Cohen real and disjoint refinement of perfect sets

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Abstract. We prove that if there exists a Cohen real over a model, then the family of perfect sets coded in the model has a disjoint refinement by perfect sets.

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It is a well known fact that in iterated generic extensions which add Cohen reals Sacks forcing collapses the continuum (see Corollary 3). This was a motivation for the present paper to find out what kind of phenomenon is behind this behaviour of Cohen reals. Theorem 1 gives an answer to this question. H. Judah, A.W. Miller, and S. Shelah [2] have proved that under Martin's axiom the additivity of Marczewski ideal is the cofinality of the continuum of the ground model in generic extensions over Sacks forcing. We use some ideas of this proof in the proof of Theorem 1.

A nonempty set $p \subseteq {}^{<\omega}2$ is a perfect tree if $s \in p$ and $t \subseteq s$ implies $t \in p$, and for every $s \in p$ there exists $t \supseteq s$ such that $t^\frown 0, t^\frown 1 \in p$. The set of all branches of a tree $p \subseteq {}^{<\omega}2$ is the perfect set $[p] = \{x \in {}^\omega 2 : (\forall\, n)\, x{\restriction}n \in p\}$. Conversely, for every perfect set $r \subseteq {}^\omega 2$ the tree $p = \{s \in {}^{<\omega}2 : [s] \cap r \neq \emptyset\}$ is perfect and $r = [p]$. We say that the perfect tree p is the code of the perfect set r.

Let us recall that the sets $[s] = \{x \in {}^\omega 2 : s \subseteq x\}$ for $s \in {}^{<\omega}2$ are the basic open sets in the Cantor space ${}^\omega 2$. We consider the space ${}^\omega 2$ as the group with the operation of addition defined by $(x + y)(n) = x(n) + y(n) \mod 2$ for $x, y \in {}^\omega 2$, and we define $A + x = \{a + x : a \in A\}$ for $A \subseteq {}^\omega 2$ and $x \in {}^\omega 2$. Conditions in Sacks forcing $\mathbb{S}$ are perfect subsets of ${}^\omega 2$, where a condition p is stronger than a condition q, if $p \subseteq q$. In such a case we write $p \leq q$.

For some information on disjoint refinement we refer to [1].

Theorem 1. *If there is a Cohen real over V, then for every sequence of perfect sets $\langle p_\alpha : \alpha < (2^\omega)^V \rangle$ with sets p_α coded in V there is a disjoint sequence of perfect sets $\langle q_\alpha : \alpha < (2^\omega)^V \rangle$ such that $q_\alpha \subseteq p_\alpha$ for every $\alpha < (2^\omega)^V$.*

We do not know whether in Theorem 1 it is possible to assume the existence of another kind of a new real instead of a Cohen real.

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Lemma 2. *If there is a Cohen real (a new real) over V, then there exists a sequence of pairwise disjoint perfect sets of Cohen reals (of new reals) $\langle r_\alpha : \alpha < (2^\omega)^V\rangle$ such that for every open set $U \subseteq {}^\omega 2$ the set $\{\alpha : r_\alpha \subseteq U\}$ has size $(2^\omega)^V$.*

Proof: Let $\mathbb{Q}$ be the set of all finite trees $\sigma \subseteq {}^{<\omega}2$ for which there exists $n_\sigma \in \omega$ such that $(\forall s \in \sigma)(\exists t \in \sigma)(s \subseteq t$ and $|t| = n_\sigma)$. For $\sigma \in \mathbb{Q}$ we denote $[\sigma] = \{x \in {}^\omega 2 : x{\restriction}n_\sigma \in \sigma\}$. $\mathbb{Q}$ is ordered by $\sigma \leq \tau$ if and only if $\tau = \sigma \cap {}^{\leq n(\tau)}2$. We prove that if $y \in {}^\omega 2$ is an irrational number, then the set $F_y = \{\sigma \in \mathbb{Q} : [\sigma] \cap ([\sigma]+y) = \emptyset\}$ is a dense subset of $\mathbb{Q}$. Let $\tau \in \mathbb{Q}$ be arbitrary and let ${}^{n_\tau}2 \cap \tau = \{s_i : i < m\}$ for some $m \in \omega$ and let $y_\tau(n) = y(n_\tau + n)$. The function $f(x) = x + y_\tau$ for $x \in {}^\omega 2$ is a one-to-one function and $f(x) \neq x$ for every $x \in {}^\omega 2$. We can find distinct reals $x_i \in {}^\omega 2$ for $i < m$ such that $f(x_i) \neq x_j$ for every $i, j < m$. Let $n \in \omega$ be minimal such that $x_i{\restriction}n$ for $i < m$ are pairwise distinct. Now let $\sigma \in \mathbb{Q}$ be the extension of τ such that $n_\sigma = n_\tau + n$ and ${}^{n_\sigma}2 \cap \sigma = \{s_i^\frown(x_i{\restriction}n) : i < m\}$. Clearly, $\sigma \in F_y$.

As $\mathbb{Q}$ is countable it is equivalent to Cohen's forcing. Let G be a V-generic subset of $\mathbb{Q}$. Then $r_G = \bigcup_{\sigma \in G} \sigma$ is a perfect tree. Using density arguments we can easily prove that $[r_G]$ is a perfect set of Cohen reals over V and $[r_G] \cap ([r_G]+y) = \emptyset$ for every irrational $y \in V$.

Let $\{s_n : n \in \omega\}$ be an enumeration of ${}^{<\omega}2$. Let us fix a sequence of irrational reals $\langle y_{\alpha,n} \in V : \alpha < (2^\omega)^V \;\&\; n \in \omega\rangle$ such that $y_{\alpha,n} - y_{\beta,m}$ is irrational whenever $(\alpha, n) \neq (\beta, m)$. By changing first $|s_n|$ values of $y_{\alpha,n}$ we can assume that $[s_n] \cap ([r_G] + y_{\alpha,n}) \neq \emptyset$ for every $n \in \omega$ and $\alpha < (2^\omega)^V$. Now let $\langle r_\alpha : \alpha < (2^\omega)^V\rangle$ be an enumeration of the family of disjoint perfect sets $\{[s_n] \cap ([r_G] + y_{\alpha,n}) : \alpha < (2^\omega)^V \;\&\; n \in \omega\}$.

Now we prove the second part of the lemma. Let us assume that there exists a new real over V. Then there are continuum new reals over V. The equivalence relation E on ${}^\omega 2$ defined by xEy if and only if $(\forall^\infty n)\, x(n) = y(n)$ has countable equivalence classes. Therefore we can find a sequence of pairwise E-inequivalent new reals $\langle y_{\alpha,n} : \alpha < (2^\omega)^V \;\&\; n \in \omega\rangle$. The perfect sets $p_{\alpha,n} = \{x \in [s_n] : (\forall k \geq |s_n|)\, x(2k) = y_{\alpha,n}(k)\}$ are pairwise disjoint and consist of new reals. $\square$

Proof of Theorem 1: Let $\langle p_\alpha : \alpha < (2^\omega)^V\rangle$ be a sequence of perfect sets coded in V. Let $A \subseteq (2^\omega)^V$ be a maximal set of ordinals with the property that for every distinct $\alpha, \alpha' \in A$, $p_\alpha \cap p_{\alpha'}$ is nowhere dense relatively in p_α, or in $p_{\alpha'}$. So if $\alpha \notin A$, then there exists $\alpha' \in A$ such that $p_\alpha \cap p_{\alpha'}$ is somewhere dense in $p_{\alpha'}$ and hence $p_\alpha \cap p_{\alpha'} \cap [s] = p_{\alpha'} \cap [s] \neq \emptyset$ for some $s \in {}^{<\omega}2$. Let $f_\alpha : {}^\omega 2 \to p_\alpha$ be a homeomorphism defined in V. Let $\langle r_\alpha : \alpha < (2^\omega)^V\rangle$ be the disjoint sequence of perfect sets of Cohen reals from Lemma 2. We claim that the family of perfect sets $\mathcal{F} = \{f_\alpha(r_\beta) : \alpha \in A \;\&\; \beta < (2^\omega)^V\}$ is disjoint and for every $\alpha < (2^\omega)^V$ the set $\{p \in \mathcal{F} : p \subseteq p_\alpha\}$ has size $(2^\omega)^V$. Using this claim we easily can construct a one-to-one mapping $\varphi : (2^\omega)^V \to \mathcal{F}$ such that $\varphi(\alpha) \subseteq p_\alpha$ for every α and we set $q_\alpha = \varphi(\alpha)$.

Clearly, $f_\alpha(r_\beta) \cap f_\alpha(r_{\beta'}) = f_\alpha(r_\beta \cap r_{\beta'}) = \emptyset$ whenever $\beta \neq \beta'$. If $\alpha \neq \alpha'$,

$\alpha, \alpha' \in A$, then without loss of generality we can assume that $p_\alpha \cap p_{\alpha'}$ is nowhere dense in p_α. Then $f_\alpha^{-1}(p_\alpha \cap p_{\alpha'})$ is a nowhere dense set coded in V. Then for all β, since r_β is the set of Cohen reals over V, $r_\beta \cap f_\alpha^{-1}(p_\alpha \cap p_{\alpha'}) = \emptyset$ and $f_\alpha(r_\beta) \cap p_{\alpha'} = \emptyset$. It follows that $f_\alpha(r_\beta) \cap f_{\alpha'}(r_{\beta'}) = \emptyset$ for all β, β'. We have proved that $\mathcal{F}$ is disjoint. Clearly, $f_\alpha(r_\beta) \subseteq p_\alpha$ holds true for every $\alpha \in A$ and $\beta < 2^\omega$. If $\alpha \notin A$, then there exist $\alpha' \in A$ and $s \in {}^{<\omega}2$ such that $\emptyset \neq p_{\alpha'} \cap [s] \subseteq p_\alpha \cap [s]$. Since the set $f_{\alpha'}^{-1}(p_{\alpha'} \cap [s])$ is open, the set $\{\beta : f_{\alpha'}(r_\beta) \subseteq p_\alpha\} \supseteq \{\beta : r_\beta \subseteq f_{\alpha'}^{-1}(p_{\alpha'} \cap [s])\}$ has size $(2^\omega)^V$. □

A special case to which next corollary can be applied is a finite support iterated forcing about which it is well known that adds Cohen reals at every limit step of the iteration with countable cofinality.

Corollary 3. *In a generic extension via iteration of length κ with $\mathrm{cf}(\kappa) \geq \omega_1$ which adds Cohen reals, Sacks forcing collapses continuum to $\mathrm{cf}(\kappa)$.*

Proof: Let V be the generic extension. Let $\{\alpha_\xi : \xi < \mathrm{cf}(\kappa)\}$ be an increasing sequence of ordinals cofinal with κ. Since over every level α_ξ there are Cohen reals in V, by Theorem 1 there is a maximal antichain $\mathcal{A}_\xi$ of perfect sets coded in V such that for some $\mathcal{A} \subseteq \mathcal{A}_\xi$, $\mathcal{A}$ refines the system of all perfect sets coded in the α_ξ level of the iteration. For every perfect set p let us fix a maximal antichain of perfect sets bellow p of size continuum which we denote by $\{(p)_\alpha : \alpha < 2^\omega\}$. Then the Sacks name $\dot{f}$ defined by

$$\|\dot{f}(\xi) = \alpha\| = \bigvee \{(p)_\alpha : p \in \mathcal{A}_\xi\}, \qquad \xi < \mathrm{cf}(\kappa)$$

is a name of a function from $\mathrm{cf}(\kappa)$ onto 2^ω. □

Remark. *Corollary 3 is equivalent to the assertion that $\mathbb{S}$ is $(\mathrm{cf}(\kappa), 2^\omega, 2^\omega)$-nowhere distributive in the generic extension. In fact the matrix $\{\mathcal{A}'_\xi : \xi < \mathrm{cf}(\kappa)\}$, where $\mathcal{A}'_\xi = \{(p)_\alpha : p \in \mathcal{A}_\xi \ \&\ \alpha < 2^\omega\}$ witnesses this (see [1] for the distributivity definition).*

References

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[2] Judah H., Miller A. W., Shelah S., *Sacks forcing, Laver forcing, and Martin's axiom*, Arch. Math. Logic **31** (1992), 145–161.

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