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Weighted inequalities for commutators of one-sided singular integrals

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Abstract. We prove weighted inequalities for commutators of one-sided singular integrals (given by a Calderón-Zygmund kernel with support in $(-\infty, 0)$) with BMO functions. We give the one-sided version of the results in [C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. Appl., vol. 3 (6), 1997, pages 743–756] and [C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal., vol 128 (1), 1995, pages 163-185]. We improve these results for one-sided singular integrals by putting in the right hand side of the inequalities a smaller operator and a wider class of weights.

Keywords: one-sided weights, one-sided singular integrals

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1. Introduction

In this paper we obtain non standard weighted inequalities for commutators of singular integral operators given by a Calderón-Zygmund kernel K with support in $(-\infty, 0)$. This estimates will reflect a higher degree of singularity compared with the standard Calderón-Zygmund singular integral operators.

Let T denote a Calderón-Zygmund singular integral operator and M denote the Hardy-Littlewood maximal operator. Coifman proved in [C] that T and M satisfy

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C \int_{\mathbb{R}^n} |Mf|^p w, \tag{1.1}$$

for $0 < p < \infty$, $w \in A_\infty(\mathbb{R}^n)$ and f such that the left hand side is finite. This is a very important estimate in weighted theory since it implies the boundedness of T from $L^p(w)$ into $L^p(w)$, for $p > 1$, when $w \in A_p$.

Combining (1.1) with certain sharp two weighted inequalities for M one can derive a two weighted estimate for T with no assumption on the weight w: If

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T is a Calderón-Zygmund singular integral operator, Pérez [P1] proves that for $1 < p < \infty$,

$$\int_{\mathbb{R}^n} |Tf|^p w \le C \int_{\mathbb{R}^n} |f|^p M^{[p]+1} w, \tag{1.2}$$

where M^k is the k-times iterated of the Hardy-Littlewood maximal operator. The case $1 < p \le 2$ was first obtained in [W], but for singular integral operators with much stronger conditions on the kernel, namely they must be of convolution type with C^∞ kernel.

It is possible to generalize inequalities (1.1) and (1.2) for a large family of singular integral operators, i.e., the higher order commutators introduced by Coifman, Rochberg and Weiss in [CRcW]. Let K be a Calderón-Zygmund kernel. For appropriate b and f we define

$$T_b^k f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k K(x-y) f(y)\, dy,$$

$k = 0, 1, 2 \ldots$ (in the principal value sense). For $k = 1$ the operator is usually denoted by $[M_b, T] = M_b \circ T - T \circ M_b$, where M_b is the operator $M_b f = bf$, and b is called the symbol of the operator. These generalizations were given by Pérez in [P2]:

Theorem A ([P2])**.** *Let $0 < p < \infty$, $w \in A_\infty$ and $b \in \mathrm{BMO}$. Then there exists a constant C such that*

$$\int_{\mathbb{R}^n} |T_b^k f|^p w \le C \|b\|_{\mathrm{BMO}}^{kp} \int_{\mathbb{R}^n} \left(M^{k+1} f\right)^p w,$$

for all f such that the left hand side is finite.

Theorem B ([P2])**.** *Let $1 < p < \infty$ and $b \in \mathrm{BMO}$. Then for each weight w there exists a constant C such that*

$$\int_{\mathbb{R}^n} |T_b^k f|^p w \le C \|b\|_{\mathrm{BMO}}^{kp} \int_{\mathbb{R}^n} |f|^p M^{[(k+1)p]+1} w.$$

Recently, Aimar, Forzani and Martín-Reyes [AFM] have studied singular integral operators associated to a Calderón-Zygmund kernel with support in $(-\infty, 0)$ or $(0, \infty)$. They prove that the maximal operators which control these singular integrals are the one-sided Hardy-Littlewood maximal operators M^+ and M^- defined for locally integrable functions f by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|,$$

and the good weights for these operators are the one-sided weights introduced by Sawyer [S]. Their result improves (1.1) for singular integrals with kernel supported in $(-\infty,0)$ in two ways, by putting in the right hand side a smaller operator and by allowing a wider class of weights for which the inequality holds. More precisely, they prove that if T is a singular integral operator given by a kernel with support in $(-\infty,0)$ then there exists C such that

$$\int_{\mathbb{R}} |Tf|^p w \le C \int_{\mathbb{R}} |M^+ f|^p w, \tag{1.3}$$

for $0<p<\infty$ and $w\in A_\infty^+(\mathbb{R})$ (see [MPT] for the definition of $A_\infty^+(\mathbb{R})$).

The aim of this paper is to study the results of C. Pérez for this kind of singular integrals and to extend them in the double sense as in [AFM]. Our results are the following:

Theorem 1. *Let $0<p<\infty$, $k=0,1,\dots$, $w\in A_\infty^+$ and $b\in \mathrm{BMO}$. Let K be a Calderón-Zygmund kernel with support in $(-\infty,0)$ and let $T_b^{+,k}$ be defined (in the principal value sense) by*

$$T_b^{+,k} f(x) = \int_x^\infty (b(x)-b(y))^k K(x-y) f(y)\,dy.$$

Then there exists C such that

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w \le C\|b\|_{\mathrm{BMO}}^{kp} \int_{\mathbb{R}} \Big((M^+)^{k+1} f\Big)^p\, w$$

for all bounded functions f with compact support.

Corollary 1. *Under the same hypotheses as in Theorem 1, if $1<p<\infty$ and $w\in A_p^+$ then there exists C such that*

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w \le C\|b\|_{\mathrm{BMO}}^{kp} \int_{\mathbb{R}} |f|^p\, w$$

for all bounded functions f with compact support.

We also give a weak type result that generalizes the result in [P3] for this kind of singular integrals:

Theorem 2. *Let $w\in A_\infty^+$, $b\in\mathrm{BMO}$ and $T_b^{+,k}$ be as in Theorem 1. Then there exists C such that*

$$\begin{aligned} &w(\{x : |T_b^{+,k} f(x)| > \lambda\}) \\ &\qquad \le C\phi_k(\|b\|_{\mathrm{BMO}}^k) \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \big(1+\log^+(|f(x)|/\lambda)\big)^k\, M^- w(x)\,dx \end{aligned}$$

for all bounded functions f with compact support, where $\phi_k(t) = t(1+\log^+ t)^k$.

Corollary 2. *Under the same hypotheses as in Theorem* 2*, if* $w \in A_1^+$ *then there exists* C *such that*

$$w(\{x : |T_b^{+,k} f(x)| > \lambda\}) \\ \leq C\phi_k(\|b\|_{\mathrm{BMO}}^k) \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \left(1 + \log^+(|f(x)|/\lambda)\right)^k w(x)\, dx$$

for all bounded functions f *with compact support.*

Theorem 3. *Let* $1 < p < \infty$*,* $b \in \mathrm{BMO}$ *and* $T_b^{+,k}$ *be as in Theorem* 1*. Then, for each weight* w *there exists* C *such that*

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w \leq C\|b\|_{\mathrm{BMO}}^{kp} \int_{\mathbb{R}} |f|^p (M^-)^{[(k+1)p]+1} w \tag{1.4}$$

for all bounded functions f *with compact support.*

The case $k = 0$, i.e., the generalization of the result in [P1] for these singular integrals, can be found in [RRoT].

Clearly, every theorem has its analogue reversing the orientation of $\mathbb{R}$.

2. Definitions and preliminaries

We introduce some definitions and tools that we need for proving the main results.

Definition 2.1. We shall say that a function K in $L^1_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ is a Calderón-Zygmund kernel if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\left| \int_{\epsilon<|x|<N} K(x)\, dx \right| \leq B_1,$$

for all ϵ and all N with $0 < \epsilon < N$ and, furthermore, $\lim_{\epsilon \to 0^+} \int_{\epsilon<|x|<1} K(x)\, dx$ exists;

(b) there exists a finite constant B_2 such that

$$|K(x)| \leq \frac{B_2}{|x|}$$

for all $x \neq 0$;

(c) there exists a finite constant B_3 such that

$$|K(x-y) - K(x)| \leq B_3 |y| |x|^{-2}$$

for all x and y with $|x| > 2|y|$.

A one-sided singular integral T^+ is a singular integral associated to a Calderón-Zygmund kernel with support in $(-\infty, 0)$; therefore, in that case,

$$T^+ f(x) = \lim_{\epsilon\to 0^+} \int_{x+\epsilon}^{\infty} K(x-y) f(y)\, dy.$$

Examples of such kernels are given in [AFM].

F.J. Martín-Reyes and A. de la Torre introduced the one-sided sharp functions in [MT].

Definition 2.2. Let f be a locally integrable function. The one-sided sharp maximal function is defined by

$$M^{+,\#} f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy.$$

It is proved in [MT] that

$$M^{+,\#} f(x) \le \sup_{h>0} \inf_{a\in\mathbb{R}} \frac{1}{h} \int_x^{x+h} (f(y)-a)^+ \, dy + \frac{1}{h} \int_{x+h}^{x+2h} (a-f(y))^+ dy \le \|f\|_{\mathrm{BMO}}.$$

See [MT] for other results and definitions.

We shall also need the following maximal operators:

$$M_\epsilon^+ f(x) = (M^+ |f|^\epsilon(x))^{1/\epsilon} \quad \text{and} \quad M_\delta^{+,\#} f(x) = \left(M^{+,\#} |f|^\delta(x) \right)^{1/\delta}.$$

Now we give definitions and results about Young functions. A function $B : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. The Luxemburg norm of a function f associated to B is

$$\|f\|_B = \inf \left\{ \lambda > 0 : \int B\left(\frac{|f|}{\lambda} \right) \le 1 \right\},$$

and so the B-average of f over I is

$$\|f\|_{B,I} = \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int_I B\left(\frac{|f|}{\lambda} \right) \le 1 \right\}.$$

We will denote by $\overline{B}$ the complementary function associated to B (see [BS]). Then the generalized Hölder's inequality

$$\frac{1}{|I|} \int_I |f\, g| \le \|f\|_{B,I} \|g\|_{\overline{B},I},$$

holds. There is a further generalization that turns out to be useful for our purposes (see [O]). If A, B, C are Young functions such that

$$A^{-1}(t) B^{-1}(t) \le C^{-1}(t),$$

then

$$\|fg\|_{C,I} \le 2 \|f\|_{A,I} \|g\|_{B,I}.$$

Definition 2.3. For each locally integrable function f, the one-sided maximal operators associated to the Young function B are defined by

$$M_B^+ f(x) = \sup_{x<b} \|f\|_{B,(x,b)} \quad \text{and} \quad M_B^- f(x) = \sup_{a<x} \|f\|_{B,(a,x)}.$$

Definition 2.4. Let B be a Young function. We say that B satisfies the B_p condition, or that $B \in B_p$, $p > 1$, if there exists $c > 0$ such that

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} \approx \int_c^\infty \left(\frac{t^{p'}}{\overline{B}(t)}\right)^{p-1}\frac{dt}{t} < \infty.$$

The B_p condition appears for the first time in [P4]. The point of Definition 2.4 is that it implies the boundedness of M_B^+ from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$. In fact one has

Theorem C ([RRoT]). *Let $1 < p < \infty$, w be a weight and B be a Young function. Then the following statements are equivalent:*

(a) *$B \in B_p$;*

(b) *there exists C such that* $\displaystyle\int (M_B^+ f)^p w \le C \int |f|^p M^- w.$

We will be working most of the time with $B(t) = t(1+\log^+ t)^k$, $k \ge 0$ and for this B, it is proved in [RRoT] that

$$M_B^+ f \approx (M^+)^{k+1} f. \tag{2.1}$$

3. Proofs

To prove Theorem 1 we need the following lemma:

Lemma 1. *Let $0 < \delta < 1$. Then*

(a) *there exists $C = C_\delta > 0$ such that*

$$M_\delta^{+,\#}\left(T^+ f\right)(x) \le C M^+ f(x);$$

(b) *for each $b \in \mathrm{BMO}$, $\delta < \epsilon < 1$ and $k = 1, 2, \ldots$, there exists $C = C_{\delta,\epsilon} > 0$ such that*

$$M_\delta^{+,\#}\left(T_b^{+,k} f\right)(x) \le C \sum_{j=0}^{k-1} \|b\|_{\mathrm{BMO}}^{k-j} M_\epsilon^+ (T_b^{+,j} f)(x) + C\|b\|_{\mathrm{BMO}}^k (M^+)^{k+1} f(x).$$

Proof: We start by proving (b). Let λ be an arbitrary constant. Then $b(x) - b(y) = (b(x)-\lambda) - (b(y)-\lambda)$ and

(3.1)
$$\begin{aligned}
T_b^{+,k} f(x) &= \int_{\mathbb{R}} (b(x)-b(y))^k K(x-y) f(y)\, dy \\
&= \sum_{j=0}^{k} C_{j,k} (b(x)-\lambda)^j \int_{\mathbb{R}} (b(y)-\lambda)^{k-j} K(x-y) f(y)\, dy \\
&= T^+((b-\lambda)^k f)(x) \\
&\quad + \sum_{j=1}^{k} C_{j,k} (b(x)-\lambda)^j \int_{\mathbb{R}} (b(y)-\lambda)^{k-j} K(x-y) f(y)\, dy \\
&= T^+((b-\lambda)^k f)(x) \\
&\quad + \sum_{j=1}^{k} \sum_{s=0}^{k-j} C_{j,k,s} (b(x)-\lambda)^{s+j} \int_{\mathbb{R}} (b(x)-b(y))^{k-j-s} K(x-y) f(y)\, dy \\
&= T^+((b-\lambda)^k f)(x) \\
&\quad + \sum_{m=0}^{k-1} C_{k,m} (b(x)-\lambda)^{k-m} T_b^{+,m} f(x),
\end{aligned}$$

where $m = k-j-s$. Let us fix x and $h > 0$ and let $I = [x, x+8h]$. Then we write $f = f_1 + f_2$ where $f_1 = f\chi_I$. Taking into account (3.1), for all $a \in \mathbb{R}$, we have the following:

(3.2)
$$\begin{aligned}
&\left(\frac{1}{h} \int_x^{x+h} \left| |T_b^{+,k} f(y)|^\delta - |a|^\delta \right| dy \right)^{\frac{1}{\delta}} + \left(\frac{1}{h} \int_{x+h}^{x+2h} \left| |T_b^{+,k} f(y)|^\delta - |a|^\delta \right| dy \right)^{\frac{1}{\delta}} \\
&\le \left(\frac{1}{h} \int_x^{x+h} |T_b^{+,k} f(y) - a|^\delta\, dy \right)^{\frac{1}{\delta}} + \left(\frac{1}{h} \int_{x+h}^{x+2h} |T_b^{+,k} f(y) - a|^\delta\, dy \right)^{\frac{1}{\delta}} \\
&\le C \left[\sum_{m=0}^{k-1} \left(\frac{1}{h} \int_x^{x+2h} |b(y)-\lambda|^{(k-m)\delta} |T_b^{+,m} f(y)|^\delta\, dy \right)^{\frac{1}{\delta}} \right. \\
&\quad \left. + \left(\frac{1}{h} \int_x^{x+2h} |T^+((b-\lambda)^k f)(y) - a|^\delta\, dy \right)^{\frac{1}{\delta}} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left[\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-\lambda|^{(k-m)\delta}|T_b^{+,m}f(y)|^\delta\,dy\right)^{\frac{1}{\delta}}\right.\\
&\quad+\left(\frac{1}{h}\int_x^{x+2h}|T^+((b-\lambda)^kf_1)(y)|^\delta\,dy\right)^{\frac{1}{\delta}}\\
&\quad\left.+\left(\frac{1}{h}\int_x^{x+2h}|T^+((b-\lambda)^kf_2)(y)-a|^\delta\,dy\right)^{\frac{1}{\delta}}\right]\\
&=(I)+(II)+(III).
\end{aligned}
$$

Let $\lambda = b_I = \frac{1}{8h}\int_x^{x+8h} b(y)\,dy$. Since $0<\delta<\epsilon<1$, we can choose q such that $1<q<\frac{\epsilon}{\delta}$. Then, using Hölder's inequality for q and q', we get

$$
\begin{aligned}
(I) &\leq C\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_I|^{(k-m)\delta q'}\,dy\right)^{\frac{1}{\delta q'}}\times\\
&\qquad\times\left(\frac{1}{h}\int_x^{x+2h}|T_b^{+,m}f(y)|^{\delta q}\,dy\right)^{\frac{1}{\delta q}} \qquad (3.3)\\
&\leq C\sum_{m=0}^{k-1}\left[\left(\frac{1}{h}\int_x^{x+8h}|b(y)-b_I|^{(k-m)\delta q'}\,dy\right)^{\frac{1}{\delta q'(k-m)}}\right]^{k-m}\times\\
&\qquad\times\left(\frac{1}{h}\int_x^{x+2h}|T_b^{+,m}f(y)|^{\delta q}\,dy\right)^{\frac{1}{\delta q}}\\
&\leq C\sum_{m=0}^{k-1}\|b\|_{\mathrm{BMO}}^{k-m}M_{\delta q}^+(T_b^{+,m}f)(x)\\
&\leq C\sum_{m=0}^{k-1}\|b\|_{\mathrm{BMO}}^{k-m}M_{\epsilon}^+(T_b^{+,m}f)(x).
\end{aligned}
$$

Using that T^+ is of weak type (1,1), Kolmogorov's inequality gives that

$$(II)\leq C\frac{1}{h}\int_x^{x+2h}|b-b_I|^k|f|\chi_I(y)\,dy.$$

And by the generalized Hölder's inequality for $B(t)=t(1+\log^+t)^k$ and $\overline{B}(t)\approx e^{t^{1/k}}$ we get,

$$(II)\leq C\|b-b_I\|_{\overline{B},I}\|f\chi_I\|_{B,I}.$$

Now if $D(t) = e^t$, using the John-Nirenberg's inequality, we have

$$(3.4)\quad \begin{aligned}(II) &\le C\|b-b_I\|^k_{D,I}\|f\chi_I\|_{B,I} \le C\|b\|^k_{\mathrm{BMO}}M^+_B f(x)\\ &\le C\|b\|^k_{\mathrm{BMO}}(M^+)^{k+1}f(x).\end{aligned}$$

For (III) we take $a = T^+((b-b_I)^k f_2)(x+2h)$. Then, by Jensen's inequality,

$$(3.5)\quad (III) \le C\frac{1}{h}\int_x^{x+2h} |T^+((b-b_I)^k f_2)(y) - T^+((b-b_I)^k f_2)(x+2h)|\,dy.$$

For $j \ge 3$, let $I_j = [x+2^j h, x+2^{j+1}h]$ and $\tilde{I_j} = [x, x+2^{j+1}h]$. Using property (c) of the kernel K, for every $y \in [x, x+2h]$, we have

$$(3.6)\quad \begin{aligned}&|T^+((b-b_I)^k f_2)(y) - T^+((b-b_I)^k f_2)(x+2h)|\\ &\quad\le \int_{x+8h}^{\infty} \frac{x+2h-y}{(t-(x+2h))^2}|b(t)-b_I|^k|f(t)|\,dt\\ &\quad\le Ch\sum_{j=3}^{\infty}\int_{x+2^j h}^{x+2^{j+1}h} \frac{|b(t)-b_I|^k}{(t-(x+2h))^2}|f(t)|\,dt\\ &\quad\le Ch\sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j-2)^2 h}\frac{1}{2^{j+1}h}\int_{\tilde{I_j}} |b(t)-b_I|^k|f(t)|\,dt.\end{aligned}$$

Observe that by the generalized Hölder's inequality and using again the John-Nirenberg's inequality, we obtain

$$(3.7)\quad \begin{aligned}&\frac{1}{2^{j+1}h}\int_{\tilde{I_j}} |b(t)-b_I|^k|f(t)|\,dt\\ &\quad\le \frac{C}{2^{j+1}h}|b_{\tilde{I_j}} - b_I|^k\int_{\tilde{I_j}}|f(t)|\,dt + \frac{C}{2^{j+1}h}\int_{\tilde{I_j}}|b(t)-b_{\tilde{I_j}}|^k|f(t)|\,dt\\ &\quad\le C(2j)^k\|b\|^k_{\mathrm{BMO}}M^+f(x) + C\|b-b_{\tilde{I_j}}\|_{\overline{B},\tilde{I_j}}\|f\chi_{\tilde{I_j}}\|_{B,\tilde{I_j}}\\ &\quad\le C(2j)^k\|b\|^k_{\mathrm{BMO}}M^+f(x) + C\|b\|^k_{\mathrm{BMO}}(M^+)^{k+1}f(x).\end{aligned}$$

So inequalities (3.5), (3.6) and (3.7) give

$$(3.8)\quad \begin{aligned}(III) &\le C\sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j-2)^2}(2j)^k\|b\|^k_{\mathrm{BMO}}M^+f(x)\\ &\qquad + C\sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j-2)^2}\|b\|^k_{\mathrm{BMO}}(M^+)^{k+1}f(x)\\ &\le C\|b\|^k_{\mathrm{BMO}}(M^+)^{k+1}f(x).\end{aligned}$$

Putting together inequalities (3.2), (3.3), (3.4) and (3.8), we obtain that

$$M_\delta^{+,\#}\left(T_b^{+,k} f\right)(x) \leq C\|b\|_{\mathrm{BMO}}^k (M^+)^{k+1} f(x) + C \sum_{m=0}^{k-1} \|b\|_{\mathrm{BMO}}^{k-m} M_\epsilon^+ (T_b^{+,m} f)(x).$$

The proof of part (a) follows the same pattern as the proof of (b) but it is easier and therefore we omit it. □

We will now prove Theorem 1.

Proof of Theorem 1: Observe that the case $k = 0$ is the inequality for singular integrals with support in $(-\infty, 0)$ (see [AFM]). We will proceed by induction on k. So assume that the theorem is true for all $j \leq k$ and let us see how it follows the case $k+1$. Since $w \in A_\infty^+$, there exists $r > 1$ such that $w \in A_r^+$. Observe that for all $\delta > 0$ small enough, we have that $r < \frac{p}{\delta}$ and thus, $w \in A_{\frac{p}{\delta}}^+$. To apply Theorem 4 in [MT] we need $\|M_\delta^+(T_b^{+,k+1} f)\|_{L^p(w)}$ to be finite. Suppose this for the moment. Then, by Lemma 1, for all ϵ with $\delta < \epsilon < 1$, we have

$$\begin{aligned}
\|T_b^{+,k+1} f\|_{L^p(w)} &\leq \|M_\delta^+(T_b^{+,k+1} f)\|_{L^p(w)} \\
&\leq C\|M_\delta^{+,\#}(T_b^{+,k+1} f)\|_{L^p(w)} \\
&\leq C \sum_{j=0}^{k} \|b\|_{\mathrm{BMO}}^{k+1-j} \|M_\epsilon^+(T_b^{+,j} f)\|_{L^p(w)} \\
&\quad + C\|b\|_{\mathrm{BMO}}^{k+1} \|(M^+)^{k+2} f\|_{L^p(w)}.
\end{aligned}$$

We choose $\epsilon > 0$ such that $r < \frac{p}{\epsilon}$. Then $w \in A_{\frac{p}{\epsilon}}^+$ and we obtain

$$\begin{aligned}
\|M_\epsilon^+(T_b^{+,j} f)\|_{L^p(w)}^p &= \int_{\mathbb{R}} (M^+(|T_b^{+,j} f|^\epsilon)^{\frac{p}{\epsilon}} w \\
&\leq C \int_{\mathbb{R}} (|T_b^{+,j} f|^\epsilon)^{\frac{p}{\epsilon}} w = C\|T_b^{+,j} f\|_{L^p(w)}^p.
\end{aligned}$$

Then, by recurrence

$$\begin{aligned}
\|T_b^{+,k+1} f\|_{L^p(w)} &\leq C \sum_{j=0}^{k} \|b\|_{\mathrm{BMO}}^{k+1-j} \|T_b^{+,j} f\|_{L^p(w)} \\
&\quad + C\|b\|_{\mathrm{BMO}}^{k+1} \|(M^+)^{k+2} f\|_{L^p(w)} \\
&\leq C \sum_{j=0}^{k} \|b\|_{\mathrm{BMO}}^{k+1-j} \|b\|_{\mathrm{BMO}}^{j} \|(M^+)^{j+1} f\|_{L^p(w)} \\
&\quad + C\|b\|_{\mathrm{BMO}}^{k+1} \|(M^+)^{k+2} f\|_{L^p(w)} \\
&\leq C\|b\|_{\mathrm{BMO}}^{k+1} \|(M^+)^{k+2} f\|_{L^p(w)}.
\end{aligned}$$

If w is bounded, then

$$\begin{aligned}\|M_\delta^+(T_b^{+,k+1}f)\|_{L^p(w)} &\le C\|M_\delta^+(T_b^{+,k+1}f)\|_{L^p(dx)}\\ &\le C\|T_b^{+,k+1}f\|_{L^p(dx)} \le C\|b\|_{\mathrm{BMO}}^{k+1}\|f\|_{L^p(dx)} < \infty.\end{aligned}$$

Then the theorem is proved if w is bounded. For the general case, we consider $w_N = \min\{w, N\}$. It is not hard to prove that $w_N \in A_\infty^+$ (A_p^+ is a lattice) with constant independent of N. Therefore, we have

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w_N \le C\|b\|_{\mathrm{BMO}}^{kp} \int_{\mathbb{R}} \left((M^+)^{k+1} f\right)^p w_N.$$

Now, we obtain the desired result after applying the monotone convergence theorem.

□

To prove Theorem 2 we need the following two lemmas.

Lemma 2. *Let $f \in L^1_{loc}(\mathbb{R})$ and $\lambda > 0$. Then for every weight w there exists $C > 0$ such that*

$$w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > \lambda\}) \le C \int_{\mathbb{R}} \frac{|f(y)|}{\lambda}\left(1 + \log^+ \frac{|f(y)|}{\lambda}\right)^k M^- w(y)\, dy.$$

Proof: This lemma is a consequence of (2.1) and Theorem 2.5 in [RRoT] with $B(t) = t(1 + \log^+ t)^k$, since $(w, M^- w) \in A_1^+$. □

Lemma 3. *Let $\phi_k(t) = t(1 + \log^+ t)^k$, $k = 0, 1, \dots$, $b \in \mathrm{BMO}$ and $w \in A_\infty^+$. Then there exists $C > 0$ such that*

$$\begin{aligned}&\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\})\\ &\qquad \le C\phi_k(\|b\|_{\mathrm{BMO}}^k) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\})\end{aligned}$$

for all bounded functions f with compact support.

Proof: We first suppose that $\|b\|_{\mathrm{BMO}} = 1$. We shall prove the following,

$$\begin{aligned}&\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\})\\ &\qquad \le C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).\end{aligned}$$

Now, set $b_m = b$ if $-m \le b \le m$, $b_m = m$ if $b \ge m$ and $b_m = -m$ if $b \le -m$. Also, set $w_N = \inf\{w, N\}$. As we have said before, $w_N \in A^+_\infty$ with constant independent of N. On the other hand $\|b_m\|_{\mathrm{BMO}} \le C'\|b\|_{\mathrm{BMO}} = C'$ with C' independent of m. In order to simplify notation, rename $b = b_m$ and $w = w_N$. Observe that for all $\delta > 0$ we have

$$w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\}) \le w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}).$$

Let us consider the functional

$$L_{b,w,\phi_k,\delta}(f) = L_\delta(f) = \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}).$$

We claim that for some $\gamma > 0$ and every $0 < \epsilon < 1$ we have

$$L_\delta(f) \le \epsilon^\gamma C L_\delta(f) + C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}). \tag{3.9}$$

If $L_\delta(f) < \infty$ then the result (for b_m and w_N) follows from (3.9), choosing ϵ small enough.

In what follows we prove that $L_\delta(f) < \infty$. In [MT] it was proved that if $w \in A^+_\infty$ and $M^+ f \in L^{p_0}(w)$ for some p_0, then

$$\begin{aligned} w(\{x \in \mathbb{R} : M^+ f(x) > t, M^{+,\#} f(x) \le t\epsilon\}) \\ \le C\epsilon^\gamma w(\{x \in \mathbb{R} : M^+ f(x) > \frac{t}{2}\}) \end{aligned} \tag{3.10}$$

for some $\gamma > 0$. Observe that we have $M_\delta^+(T_b^{+,k} f) \in L^{p_0}(w)$ for some p_0 since f is bounded with compact support, $w \le N$ and $|b| \le m$. Then

$$\begin{aligned} & w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}) \\ &\quad = w(\{x \in \mathbb{R} : M^+(|T_b^{+,k} f|^\delta)(x) > t^\delta, M^{+,\#}(|T_b^{+,k} f|^\delta)(x) \le t^\delta \epsilon\}) \\ &\qquad + w(\{x \in \mathbb{R} : M^+(|T_b^{+,k} f|^\delta)(x) > t^\delta, M^{+,\#}(|T_b^{+,k} f|^\delta)(x) > t^\delta \epsilon\}) \\ &\quad \le C\epsilon^\gamma w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) \ge t/2^{\frac{1}{\delta}}\}) \\ &\qquad + w(\{x \in \mathbb{R} : M_\delta^{+,\#}(T_b^{+,k} f)(x) > t\epsilon^{1/\delta}\}) \\ &\quad = I + II. \end{aligned} \tag{3.11}$$

Using Lemma 1 for $\epsilon = \delta r$ and $1 < r < \frac{1}{\delta}$, we have

$$\begin{aligned} II \le w(\{x \in \mathbb{R} : \sum_{j=0}^{k-1} (C')^{k-j} M_{\delta r}^+(T_b^{+,j} f)(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2C}\}) \\ + w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2C(C')^k}\}). \end{aligned} \tag{3.12}$$

Bearing in mind (3.11) and (3.12) we obtain

$$
\begin{aligned}
&\frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}) \\
&\leq \frac{C\epsilon^\gamma}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k}) f(x) > \frac{t}{2^{\frac{1}{\delta}}}\}) \\
&\quad + \sum_{j=0}^{k-1} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^+(T_b^{+,j} f)(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2Ck(C')^{k-j}}\}) \qquad (3.13)\\
&\quad + \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2C(C')^k}\}) \\
&= I' + II' + III'.
\end{aligned}
$$

Observe that there exists C such that $\phi_k(2t) \leq C\phi_k(t)$ for all $t > 0$ (*i.e.* ϕ_k is doubling). Let $l \in \mathbb{N}$ be such that $2^{\frac{1}{\delta}} < 2^l$. Using that ϕ_k is non-decreasing, we get

$$\phi_k\left(\frac{2^{\frac{1}{\delta}}}{t}\right) \leq \phi_k\left(\frac{2^l}{t}\right) \leq C\phi_k\left(\frac{1}{t}\right).$$

Then

$$I' \leq \frac{C\epsilon^\gamma}{\phi_k(\frac{2^{\frac{1}{\delta}}}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > \frac{t}{2^{\frac{1}{\delta}}}\}) \leq C\epsilon^\gamma L_\delta(f).$$

Now let $a_j = \frac{2Ck(C')^{k-j}}{\epsilon^{\frac{1}{\delta}}}$ and $h \in \mathbb{Z}$ be such that $a_j \leq 2^h$, for all j. Therefore

$$\phi_k\left(\frac{a_j}{t}\right) \leq \phi_k\left(\frac{2^h}{t}\right) \leq C\phi_k\left(\frac{1}{t}\right).$$

As a consequence,

$$
\begin{aligned}
II' &\leq C \sum_{j=0}^{k-1} \frac{1}{\phi_k(\frac{a_j}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^+(T_b^{+,j} f)(x) > \frac{t}{a_j}\}) \qquad (3.14)\\
&\leq C \sum_{j=0}^{k-1} \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^+(T_b^{+,j} f)(x) > t\}).
\end{aligned}
$$

Now for each $j = 0, 1 \ldots, k-1$, let us estimate $\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^+(T_b^{+,j} f)(x) > t\})$.

Using that ϕ_k is doubling and non-decreasing, it follows from (3.10) and Lemma 1(a) that, for all $0<\epsilon<1$,

$$\begin{aligned}\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:M_\epsilon^+(T^+f)(x)>t\}) &\le \sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:M_\epsilon^{+,\#}(T^+f)(x)>t\})\\ &\le C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:M^+f(x)>t\})\\ &\le C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:(M^+)^{k+1}f(x)>t\}).\end{aligned}$$

Fix $J<k-1$ and suppose that, for every $0\le j\le J$ and for all $0<\epsilon<1$, there exists C such that

$$\begin{aligned}(3.15)\qquad \sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}&w(\{x\in\mathbb{R}:M_\epsilon^+(T_b^{+,j}f)(x)>t\})\\ &\le C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x\in\mathbb{R}:(M^+)^{k+1}f(x)>t\}).\end{aligned}$$

We will prove, that (3.15) holds for $j=J+1$. Using again that ϕ_k is doubling, non-decreasing, (3.10) and Lemma 1(b) we obtain

$$\begin{aligned}&\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:M_\epsilon^+(T_b^{+,J+1}f)(x)>t\})\\ &\le C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:M_\epsilon^{+,\#}(T_b^{+,J+1}f)(x)>t\})\\ &\le C\left[\sum_{i=0}^{J}\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:M_{\epsilon'}^+(T_b^{+,i}f)(x)>t\})+w(\{x:(M^+)^{J+1}f(x)>t\})\right]\\ &\le C\sum_{i=0}^{J}\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:(M^+)^{k+1}f(x)>t\})\\ &\qquad +C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:(M^+)^{J+1}f(x)>t\})\\ &\le C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x:(M^+)^{k+1}f(x)>t\}),\end{aligned}$$

where $\epsilon<\epsilon'<1$. As a consequence, for $\epsilon=\delta r$, (3.15) together with (3.14) gives

$$II'\le C\sup_{t>0}\frac{1}{\phi_k(\frac{1}{t})}w(\{x\in\mathbb{R}:(M^+)^{k+1}f(x)>t\}).$$

Finally, let $a = \frac{\epsilon^{\frac{1}{\delta}}}{2C(C')^k}$. Then

$$\begin{aligned} III' &\leq \frac{C}{\phi_k(\frac{1}{at})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > at\}) \\ &\leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}). \end{aligned}$$

Putting all these estimates together we get (3.9).

Therefore if we prove that $L_{b,w,\phi_k,\delta} f < \infty$, using (3.9) we obtain

$$L_{b,w,\phi_k,\delta}(f) \leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).$$

Assume now that $\operatorname{supp} f \subset (-R, R)$, for some $R > 0$. Then for $x \leq -2R$ we have

$$\begin{aligned} |T_b^{+,k} f(x)| &\leq C \int_{-R}^{R} \frac{|b(x) - b(y)|^k}{|x-y|} |f(y)|\, dy \\ &\leq \frac{2Cm^k}{|x|} \int_{x}^{R} |f(y)|\, dy \\ &\leq C m^k M^+ f(x). \end{aligned} \tag{3.16}$$

Using that $0 < \delta < 1$, the fact that M^+ is of weak type $(1,1)$ with respect to the pair $(w, M^- w) \in A_1^+$, Lemma 2 and (3.16), we get

$$\begin{aligned} &\frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}) \\ &\leq \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(\chi_{(-2R,2R)} T_b^{+,k} f)(x) > t/2\}) \\ &\quad + \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(\chi_{(-\infty,-2R)} T_b^{+,k} f)(x) > t/2\}) \\ &\leq \frac{1}{\phi_k(\frac{1}{t})} \frac{C}{t} \int_{-2R}^{2R} |T_b^{+,k} f(x)| M^- w(x)\, dx \\ &\quad + \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > C_m t\}) \\ &\leq C4NR \left(\frac{1}{4R} \int_{-2R}^{2R} |T_b^{+,k} f(x)|^2\, dx \right)^{\frac{1}{2}} + \frac{C}{\phi_k(\frac{1}{t})} \int_{\mathbb{R}} \phi_k\left(\frac{|f(x)|}{C_m t} \right) M^- w(x)\, dx \\ &\leq C4NR \left(\frac{1}{4R} \int_{-R}^{R} |f(x)|^2\, dx \right)^{\frac{1}{2}} + CN \int_{-R}^{R} \phi_k(|f(x)|)\, dx. \end{aligned}$$

Since f is bounded and with compact support the last expression is finite.

Then, we have obtained the following:

$$\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : |T^{+,k}_{b_m} f(x)| > t\})$$
$$\leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).$$

Observe that $\left\{b^j_m f\right\}$ converges to $b^j f$ in $L^1(dx)$, since f is bounded with compact support and $b \in \mathrm{BMO}$ implies that b is locally in $L^p(dx)$ for all $p \geq 1$. Then, taking into account that T^+ is of weak type (1,1) with respect to the Lebesgue measure, we obtain that $\left\{T^+(b^j_m f)\right\}$ converges to $T^+(b^j f)$ in measure. This implies that, for a subsequence, we have almost everywhere convergence. On the other hand, $\left\{b^j_m T^+ f\right\}$ converges to $b^j T^+ f$ almost everywhere. As a consequence, a subsequence of $\left\{|T^{+,k}_{b_m} f|\right\}$ converges to $|T^{+,k}_b f|$ almost everywhere. We shall continue denoting this subsequence by $\left\{|T^{+,k}_{b_m} f|\right\}$. Then, by Fatou's lemma,

$$\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : |T^{+,k}_b f(x)| > t\})$$
$$= \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} \int_{\mathbb{R}} \lim_{m\to\infty} w_N(x) \chi_{\{x \in \mathbb{R}: |T^{+,k}_{b_m} f(x)| > t\}}\, dx$$
$$\leq \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} \liminf_{m\to\infty} w_N(\{x \in \mathbb{R} : |T^{+,k}_{b_m} f(x)| > t\})$$
$$\leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).$$

Letting N go to infinity we obtain the desired result.

Now, for general $b \in \mathrm{BMO}$ ($\|b\|_{\mathrm{BMO}} > 0$), we consider $h = \frac{b}{\|b\|_{\mathrm{BMO}}}$. Then, since $T^{+,k}_h f = \frac{1}{\|b\|^k_{\mathrm{BMO}}} T^{+,k}_b f$ and taking into account that ϕ_k is submultiplicative, we have

$$\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T^{+,k}_b f(x)| > t\})$$
$$= \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T^{+,k}_h f(x)| > \frac{t}{\|b\|^k_{\mathrm{BMO}}}\})$$
$$\leq \phi_k(\|b\|^k_{\mathrm{BMO}}) \sup_{t>0} \frac{1}{\phi_k\left(\frac{\|b\|^k_{\mathrm{BMO}}}{t}\right)} w(\{x \in \mathbb{R} : T^{+,k}_h f(x) > \frac{t}{\|b\|^k_{\mathrm{BMO}}}\})$$

$$\leq C\phi_k(\|b\|^k_{\mathrm{BMO}}) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M^{k+1} f(x) > t\}).$$

□

Proof of Theorem 2: It suffices to consider the case $\lambda = 1$. (For $\lambda > 0$ the result follows by considering $\frac{f}{\lambda}$). By Lemma 3, the fact that ϕ_k is submultiplicative and by Lemma 2 we get,

$$\begin{aligned} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > 1\}) &\leq \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\}) \\ &\leq C\phi_k(\|b\|^k_{\mathrm{BMO}}) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}) \\ &\leq C\phi_k(\|b\|^k_{\mathrm{BMO}}) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} \phi_k(\frac{1}{t}) \int_{\mathbb{R}} \phi_k(|f(x)|) M^- w(x)\, dx \\ &= C\phi_k(\|b\|^k_{\mathrm{BMO}}) \int_{\mathbb{R}} |f(x)|(1 + \log^+ |f(x)|)^k M^- w(x)\, dx. \end{aligned}$$

□

Proof of Theorem 3: By duality, (1.4) is equivalent to

$$\int_{\mathbb{R}} |T_b^{-,k} f|^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'} w^{1-p'}.$$

Observe that $((M^-)^{[(k+1)p]+1} w)^{1-p'} \in A_\infty^-$, and by Theorem 1, we get

$$\int_{\mathbb{R}} |T_b^{-,k} f|^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} ((M^-)^{k+1} f)^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'}.$$

Therefore it suffices to prove that

$$\int_{\mathbb{R}} ((M^-)^{k+1} f)^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'} w^{1-p'}. \tag{3.17}$$

Now observe that proving (3.17) is equivalent to

$$\int_{\mathbb{R}} ((M^-)^{k+1} (f w^{\frac{1}{p}}))^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'}. \tag{3.18}$$

If $\phi_k(t) = t(1 + \log^+ t)^k$, then (3.18) is equivalent to

$$\int_{\mathbb{R}} ((M^-_{\phi_k})(f w^{\frac{1}{p}}))^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'}. \tag{3.19}$$

For large t, $\phi_k^{-1}(t) \approx \dfrac{t}{\log(t)^k}$. Then, for $\epsilon > 0$,

$$\phi_k^{-1}(t) \approx \frac{t^{\frac{1}{p}}}{\log(t)^{k+\frac{p-1+\epsilon}{p}}} \times t^{\frac{1}{p'}} \log(t)^{\frac{p-1+\epsilon}{p}} = A^{-1}(t) \times B^{-1}(t),$$

where $A(t) \approx t^p \log(t)^{(k+1)p-1+\epsilon}$ and $B(t) \approx \dfrac{t^{p'}}{\log(t)^{1+(p'-1)\epsilon}}$. Then, by the generalized Hölder's inequality, we have

$$(M^-_{\phi_k})(fw^{\frac{1}{p}}) \le CM^-_{\overline{B}}(f)M^-_{\overline{A}}(w^{\frac{1}{p}}) \le CM^-_{\overline{B}}(f)(M^-_{\overline{D}}(w))^{\frac{1}{p}},$$

where $D(t) = t(\log t)^{(k+1)p-1+\epsilon}$. We choose ϵ such that $(k+1)p-1+\epsilon = [(k+1)p]$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} ((M^-_{\phi_k})(fw^{\frac{1}{p}}))^{p'}((M^-)^{[(k+1)p]+1}w)^{1-p'} \\
&\qquad \le C\int_{\mathbb{R}} (M^-_{\overline{B}}(f))^{p'}((M^-_{\overline{D}}(w))^{\frac{p'}{p}}((M^-)^{[(k+1)p]+1}w)^{1-p'} \\
&\qquad \le C\int_{\mathbb{R}} (M^-_{\overline{B}}(f))^{p'}((M^-_{\overline{D}}(w))^{p'-1}((M^-_{\overline{D}}(w))^{1-p'} \\
&\qquad \le C\int_{\mathbb{R}} |f|^{p'},
\end{aligned}$$

where the last inequality follows from Theorem C, since $B \in B_{p'}$. □

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