Forcing with ideals generated by closed sets

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Abstract. Consider the poset $P_I = \mathrm{Borel}(\mathbb{R}) \setminus I$ where I is an arbitrary σ-ideal σ-generated by a projective collection of closed sets. Then the P_I extension is given by a single real r of an almost minimal degree: every real $s \in V[r]$ is Cohen-generic over V or $V[s] = V[r]$.

Keywords: forcing, descriptive set theory, large cardinals

Classification: 03E17, 03E55, 03E60

0. Introduction

Under suitable large cardinal hypotheses, every proper definable forcing for adding a single real is forcing-equivalent to the poset $\mathrm{Borel}(\mathbb{R}) \setminus I$ ordered by inclusion, for a suitable σ-ideal I ([Z2]). In this paper I will analyze the case of σ-ideals I σ-generated by a projective collection of closed sets. For such an ideal the forcing $\mathrm{Borel}(\mathbb{R}) \setminus I$ is always proper. The representatives include some familiar posets (Sacks real = $\mathrm{Borel}(\mathbb{R})$ minus the countable sets, Miller real = $\mathrm{Borel}(\omega^\omega)$ minus the modulo finite bounded sets, Cohen real = $\mathrm{Borel}(\mathbb{R})$ minus the meager sets) as well as posets as yet not used nor understood. Consider for example the forcing associated with the σ-ideal σ-generated by the closed measure zero sets ([B]), or with the σ-ideal σ-generated by closed sets of some fixed Hausdorff dimension.

The main result of this paper is

0.1 Theorem (ZFC+large cardinals)**.** *Let I be a σ-ideal σ-generated by a projective collection of closed sets. The poset $P_I = \mathrm{Borel}(\mathbb{R}) \setminus I$ is proper and adds a single real r_{gen} of an almost minimal degree: If $V \subseteq V[s] \subseteq V[r_{gen}]$ is an intermediate model for some real s, then $V[s]$ is a Cohen extension of V or else $V[s] = V[r_{gen}]$.*

However, the point of the paper is not exactly to prove this theorem. Rather, the point is to expose certain technologies that connect the descriptive set theory with the practice of definable proper forcing. Another point is to show that there are certain posets about which one can prove quite a bit by virtue of the

The author is partially supported by grants GA ČR 201-00-1466 and NSF DMS-0071437.

syntax of their definition, but (to date) one can prove absolutely nothing from the combinatorics of the related objects.

The notation in the paper follows the set theoretic standard of [J]. AD denotes the Axiom of Determinacy. If I is a σ-ideal on the reals the symbol P_I denotes the poset $\mathrm{Borel}(\mathbb{R}) \setminus I$ ordered by inclusion. The real line $\mathbb{R}$ is construed to be the set of all total functions from ω to ω. The large cardinal hypothesis needed for the proof of Theorem 0.1 can be specified to be "infinitely many Woodin cardinals" or less, depending on the descriptive complexity of the ideal I.

1. The results

1.1 Lemma. *If I is a σ-ideal on the real line then P_I forces "for some unique real r the generic filter is just the set $\{B^V : B$ is a Borel ground model coded set of reals with $r \in B\}$".*

Proof: Let $\dot{r}$ be the P_I-name for a real defined by $\dot{r}(\check{n}) \in \check{m}$ if the set $\{s \in \mathbb{R} : s(n) = m\}$ belongs to the generic filter. Note that this indeed defines a name for a total function from ω to ω since the collections $\{\{s \in \mathbb{R} : s(n) = m\} : m \in \omega\}$ are maximal antichains in the poset P_I for each integer $n \in \omega$. Also note that if any real is to be in the intersection of all Borel sets in the generic filter, it must be $\dot{r}$. In order to complete the proof, it is enough to argue by induction on the complexity of the Borel set $B \notin I$ that $B \Vdash \dot{r} \in \dot{B}$ where $\dot{B}$ is the Borel set in the extension with the same Borel definition as B.

Now this is clearly true if B is a closed or a basic open set. Suppose $B = \bigcap_n A_n$ and for all $n \in \omega$, $A_n \Vdash \dot{r} \in \dot{A}_n$ has been proved. Then for all $n \in \omega$ $B \subset A_n$, so $B \Vdash \forall n \in \omega\ \dot{r} \in \dot{A}_n$ and $B \Vdash \dot{r} \in \bigcap_n \dot{A}_n = \dot{B}$ as desired. And suppose that $B = \bigcup_n A_n$ and for all $n \in \omega$, $A_n \Vdash \dot{r} \in \dot{A}_n$ has been proved. The collection $\{A_n : A_n \notin I\}$ is a predense set in P_I below B — this simple observation uses the σ-completeness of the ideal I. So $B \Vdash \exists n \in \omega\ \dot{r} \in \dot{A}_n$, and $B \Vdash \dot{r} \in \bigcup_n \dot{A}_n = \dot{B}$ as desired. Since all Borel sets are obtained from closed sets and basic open sets by iterated applications of a countable intersection and union, the proof is complete. □

The unique real from the statement of the previous lemma will be called the P_I-generic real and denoted by $\dot{r}_{gen}$. This real is forced to fall out of all ground model Borel sets in the ideal I. Another standard piece of terminology: if M is an elementary submodel of some large structure and r is a real such that the set $\{B \in P_I \cap M : r \in B\}$ is an M-generic filter on P_I, then I will call the real r M-generic. The following basic complexity calculation will be used in many places in this paper.

1.2 Lemma. *Let I be a σ-ideal, $A \in P_I$ a positive Borel set, $\dot{s}$ a P_I-name for a real and M a countable elementary submodel of a large structure containing all of the heretofore mentioned objects. Then*

(1) *the set $A^* = \{r \in A : r$ is an M-generic real$\}$ is Borel,*

(2) *the function $r \in A^* \mapsto \dot{s}/r$ is Borel.*

Proof: Let $B_n : n \in \omega$ be an enumeration of the set $P_I \cap M$. Let also $O_m : m \in \omega$ be an enumeration of all open dense subsets of the poset P_I in the model M, and let $x_m \in 2^\omega$ be the numbers defined by $x_m(n) = 1 \leftrightarrow B_m \in O_n$. Clearly, the set $\{\langle r, x\rangle : r \in A \text{ and } \forall n\; r \in B_n \leftrightarrow x(n) = 1 \text{ and } \forall m \exists n\; x_m(n) = 1 \wedge x(n) = 1\}$ is Borel, and the set A^* is a one-to-one continuous image of this set — the image under the projection to the first coordinate. Thus the set A^* is Borel.

For (2) let $y_m \in 2^\omega$ be the function defined by $y_m(n) = 1 \leftrightarrow B_n \Vdash \dot{s}(\check{m}) = 1$. The set $\{\langle r, x, s\rangle : r \in A^* \text{ and } \forall n\; r \in B_n \leftrightarrow x(n) = 1 \text{ and } \forall m\; s(m) = 1 \leftrightarrow \exists n\; x(n) = 1 \wedge y_m(n) = 1\}$ is Borel and the function in question is the one-to-one continuous image of this set, namely the image under the projection to the first and third coordinate. Thus the function is Borel. □

1.3 Lemma. *If I is a σ-ideal σ-generated by closed sets then the poset P_I is $< \omega_1$-proper.*

Proof: Let me first show that the poset P_I is proper. Suppose $A \in P_I$ is an arbitrary condition and M is a countable elementary submodel of a large enough structure containing all the relevant information. I must produce a master condition $A^* \subset A$ for the model M. Consider the set A^* of all M-generic reals in A. This set is Borel by the previous lemma, and if it is I-positive, by Lemma 1.1 it forces $r_{gen} \in \dot{A}^*$, which is to say "$\dot{G} \cap \check{M} \subset \check{P}_I \cap \check{M}$ is $\check{M}$-generic", which is to say that A^* is a master condition for the model M. Note also that if $B \subset A$ is any other master condition for the model M, necessarily $B \setminus A^* \in I$. So A^* is really the only candidate for the required master condition. The only thing left to verify is $A^* \notin I$.

Suppose that $\{C_n : n \in \omega\}$ is a collection of closed sets in the ideal I. I must produce a real $r \in A^* \setminus \bigcup_n C_n$. Let $D_n : n \in \omega$ be a list of all open dense subsets of the poset P_I in the model M, and by induction on $n \in \omega$ build sets $A = A_0 \supset A_1 \supset A_2 \supset \dots$ in the model M so that for every $n \in \omega$, $A_{n+1} \in D_n$ and $A_{n+1} \cap C_n = 0$. To perform the inductive step, first choose a set $A_{n+0.5} \subset A_n$ in $M \cap D_n$ and then note that since the set $A_{n+0.5} \setminus C_n$ is I-positive and the set C_n is closed, there must be a basic open set O_n such that $O_n \cap C_n = 0$ and $A_{n+0.5} \cap O_n$ is still I-positive. But then the set $A_n = A_{n+0.5} \cap O_n$ is in the model M and satisfies the inductive assumptions. Once the induction is complete, look at the M-generic filter on $P_I \cap M$ generated by the sequence of A_n's. By the previous lemma applied in the model M, the intersection of all the sets in this filter is a singleton containing a real r. By the construction, $r \in A^* \setminus \bigcup_n C_n$ as desired.

The attentive reader will have noticed that the previous argument gives even strong properness of the poset P_I, see [S]. A slight variation of the argument will give $< \omega_1$-properness.

By induction on $\alpha \in \omega_1$ prove that the poset P_I is α-proper. The successor step is trivial on the account of the previously proved properness of P_I. So suppose that α is a limit ordinal, a limit of an increasing sequence $\alpha_0 \in \alpha_1 \in \alpha_2 \in \dots$ and for all $n \in \omega$ the α_n-properness of the poset P_I has been proved. Let $A \in P_I$ be an arbitrary condition and let $\langle M_\beta : \beta \in \alpha \rangle$ be a continuous $\in$-tower of countable elementary submodels of a large enough structure such that $A \in M_0$. As before, it is enough to show that the set $A^* = \{r \in A :$ for all $\beta \in \alpha$ the real r is M_β-generic$\}$ is I-positive, since it will be the required master condition for the tower.

Suppose that $\{C_n : n \in \omega\}$ is a collection of closed sets in the ideal I. I must produce a real $r \in A^* \setminus \bigcup_n C_n$. Let $M = \bigcup_{\beta \in \alpha} M_\beta$ and let $D_n : n \in \omega$ be a list of all open dense subsets of the poset P_I in the model M such that $D_n \in M_{\alpha_n+1}$, and by induction on $n \in \omega$ build sets $A = A_0 \supset A_1 \supset A_2 \supset \dots$ so that for every $n \in \omega$, $A_{n+1} \in D_n \cap M_{\alpha_n+1}$, $A_{n+1} \subset \{r \in A :$ the set $\{B \in P_I \cap M_\beta : r \in B\}$ is an M_β-generic filter, for all $\beta \in \alpha_n\}$, and $A_{n+1} \cap C_n = 0$. To perform the inductive step, first look at the set $A_n^* = \{r \in A_n : r$ is M_β-generic for all $\alpha_{n-1} \in \beta \in \alpha_n\}$. This set is Borel, it is I-positive by the α_n-properness of the poset P_I (it is the only candidate for the master condition for the tower $\langle M_\beta : \alpha_{n-1} \in \beta \in \alpha_n \rangle$) and it is in the model M_{α_n+1}. As in the second paragraph of this proof it is now possible to choose a set $A_{n+1} \subset A_n^*$ in $D_n \cap M_{\alpha_n+1}$ with $A_{n+1} \cap C_n = 0$. Such a set satisfies the inductive assumptions. Once the induction is complete, look at the M-generic filter on $P_I \cap M$ generated by the sequence of A_n's. By the previous lemma applied in the model M, the intersection of all the sets in this filter is a singleton containing a real r. By the construction, $r \in A^* \setminus \bigcup_n C_n$ as desired. □

It is well known that if I is a σ-ideal such that the forcing P_I is proper, and $B \Vdash \dot{s}$ is a real, then by using a stronger condition $C \subset B$ if necessary the name $\dot{s}$ can be reduced to a Borel function $f : C \to \mathbb{R}$ such that $C \Vdash \dot{s} = \dot{f}(\dot{r}_{gen})$. To see how this can be done, choose a countable elementary submodel M of a large enough structure, let C be the set of all M-generic reals in the set B and define $f : C \to \mathbb{R}$ by $f(r) = \dot{s}/r$. The function f is Borel by Lemma 1.2 and an absoluteness argument just like in the proof of the previous lemma shows that this function will work.

The following theorem is assembled from results of Martin and Solecki and appears in [S].

1.4 Lemma (ZFC+large cardinals)**.** *If I is a σ-ideal generated by closed sets, and if $A \subset \mathbb{R}$ is an I-positive projective set of reals then A has a Borel I-positive subset.*

And the key tool for establishing Theorem 0.1 is

1.5 Lemma (ZFC+projective uniformization)**.** *Suppose I is a σ-ideal such that*

(1) *I is generated by a projective collection of projective sets,*

(2) *every projective I-positive set has an I-positive Borel subset,*
(3) *the forcing P_I is $< \omega_1$-proper in all forcing extensions.*

If $G \subset P_I$ is a generic filter and $V \subseteq V[H] \subseteq V[G]$ is an intermediate model, then $V[H]$ is a c.c.c. extension of V or else $V[G] = V[H]$.

Here (1) means that there is an integer n such that the ideal is generated by boldface Σ^1_n sets and the set of all codes for boldface Σ^1_n sets in the ideal is itself projective.

Proof: Let I be a σ-ideal satisfying the assumptions of the lemma. On the account of (3) I can assume that the continuum hypothesis holds, because it can be forced by a σ-closed notion of forcing, not changing the poset P_I. Suppose that $\mathbb{B}$ is a nowhere c.c.c. complete subalgebra of the completion of the poset P_I. I will prove that the generic real $\dot{r}$ for the poset P_I can be recovered from the generic filter $\dot{H} \subset \mathbb{B}$.

First, a piece of notation: Suppose M is a countable elementary submodel of a large enough structure and r is a real. I will write $\dot{H} \cap M/r$ to denote the set $\{b \in \mathbb{B} \cap M :$ for some set $A \in P_I \cap M$, $A \leq b$ and $r \in A\}$. It is not difficult to see that if the real r falls out of all I-small sets in the model M, then $\dot{H} \cap M/r$ will be a filter on $\mathbb{B} \cap M$, and if the real r is M-generic for the poset P_I then this filter will be actually M-generic. The distinction between these two cases is the key point in the argument.

1.6 Claim. *There is an $\in$-sequence $\langle M_k : k \in \omega \rangle$ of countable elementary submodels of a sufficiently large structure such that for every infinite set $x \subset \omega$ the following set A_x is I-positive: $A_x = \{r \in \mathbb{R} : k \in x \leftrightarrow$ the set $\dot{H} \cap M_{k+1}/r$ is an M_{k+1}-generic filter on $\mathbb{B} \cap M_{k+1}\}$.*

Suppose that the claim has been proved and $M_k : k \in \omega$ are the ascertained models and M their union. Then for distinct infinite sets $x, y \subset \omega$, the sets A_x and A_y are disjoint and even more than that, if $r \in A_x$ and $s \in A_y$ are reals then the filters $\dot{H} \cap M/r$ and $\dot{H} \cap M/s$ are distinct subsets of the poset $B \cap M$. I am going to find a I-positive Borel set $B \subset \mathbb{R}$ such that $B \subset \bigcup_x A_x$ and for every infinite set $x \subset \omega$ the intersection $B \cap A_x$ contains at most one element. This will complete the proof since by an absoluteness argument between V and $V[G]$, $B \Vdash \dot{r}_{gen}$ is the unique real $r \in \dot{B}$ such that $\dot{H} \cap \check{M} = (\dot{H} \cap M)/r$. By another absoluteness argument between $V[H]$ and $V[G]$, this unique real must belong to the model $V[H]$. In other words B forces that $\dot{r}_{gen}$ can be reconstructed from $\dot{H} \cap \check{M}$ and so $V[G] = V[H]$.

To find the set $B \subset \mathbb{R}$, use the argument of Lemma 1.2 to note that the relation $r \in A_x$ is Borel. Let $U \subset [\omega]^{\aleph_0} \times \mathbb{R}$ be a Σ^1_n universal set and using the projective uniformization find a projective function f such that $\mathrm{dom}(f) = \{x \subset \omega :$ the vertical section U_x of U belongs to the ideal $I\}$ and for each $x \in \mathrm{dom}(f)$ the value $f(x)$ is an element of the I-positive set $A_x \setminus U_x$. Look at the projective set

$\mathrm{rng}(f) \subset \mathbb{R}$. This set must be I-positive since every generator of the ideal I is of the form U_x for some infinite set $x \subset \omega$ and then $f(x) \in \mathrm{rng}(f)$ is a real that does not belong to that generator. Now just let $B \subset \mathbb{R}$ be any Borel I-positive subset of $\mathrm{rng}(f)$ using the assumption (2).

All that remains to be done is the verification of the claim. First construct an $\in$-tower $\langle N_\alpha : \alpha \in \omega_1 \rangle$ of countable elementary submodels of a large enough structure so that

(1) at successor ordinals α $\langle N_\beta : \beta \in \alpha \rangle \in N_\alpha$ and at limit ordinals α $\omega_1 \cap N_\alpha = \bigcup_{\beta \in \alpha}(\omega_1 \cap N_\beta)$;
(2) if $\alpha = \omega_1 \cap N_\alpha$ then whenever possible subject to (1) the model N_α is such that there is a sequence $\langle N'_\beta : \beta \in \omega_1 \rangle$ in the model N_α with $N_\beta = N'_\beta$ for all $\beta \in \alpha$. The tower will necessarily be discontinuous at such ordinals;
(3) at limit ordinals α where (2) does not happen $N_\alpha = \bigcup_{\beta \in \alpha} N_\beta$.

Let me denote the set of all points $\alpha \in \omega_1$ at which (2) happens by D. The set D is uncountable. For if it were not, consider a countable elementary submodel M of the structure containing the sequence $\langle N_\alpha : \alpha \in \omega_1 \rangle$. Letting $\alpha = \omega_1 \cap M$ it should be that α is greater than all points in D but at the same time, $\alpha \in D$ as witnessed by the model M. Contradiction.

Now let $\langle \alpha_k : k \in \omega \rangle$ be the first ω many ordinals in the set D, and let $M_k = N_{\alpha_k}$ for every number $k \in \omega$. This is the required sequence of models, but why should the sets A_x be I-positive? For any infinite set $x \subset \omega$ consider the continuous $\in$-tower T_x indexed by the ordinals in the set $\{\alpha_{k+1} : k \in x\} \cup \bigcup\{(\alpha_k, \alpha_{k+1}] : k \notin x\}$ defined in the following way. If β is an ordinal in this set then the β-th model on this tower is just N_β unless $\beta = \alpha_{k+1}$ for $k \notin x$, where the β-th model is $\bigcup_{\gamma \in \beta} N_\gamma$ as dictated by the continuity requirement. Let $B_x = \{r \in \mathbb{R} : r \text{ is generic for every model on the tower } T_x\}$. This set is I-positive by the $< \omega_1$-properness of the poset P_I. The proof of the claim will be complete once I show that $B_x \subset A_x$.

Let $r \in B_x$ be a real. I must verify that $r \in A_x$. Well, if $k \in x$ then the model M_{k+1} is on the tower T_x, the real r is M_{k+1}-generic and the set $\dot{H} \cap M_{k+1}/r$ is an M_{k+1}-generic filter on $\mathbb{B} \cap M_{k+1}$ as required in the definition of the set A_x. But what if $k \notin x$? Look at the model M_{k+1} and choose in it a sequence $\langle N'_\beta : \beta \in \omega_1 \rangle$ such that for all $\beta \in \alpha_{n+1}$ $N_\beta = N'_\beta$ holds. Now since the algebra $\mathbb{B}$ is nowhere c.c.c. and of density $\aleph_1 = 2^{\aleph_0}$, it must be that $\mathbb{B} \Vdash$ for cofinally many ordinals $\beta \in \omega_1$ the filter $\dot{H} \cap \check{N}'_\beta$ is not $\check{N}'_\beta$-generic. But for all ordinals β between α_k and $\alpha_{k+1} = \omega_1 \cap M_{k+1}$ the models $N'_\beta = N_\beta$ are on the tower T_x and so both the real r and the filter $\dot{H} \cap N_\beta/r$ are N_β-generic. This means that the filter $\dot{H} \cap M_{k+1}/r$ cannot be M_{k+1}-generic by the elementarity of the model M_{k+1}. Thus $r \in A_x$ as required. $\square$

The assumptions of the Lemma feel somewhat ad hoc. I do not have any example of an ideal I satisfying the assumptions that would not be generated by analytic sets. I do not have an example of a definable ideal I such that the properness of the poset P_I would not be absolute throughout forcing extensions. I also do not have an example of a definable ideal I such that the forcing P_I is proper but not $< \omega_1$-proper. The problem here is that properness or $< \omega_1$-properness do not seem to be expressible as projective properties of the ideals. The conjecture though is that even in the presence of large cardinals there are such ideals.

The Lemma can be applied to posets like Laver forcing, if there is a suitable determinacy argument that verifies (1) and (2) of the Lemma for the poset. In the case of Laver forcing this has been done in [Z1, Section 3.2]. There is a fine line dividing the definable forcings into two groups: The P_I's for simply generated ideals I, and P_I's for ideals I for which no generating family consisting of simple sets can be found.

1.7 Example. *Assume that suitable large cardinals exist. Let I be the ideal of sets of subsets of ω which are nowhere dense in the algebra* Power(ω) *modulo finite. Then for every $n \in \omega$ there is $m \in \omega$ and a boldface Σ^1_m set in I which is not a subset of a Σ^1_n set in I.*

Proof: Consider the Mathias forcing. The results of [Z1, Section 3.4] show that this forcing is equivalent to P_I, and that every projective I-positive set has a Borel I-positive subset. Also the Mathias forcing is $< \omega_1$-proper. If the statement in 1.6 failed then Lemma 1.4 could be applied to say that all the intermediate extensions of the Mathias real extension are c.c.c. However, Mathias forcing can be decomposed into an iteration of a σ closed and c.c.c. forcing, and the first step in that iteration is certainly not c.c.c. A contradiction. $\square$

To argue for Theorem 0.1, fix a σ-ideal I σ-generated by a projective collection of closed sets. Lemmas 1.2 and 1.3 show that the assumptions of Lemma 1.4 are satisfied and so if r_{gen} is a V-generic real for the poset P_I and $s \in V[r_{gen}]$ is an arbitrary real, then $V[s]$ is a c.c.c. extension of V or $V[s] = V[r_{gen}]$. Let us investigate the case of $V[s]$ being a c.c.c. extension of V. Such a real s is obtained through a ground model I-positive Borel set B and a Borel function $f : B \to \mathbb{R}$ such that $B \Vdash \dot{f}(\dot{r}_{gen}) = \dot{s}$. Move back into the ground model and let $J = \{A \subset \mathbb{R}$ Borel: $B \Vdash \dot{s} \notin \dot{A}\}$. Clearly, J is a σ-ideal of Borel sets and the poset P_J is c.c.c.: an uncountable antichain in it would give an uncountable antichain in the algebra generated by the name $\dot{s}$. Since P_J is c.c.c. and the real $\dot{s}$ is forced to fall out of all J-small ground model coded Borel sets, the real $\dot{s}$ is actually forced to be P_J-generic. Now I will show that in a Cohen extension there is a generic real for the poset P_J, which will conclude the argument since all complete subalgebras of the Cohen algebra have countable density and therefore are Cohen themselves.

Let M be a countable elementary submodel of some large structure containing all the necessary information and look at the forcing $P_I \cap M$. This is a countable notion of forcing adding a canonical single real, which by an argument from Lemma 1.3 is forced to fall out of all ground model coded I-small sets. Force with the poset $P_I \cap M$ below the condition B, getting a real $d \in B$; so $V[d]$ is a Cohen generic extension of V. Look at the real $f(d)$. Whenever A is a ground model coded J-small set, the set $f^{-1}A$ is a ground model coded I-small set and so $d \notin f^{-1}A$ and $f(d) \notin A$. Thus the real $f(d)$ falls out of all ground model coded J-small sets and must be generic for P_J as required.

The last remark. Turning the history of forcing on its head, the understanding of the forcing P_I means finding a determinacy argument that will produce a dense subset of P_I consisting of combinatorially manageable sets, for example perfect sets in the case of Sacks forcing. Remarkably, in all known cases this also leads to the proof of the following proposition: for every I-positive Borel set B there is a Borel function $f : \mathbb{R} \to B$ such that the preimages of I-small sets are I-small. This property of the ideal I is critical in the proof that the covering number for I can be isolated, see [Z1]. Can such a feat be repeated for ideals like the closed measure zero ideal?

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(*Received* July 27, 2001, *revised* August 20, 2001)