)$. Being Ω bounded, $\mathbf{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism.

In this section, we study G-convergence of operators in $\mathbb{M}(K)$. Let $\mathbf{A}_h = \sum_{ij} D_i a_{ij}^h D_j \in \mathbb{M}(K)$, $h \in \mathbb{N}$, $\mathbf{A} \in \mathbb{M}(K)$; we say that $\{\mathbf{A}\}_h$ G-converges to $\mathbf{A}$, and write $\mathbf{A}_h \xrightarrow{G} \mathbf{A}$ (or even $A_h \xrightarrow{G} A$) if for all $F \in H^{-1}(\Omega)$ we have $\mathbf{A}_h^{-1} F \rightharpoonup \mathbf{A}^{-1} F$ weakly in $H_0^1(\Omega)$, that is, if u_h, u are respectively the solutions of the Dirichlet problems

$$\begin{cases} \sum_{ij} D_i(a_{ij}^h D_j u_h) = F \\ u_h \in H_0^1(\Omega) \end{cases} \qquad \begin{cases} \sum_{ij} D_i(a_{ij} D_j u) = F \\ u \in H_0^1(\Omega) \end{cases}$$

then $u_h \rightharpoonup u$ weakly in $H_0^1(\Omega)$.

The notion of G-convergence was introduced by Spagnolo [19]. For general properties we refer to [3], [17], [21]. Here we only mention some fundamental facts.

First, we recall the compactness of $\mathbb{M}(K)$ with respect to G-convergence: given a sequence $\{\mathbf{A}_h\}$ of operators in $\mathbb{M}(K)$, there exists an operator $\mathbf{A} \in \mathbb{M}(K)$ and a subsequence $\{\mathbf{A}_{h_k}\}$ G-converging to $\mathbf{A}$. We also recall that G-convergence is weaker than the L^1_{loc}-convergence of the coefficient matrices, and is not comparable with the $\sigma(L^\infty, L^1)$ weak convergence of the coefficient matrices. In this context, the result of [12] is illuminating: every operator $\mathbf{A}$ is the G-limit of a sequence of isotropic operators $\mathbf{B}_h = \sum_i D_i(\beta_h D_i)$. We conclude our summing up the properties of G-convergence by recalling that, if $\mathbf{A}_h \xrightarrow{G} \mathbf{A}$, then for every $F \in H^{-1}(\Omega)$, together with the convergence of the solutions u_h, u, we have also convergence of the momenta and of the energy densities: $\sum_{ij} a_{ij}^h D_j u_h \rightharpoonup \sum_{ij} a_{ij} D_j u$ weakly in $L^2(\Omega)$ and $\sum_{ij} a_{ij}^h D_i u_h D_j u_h \rightharpoonup \sum_{ij} a_{ij} D_i u D_j u$ in $\mathcal{D}'(\Omega)$.

We shall study G-convergence of operators $\mathbf{A}$ whose coefficient matrix is of the form

$$A = \frac{1}{K(x)} I + \left(K(x) - \frac{1}{K(x)} \right) e \otimes e \tag{5.3}$$

where $K(x) \geq 1$, $|e| = 1$. In dimension $n \geq 3$, it is easily seen that the structure (5.3) of A as sum of a scalar matrix and a rank-one matrix is not preserved under G-convergence.

Example. Here we construct a sequence of operators $\mathbf{A}_h$ with diagonal coefficient matrices of the form

$$\begin{pmatrix} a_h & 0 & \ldots\ldots & 0 \\ 0 & a_h & \ldots\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots\ldots & 0 & \dfrac{1}{a_h} \end{pmatrix}$$

G-converging to an operator which is not of this type. We assume that

$$a_h(x) = \prod_{j=1}^{n} a_h^{(j)}(x_j)$$

(where $x = (x_1, \ldots, x_n) \in \Omega$). In this case, there is a well known formula for the G-limit ([23]). The coefficient matrix is diagonal

$$\begin{pmatrix} \alpha_1 & 0 & \ldots\ldots & 0 \\ 0 & \alpha_2 & \ldots\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots\ldots & 0 & \alpha_n \end{pmatrix}$$

and the entries are given by

$$\begin{cases} \alpha_i = \nu_i \prod\limits_{j \neq i} \mu_j, \qquad i = 1, \ldots, n-1, \\[2ex] \alpha_n = \dfrac{1}{\mu_n} \dfrac{1}{\prod\limits_{j \neq n} \nu_j}, \end{cases} \tag{5.4}$$

where

$$a_h^{(j)} \rightharpoonup \mu_j, \qquad \frac{1}{a_h^{(j)}} \rightharpoonup \frac{1}{\nu_j} \qquad \sigma(L^\infty, L^1)$$

for $j = 1, \ldots, n$. Clearly we have $\nu_j \leq \mu_j$.

For simplicity, we consider the case $n = 3$. For given positive numbers α_1, α_2, α_3, relations (5.4) read as

$$\begin{cases} \nu_1\mu_2\mu_3 = \alpha_1 \\ \mu_1\nu_2\mu_3 = \alpha_2 \\ \nu_1\nu_2\mu_3 = 1/\alpha_3 \end{cases} \tag{5.5}$$

and the conditions $\nu_1 \leq \mu_1$, $\nu_2 \leq \mu_2$ are equivalent to

$$\alpha_2\alpha_3 \geq 1, \qquad \alpha_1\alpha_3 \geq 1. \tag{5.6}$$

A solution of system (5.5) is $\mu_1 = \mu_2 = \mu_3 = (\alpha_1\alpha_2\alpha_3)^{\frac{1}{3}}$, $\nu_1 = \alpha_1^{\frac{1}{3}}(\alpha_2\alpha_3)^{-\frac{2}{3}}$, $\nu_2 = \alpha_2^{\frac{1}{3}}(\alpha_1\alpha_3)^{-\frac{2}{3}}$. We remark that on the coefficients of the G-limit we have only the constrains (5.6), which do not imply the structure $\alpha_1 = \alpha_2 = 1/\alpha_3$. For example, we can approximate the operator with constant coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

by operators of type (5.3).

Now we assume that the dimension is $n = 2$. We consider a sequence $[B_h, E_h]$ of couples of vector fields on $\Omega \subset \mathbb{R}^2$, verifying for every $h \in \mathbb{N}$ the distortion inequality

$$|B_h|^2 + |E_h|^2 \geq \left(K + \frac{1}{K}\right)\langle B_h, E_h\rangle$$

and $\operatorname{div} B_h$, $\operatorname{curl} E_h$ belong to a compact set of $H^{-1}(\Omega)$. We consider the Beltrami operators associated with $[B_h, E_h]$, whose coefficient matrices are

$$\mathcal{A}_h = \mathcal{A}[B_h, E_h],$$

see Proposition 3.1. We also consider a couple $[B, E]$ verifying $\langle B, E\rangle > 0$ a.e. on Ω and consider the Beltrami operator associated with it : $\mathcal{A} = \mathcal{A}[B, E]$. We have the following

Theorem 5.1. *Under the above assumptions, if $B_h \rightharpoonup B$ and $E_h \rightharpoonup E$ weakly in L^2, then $\mathcal{A}_h \xrightarrow{G} \mathcal{A}$.*

By compactness, we can assume that $(\mathcal{A}_h)$ G-converges to an operator of class $\mathbb{M}(K)$: $\mathcal{A}_h \xrightarrow{G} A$. So we have to show that $A = \mathcal{A}$. This equality follows because the coefficient matrices of both A and $\mathcal{A}$ verify (3.1); this is clear by definition for $\mathcal{A}$, see Section 3. On the other hand, it is known that $AE = B$. The equality $\det A = 1$ a.e. is proven in the following

Lemma 5.1. *If $A_h \xrightarrow{G} A$ and $\det A_h = 1$ for any $h \in \mathbb{N}$, then also $\det A = 1$.*

This is a particular case of a result from [6].

The proof we present follows an argument due to Tartar and presented in [5].

Proof: Let $\mathcal{B}$ a ball such that $2\mathcal{B} \subset \Omega$, and let $u, v \in H_0^1(2\mathcal{B})$ satisfy

$$\nabla u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \nabla v = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = R \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{in } \mathcal{B}.$$

For each $h \in \mathbb{N}$, let $u_h, v_h \in H_0^1(2\mathcal{B})$ be the solution to

$$\operatorname{div} A_h \nabla u_h = \operatorname{div} A \nabla u,$$

$$\operatorname{div} A_h \nabla v_h = \operatorname{div} A \nabla v.$$

As $A_h \xrightarrow{G} A$, we have

$$\begin{aligned} \nabla u_h &\rightharpoonup \nabla u, & \nabla v_h &\rightharpoonup \nabla v \\ A_h \nabla u_h &\rightharpoonup A \nabla u, & A_h \nabla v_h &\rightharpoonup A \nabla u \end{aligned}$$

and then, by div-curl Lemma of Murat-Tartar ([16], [22]), we get

$$\langle A_h \nabla u_h, R^t A_h \nabla v_h \rangle \to \langle A \nabla u, R^t A \nabla v \rangle,$$
$$\langle \nabla u_h, R^t \nabla v_h \rangle \to \langle \nabla u, R^t \nabla v \rangle,$$

in $\mathcal{D}'(\Omega)$.

Finally, we note that, as $\det A_h = 1$ we have $A_h^{-1} = R^t A_h R$ and hence

$$\langle A_h \nabla u_h, R^t A_h \nabla v_h \rangle = \langle \nabla u_h, R^t \nabla v_h \rangle.$$

Therefore we find

$$\langle A \nabla u, R^t A \nabla v \rangle = \langle \nabla u, R^t \nabla v \rangle$$

which yields $\det A(x) = 1$ for a.e.$x \in \mathcal{B}$. □

We remark that Theorem 5.1 contains the result of Theorem 2 in [20].

Remark. In particular, by Theorem 5.1, $\mathcal{A}[B,E] \in \mathbb{M}(K)$, that is

$$|B|^2 + |E|^2 \geq \left(K + \frac{1}{K}\right)\langle B, E\rangle. \tag{5.7}$$

More generally [4] it can be proved that if

$$|B_h|^2 + |E_h|^2 \geq \left(K_h + \frac{1}{K_h}\right)\langle B_h, E_h\rangle,$$

$B_h \rightharpoonup B$, $E_h \rightharpoonup E$, $K_h \rightharpoonup k$ in L^1, then (5.7) holds.

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