On the convergence of certain sums of independent random elements

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Abstract. In this note we investigate the relationship between the convergence of the sequence $\{S_n\}$ of sums of independent random elements of the form $S_n = \sum_{i=1}^{n} \varepsilon_i x_i$ (where ε_i takes the values ± 1 with the same probability and x_i belongs to a real Banach space X for each $i \in \mathbb{N}$) and the existence of certain weakly unconditionally Cauchy subseries of $\sum_{n=1}^{\infty} x_n$.

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1. Preliminaries

Our notation is standard ([1], [3], [4], [9]). Throughout this note Δ will denote the Cantor space $\{-1,1\}^{\mathbb{N}}$, Σ the σ-algebra of subsets of Δ generated by the n-cylinders of Δ for each $n \in \mathbb{N}$, and ν the Borel probability $\otimes_{i=1}^{\infty} \nu_i$ on Σ, where $\nu_i : 2^{\{-1,1\}} \to [0,1]$ is defined by $\nu_i(\emptyset) = 0$, $\nu_i(\{-1\}) = \nu_i(\{1\}) = 1/2$ and $\nu_i(\{-1,1\}) = 1$ for each $i \in \mathbb{N}$. In what follows X will be a real Banach space and $L_0(\nu, X)$ will stand for the (F)-space over $\mathbb{R}$ of all [classes of] ν-measurable X-valued functions equipped with the (F)-norm

$$\|f\|_0 = \int_\Delta \frac{\|f(\varepsilon)\|}{1 + \|f(\varepsilon)\|}\, d\nu(\varepsilon)$$

of the convergence in probability. We shall represent by $P_1(\nu, X)$ the (real) normed space consisting of all those [classes of] ν-measurable X-valued Pettis integrable functions f defined on Δ provided with the semivariation norm

$$\|f\|_{P_1(\nu,X)} = \sup\left\{ \int_\Delta |x^* f(\omega)|\, d\nu(\omega) : x^* \in X^*,\ \|x^*\| \le 1 \right\}.$$

As it is well known, $P_1(\nu, X)$ is not a Banach space whenever X is infinite-dimensional. In the sequel we shall shorten by wuC the sentence 'weakly unconditionally Cauchy'.

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In [5] we have shown that if a series of independent random elements of the form $\sum_{n=1}^{\infty} f_n$, with $f_n(\omega) = \omega_n x_n$ for $\omega \in \Delta$ and $\{x_n\} \subseteq X$, converges ν-almost surely in X, then $\sum_{n=1}^{\infty} x_n$ has a subseries which is unconditionally convergent in norm. In this note we continue the investigation on the relationship among the convergence of the functional series $\sum_{n=1}^{\infty} f_n$ under different topologies and the existence of certain wuC subseries of $\sum_{n=1}^{\infty} x_n$.

2. On certain weakly unconditionally Cauchy subseries

Lemma 2.1. *If there are a closed set A in Δ with $\nu(A) > 1/2$ and a nonempty set $S \subseteq X^*$ such that $\sum_{i=1}^{\infty} x^* f_i(\omega)$ converges for $\omega \in A$ and $x^* \in S$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\sum_{i=1}^{\infty} |x^* x_{n_i}| < \infty$ for each $x^* \in S$.*

Proof: The following fact is contained in the proof of [8, Proposition] (see also [5, Claim]). We shall denote by $C_{i_1 i_2 \dots i_k}$ or $C_{i_1 i_2 \dots i_k}(\varepsilon)$ any rectangle of Δ with fixed coordinates $i_1, i_2, \dots, i_k$, i.e., $C_{i_1 i_2 \dots i_k}(\varepsilon) = \{\omega \in \Delta : \omega_{i_j} = \varepsilon_j, 1 \le j \le k\}$ for some $\varepsilon \in \Delta$. On the other hand, given a strictly increasing sequence $Q = \{n_i : i \in \mathbb{N}\}$ of positive integers, for each $\omega \in \Delta$ we shall design by ω' (as in [8]) the element of Δ defined by $\omega'_i = \omega_i$ if $i \in Q$ and $\omega'_i = -\omega_i$ if $i \notin Q$.

Fact. *Let $A \in \Sigma$. If $\nu(A) > 1/2$, there is a strictly increasing sequence $\{n_i\}$ of positive integers such that $A \cap A' \cap C_{n_1 n_2 \dots n_k} \neq \emptyset$ for each $C_{n_1 n_2 \dots n_k}$ and each $k \in \mathbb{N}$.*

By hypothesis there is a closed set A in Δ with $\nu(A) > 1/2$ such that $\sum_{n=1}^{\infty} \omega_n x^* x_n$ converges for $\omega \in A$ and $x^* \in S$. According to the preceding fact there exists a strictly increasing sequence $Q = \{n_i\}$ of positive integers such that, given $\varepsilon \in \Delta$, then $A \cap A' \cap C_{n_1 n_2 \dots n_k}(\varepsilon) \neq \emptyset$ for each $k \in \mathbb{N}$. Since $\{A \cap A' \cap C_{n_1 n_2 \dots n_k}(\varepsilon) : k \in \mathbb{N}\}$ is a decreasing sequence of nonempty closed sets in the compact space Δ, there is a point ζ (which depends of ε) in Δ which belongs to the intersection $\bigcap_{k=1}^{\infty} A \cap A' \cap C_{n_1 n_2 \dots n_k}(\varepsilon)$. Hence, for each $x^* \in S$ and each pair (r, s) of positive integers, with $s > r$, one has

$$\left| \sum_{i=r+1}^{s} \varepsilon_i x^* x_{n_i} \right| = \left| \sum_{i=r+1}^{s} \zeta_{n_i} x^* x_{n_i} \right| \le \frac{1}{2} \left(\left| \sum_{i=n_r+1}^{n_s} x^* f_i(\zeta) \right| + \left| \sum_{i=n_r+1}^{n_s} x^* f_i\left(\zeta'\right) \right| \right).$$

Since $\zeta, \zeta' \in A$ and $x^* \in S$, both series $\sum_{i=1}^{\infty} x^* f_i(\zeta)$ and $\sum_{i=1}^{\infty} x^* f_i(\zeta')$ are convergent. So, for a given $\epsilon > 0$ there is a $k \in \mathbb{N}$ such that $\left| \sum_{i=n_r+1}^{n_s} x^* f_i(\zeta) \right| < \epsilon$ and $\left| \sum_{i=n_r+1}^{n_s} x^* f_i(\zeta') \right| < \epsilon$ for $s > r \ge k$, which implies that $\left| \sum_{i=r+1}^{s} \varepsilon_i x^* x_{n_i} \right| \le \epsilon$ for $s > r \ge k$. Hence the numerical series $\sum_{i=1}^{\infty} \varepsilon_i x^* x_{n_i}$ converges. Given that this is true for each $\varepsilon \in \Delta$, it follows that $\sum_{i=1}^{\infty} |x^* x_{n_i}| < \infty$ for each $x^* \in S$ and we are done. $\square$

Theorem 2.2. *Assume that* $\|x_n\| = 1$ *for each* $n \in \mathbb{N}$ *and* X *has a dual unit ball with countably many extreme points. If*

$$\sup_{n\in\mathbb{N}} \int_\Delta |x^* S_n(\omega)| \, d\nu(\omega) < \infty$$

for each $x^* \in \operatorname{Ext} B_{X^*}$, *then* X *contains a copy of* c_0.

Proof: By hypothesis, for each $x^* \in \operatorname{Ext} B_{X^*}$ there exists $C_{x^*} > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_\Delta \left| \sum_{i=1}^n x^* f_i(\omega) \right| d\nu(\omega) < C_{x^*}. \tag{2.1}$$

Hence, given $x^* \in \operatorname{Ext} B_{X^*}$, as a consequence of (2.1) and of Khinchine's inequalities there exists a $K > 0$ such that

$$\begin{aligned} \left\{ \sum_{i=1}^n \sigma^2 (x^* f_i) \right\}^{1/2} &= \left\{ \sum_{i=1}^n (x^* x_i)^2 \right\}^{1/2} \\ &\le K \int_\Delta \left| \sum_{i=1}^n x^* f_i(\omega) \right| d\nu(\omega) < K C_{x^*} \end{aligned} \tag{2.2}$$

for each $n \in \mathbb{N}$. Considering that the sequence $\{x^* f_i\}$ consists of independent random variables such that

$$\mathbf{E}(x^* f_i) = \int_\Delta x^* f_i(\omega) \, d\nu(\omega) = 0$$

for each $i \in \mathbb{N}$, according to [7, Section 46, Theorem B] equation (2.2) ensures that $\sum_{i=1}^\infty x^* f_i(\omega)$ converges almost surely for $\omega \in \Delta$. Since $\operatorname{Ext} B_{X^*}$ is countable, it follows that there exists a ν-null set N such that $\sum_{i=1}^\infty x^* f_i(\omega)$ converges for each $\omega \in \Delta - N$ and each $x^* \in \operatorname{Ext} B_{X^*}$. So, using inner regularity we may choose a closed set A with $A \subseteq \Delta - N$ and $\nu(A) > 1/2$ such that $\sum_{i=1}^\infty x^* f_i(\omega)$ converges for each $\omega \in A$ and each $x^* \in \operatorname{Ext} B_{X^*}$. On the basis of Lemma 2.1, this implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\sum_{i=1}^\infty |x^* x_{n_i}| < \infty$ for each $x^* \in \operatorname{Ext} B_{X^*}$. Since $\sum_{n=1}^\infty x_n$ diverges, Elton's theorem guarantees that X contains a copy of c_0. □

Proposition 2.3. *If the sums* $\{S_n\}$ *are bounded inside of a complete linear subspace* L *of* $P_1(\nu, X)$, *then* $\sum_{n=1}^\infty x_n$ *has a wuC subseries.*

Proof: Since $\{S_n\}$ is bounded inside of a complete linear subspace L of $P_1(\nu, X)$ and given that the canonical inclusion map from $P_1(\nu, X)$ into $L_0(\nu, X)$ has closed graph ([6, Lemma 4]), then Banach-Schauder's theorem guarantees that $\{S_n\}$ is stochastically bounded. So, according to [9, Section 5.2.3, Theorem 2.2] the sums $\{S_n\}$ are bounded almost surely, i.e. $\nu(\{\omega \in \Delta : \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n f_i(\omega) \right\| = \infty\}) = 0$. Hence Kwapień's theorem [8, Proposition] assures the existence of a wuC subseries of $\sum_{n=1}^\infty x_n$. □

Corollary 2.4. *Assume that $\{f_n\}$ is a basic sequence in $\widehat{P_1(\nu,X)}$ equivalent to the unit vector basis of c_0. If $[f_n]$ is contained in $P_1(\nu,X)$, then there exists a subsequence $\{f_{n_i}\}$ such that $[f_{n_i}]$ is isomorphic to a complemented copy of c_0.*

PROOF: Since the series $\sum_{i=1}^{\infty} f_i$ is wuC in $P_1(\nu,X)$, there is $K>0$ such that

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} \xi_i f_i \right\|_{P_1(\nu,X)} < K\|\xi\|_\infty$$

for each $\xi \in \ell_\infty$. Hence the sums $\{S_n\}$ are bounded in the complete linear subspace $[f_i]$ of $P_1(\nu,X)$ and Proposition 2.3 guarantees that $\sum_{n=1}^{\infty} x_n$ has a wuC subseries. Since $\|x_n\| = \|f_n\|_{P_1(\nu,X)}$ for each $n\in\mathbb{N}$, then $\inf_{n\in\mathbb{N}}\|x_n\| > 0$ and the classic Bessaga-Pełczyński allows us to conclude that $\{x_n\}$ contains a subsequence $\{x_{n_i}\}$ equivalent to the unit vector basis of c_0. Therefore, there exists a bounded sequence $\{y_i^*\}$ in X^* such that $y_i^* x_{n_j} = \delta_{ij}$ for each $i,j\in\mathbb{N}$. Assuming without loss of generality that $y_i \in B_{X^*}$, set $g_i(\varepsilon) = \varepsilon_i y_i^*$ for each $i\in\mathbb{N}$ and define

$$\langle g_i, f\rangle = \int_\Delta \varepsilon_i y_i^* f(\varepsilon)\, d\nu(\varepsilon)$$

for each $f\in P_1(\nu,X)$. So we have $\langle g_i, f_{n_j}\rangle = \delta_{ij}$ for each $i,j\in\mathbb{N}$. On the other hand, denoting by C_n the rectangle of Δ formed by all those $\varepsilon\in\Delta$ with $\varepsilon_n = 1$ and noting that $\nu(E\cap C_n)\to\nu(E)/2$ for all $E\in\Sigma$, it follows that

$$\mathbf{E}_{C_n}(\varphi) = \frac{1}{\nu(C_n)}\int_{C_n}\varphi\, d\nu \to \int_\Delta \varphi\, d\nu = \mathbf{E}(\varphi)$$

for each ν-simple function $\varphi:\Delta\to\mathbb{R}$. This implies that $\mathbf{E}_{C_n}(\varphi)\to\mathbf{E}(\varphi)$ for each $\varphi\in L_1(\nu)$, which leads to $\int_\Delta \varepsilon_i\,\varphi(\varepsilon)\,d\nu \to 0$ for each $\varphi\in L_1(\nu)$. Since, in addition, (Δ,Σ,ν) is a perfect measure space, it can be shown as in [2] that $\langle g_i, f\rangle \to 0$ for each $f\in P_1(\nu,X)$. Consequently the map $P: P_1(\nu,X)\to P_1(\nu,X)$ defined by

$$Pf = \sum_{i=1}^{\infty} \langle g_i, f\rangle\; f_{n_i}$$

is a bounded linear projection operator from the barreled space $P_1(\nu,X)$ onto $[f_{n_i}]$. □

Proposition 2.5. *If there exists a complete linear subspace L in $P_1(\nu,X)$ such that $\{f_i\}\subseteq L$ and $\sum_{i=1}^{\infty} f_i$ converges in $P_1(\nu,X)$ to some separably-valued $f\in L$, then there exists a subseries of $\sum_{i=1}^{\infty} x_i$ which is unconditionally convergent in X.*

Proof: Given that $\sum_{i=1}^{\infty} f_i = f$ in $P_1(\nu, X)$ and L is complete, then $\sum_{i=1}^{\infty} f_i = f$ in L. Then, using the fact that the inclusion map from $P_1(\nu, X)$ into $L_0(\nu, X)$ has closed graph together with the Banach-Schauder theorem, we get that $\sum_{i=1}^{\infty} f_i = f$ in probability. Since the range of f is separable in norm, then [9, Section 5.2.3, Theorem 2.1] guarantees that the series $\sum_{i=1}^{\infty} f_i(\omega)$ converges in X to $f(\omega)$ almost surely for $\omega \in \Delta$. Hence [5, Theorem 2.1] establishes the existence of a subseries of $\sum_{n=1}^{\infty} x_n$ which is unconditionally convergent in X. □

Question. *We do not know whether the statement of Theorem 2.2 is true without the assumption that B_{X^*} has countable many extreme points.*

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