On concentrated probabilities on non locally compact groups

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Abstract. Let G be a Polish group with an invariant metric. We characterize those probability measures μ on G so that there exist a sequence $g_n \in G$ and a compact set $A \subseteq G$ with $\mu^{*n}(g_n A) \equiv 1$ for all n.

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In what follows we shall use the terminology and notation from [1]. However, for the convenience of the reader we briefly recall the most important ones. A metric d on the group G is said to be invariant if $\mathrm{d}(g_1 g, g_2 g) = \mathrm{d}(g g_1, g g_2) = \mathrm{d}(g_1, g_2)$ for all $g, g_1, g_2 \in G$. Given $\varepsilon > 0$ and $A \subseteq G$ by $L(A, \varepsilon)$ we denote the largest natural l (if it does not exist, then we set $L(A, \varepsilon) = \infty$) such that there exists a finite set $\{y_1, y_2, \dots, y_l\} \subseteq A$ with $\mathrm{d}(y_i, y_j) \geq \varepsilon$ if $i \neq j$. For $r > 0$ by $K(A, r)$ we denote the generalized open ball $\{g \in G : \inf_{a \in A} \mathrm{d}(a, g) < r\}$.

As usual $*$ stands for the convolution operation, which is well defined on $M(G)$, the Banach lattice of all finite signed (Borel) measures on G. If μ is a probability measure on G then $S(\mu)$ is its topological support. A measure μ is said to be adapted if the closed subgroup generated by $S(\mu)$ coincides with G. The smallest closed subgroup $H \subseteq G$ such that $gH = Hg$ and $S(\mu) \subseteq gH$ for all $g \in S(\mu)$ is denoted by $\mathfrak{h}(\mu)$. If an adapted measure μ satisfies $\mathfrak{h}(\mu) = G$ then we say that it is strictly aperiodic.

The paper is devoted to asymptotic behaviour of convolution powers μ^{*n} of a fixed probability measure μ. In particular, we examine when the concentration function does not tend to zero (i.e. $\sup_{g \in G} \mu^{*n}(gA) \geq \varepsilon$ for some $\varepsilon > 0$, compact $A \subseteq G$, and all n).

In the past this problem was studied mainly for locally compact topological groups. The reader is referred to [1], [2] and [4] for more details in this regard. It should be noted that [4] contains an affirmative answer to the so called Hofmann-Mukhereja conjecture, which says that adapted and strictly aperiodic probability

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measures on locally compact, Hausdorff and σ-compact (noncompact) groups have concentration functions tending to zero.

The aim of the present paper is to extend the main result of [1] to non locally compact groups, that is to prove the following result:

Theorem. *Let (G,d) be a Polish group with an invariant metric* d *and μ be a probability Borel measure on G. Then the following conditions are equivalent:*

(i) *there exist a sequence $g_n \in G$ and compact $A \subseteq G$ such that $\mu^{*n}(g_n A) \equiv 1$ for all n (μ is* **concentrated**),
(ii) *there exist a sequence $g_n \in G$, compact $A \subseteq G$ and $\varepsilon > 0$ such that $\mu^{*n}(g_n A) \geq \varepsilon$ for all n (μ is* **nonscatterred**),
(iii) *$\check{\mu} * \varrho * \mu = \varrho$ for some probability measure ϱ,*
(iv) *$\lim_{n\to\infty} L(S(\mu^{*n}), \varepsilon) = \ell_\varepsilon < \infty$ for all $\varepsilon > 0$,*
(v) *$\mathfrak{h}(\mu)$ is compact.*

Moreover, if the above statements hold then

$$\mathfrak{h}(\mu) = S(\omega), \quad \text{where} \quad \omega = \lim_{n\to\infty} \check{\mu}^{*n} * \mu^{*n} = \lim_{n\to\infty} \mu^{*n} * \check{\mu}^{*n}$$

is the normalized Haar measure on $\mathfrak{h}(\mu)$, and the convergence holds in the weak measure topology.

Most of the arguments used in the proof of Theorem 1 from [1] is still valid. However, we have to replace those parts of the old proof where we rely on the Haar measure. In particular, the convolution operators P_μ cannot be introduced. Because of this, the condition (iii) from [1] is scrapped. Our new proof is based on the following two lemmas:

Lemma 1 (see [3]). *Let μ be a probability measure on G and*

$$\alpha_\mu = \sup_{\substack{F\subseteq G \\ F \text{ compact}}} \lim_{n\to\infty} \sup_{g\in G} \mu^{*n}(gF).$$

Then $\alpha_\mu = 0$ or $\alpha_\mu = 1$.

Proof: For the proof the reader is referred to (3.6) Theorem 3.1 in [3]. □

Lemma 2. *If $\alpha_\mu = 1$ then there exists a probability measure ϱ on G such that $\check{\mu} * \varrho * \mu = \varrho$.*

Proof: Given $\varepsilon > 0$ there exist compact $F \subseteq G$ and a sequence $g_n \in G$ such that $\mu^{*n}(g_n F) > 1 - \varepsilon$. This implies

$$\check{\mu}^{*n} * \mu^{*n}(F^{-1}F) > (1-\varepsilon)^2.$$

Define $T_\mu(\nu) = \check{\mu} * \nu * \mu$ to be a linear positive contraction on $M(G)$. It follows from Lemma 2 and the Prohorov's criterion (see [5, Proposition 52.3]) that the sequence $\frac{1}{N}\sum_{n=0}^{N-1} T_\mu^n \delta_e$ is relatively compact for the weak measure topology. Hence

$$\varrho = \lim_{N_l\to\infty} \frac{1}{N_l}\sum_{n=0}^{N_l-1} T_\mu^n \delta_e \quad \text{for some sequence} \quad N_l \nearrow \infty .$$

Clearly, ϱ is a T_μ-invariant probability measure (in particular $\check{\mu} * \varrho * \mu = \varrho$). □

Proof (of the theorem): For implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) the reader is referred to [1, Theorem 1] and (ii) $\Rightarrow$ (iii) easily follows from Lemmas 1 and 2. To complete the proof we must show that these conditions imply

$$\omega = \lim_{n\to\infty} \check{\mu}^{*n} * \mu^{*n}$$

exists and coincides with the normalized Haar measure on $\mathfrak{h}(\mu) = \mathfrak{h}(\check{\mu})$. For this we define the Markov operator

$$Tf(g) = \iint f(xgy)\, d\check{\mu}(x)\, d\mu(y)$$

on the Banach lattice $C(\mathfrak{h}(\mu))$ of all continuous functions on $\mathfrak{h}(\mu)$. Note that T is well defined as

$$x^{-1}gy \in \mathfrak{h}(\mu) \quad \text{for all} \quad x, y \in S(\mu) \quad \text{and} \quad g \in \mathfrak{h}(\mu).$$

Clearly, the adjoint operator T^* coincides with T_μ (restricted to $M(\mathfrak{h}(\mu))$). For every $f \in C(\mathfrak{h}(\mu))$ the iterations $T^n f$ are norm (sup) relatively compact. This will follow from the Arzela theorem. In fact, let $\delta > 0$ be such that

$$|f(g_1) - f(g_2)| < \varepsilon \quad \text{whenever} \quad \mathrm{d}\,(g_1, g_2) < \delta.$$

By the invariance of d for arbitrary $x, y \in S(\mu^{*n})$ we get

$$\mathrm{d}\,(x^{-1}g_1y, x^{-1}g_2y) = \mathrm{d}\,(g_1, g_2).$$

Hence

$$|T^n f(g_1) - T^n f(g_2)| \le \iint |f(x^{-1}g_1y) - f(x^{-1}g_2y)|\, d\mu^{*n}(x)\, d\mu^{*n}(y) < \varepsilon.$$

Now we show that T is irreducible. Given a nonzero and nonnegative $f \in C(\mathfrak{h}(\mu))$ let us suppose that

$$T^n f(g_n) = 0 \quad \text{where} \quad g_n \in \mathfrak{h}(\mu).$$

We choose $\varepsilon > 0$ and a convergent subsequence

$$g_0 = \lim_{j\to\infty} g_{n_j}.$$

By continuity

$$f \equiv 0 \quad \text{on} \quad S(\check{\mu}^{*n}) g_n S(\mu^{*n}),$$

what implies

$$f(g) < \varepsilon \quad \text{for all} \quad g \in K(S(\check{\mu}^{*n_j}) g_{n_j} S(\mu^{*n_j}),\, \delta).$$

From the proof of Theorem 1 in [1] it follows that

$$\mathfrak{h}(\mu) = \overline{\bigcup_{n=1}^{\infty} S(\check{\mu}^{*n})\, S(\mu^{*n})}.$$

Hence, there are $v_j, w_j \in S(\mu^{*j})$ such that

$$\mathrm{d}\,(g_{n_j}, w_j^{-1} v_j) \xrightarrow[j\to\infty]{} 0.$$

If j is large enough we get

$$S(\check{\mu}^{*n_j}) w_j^{-1} v_j S(\mu^{*n_j}) \subseteq K(S(\check{\mu}^{*n_j}) g_{n_j} S(\mu^{*n_j}),\, \delta).$$

It is proved in [1] (see Theorem 1) that if j tends to infinity and if μ is nonscattered then the compact sets

$$S(\check{\mu}^{*n_j}) w_j^{-1} \quad \text{and} \quad v_j S(\mu^{*n_j})$$

are close in the Hausdorff metric to

$$S(\check{\mu}^{*(n_j+j)}) \quad \text{and} \quad S(\mu^{*(n_j+j)})$$

respectively. Hence

$$S(\check{\mu}^{*(n_j+j)} * \mu^{*(n_j+j)}) \subseteq K(S(\check{\mu}^{*n_j}) g_{n_j} S(\mu^{*n_j}),\, 2\delta)$$

for j large enough. Since the sequence $S(\check{\mu}^{*n} * \mu^{*n})$ is nondecreasing we obtain

$$\mathfrak{h}(\mu) \subseteq K(S(\check{\mu}^{*n_j}) g_{n_j} S(\mu^{*n_j}),\, 2\delta)$$

for some j, and we get $f(g) < \varepsilon$ for all $g \in \mathfrak{h}(\mu)$. This contradicts f being nonzero as ε may be taken as small as we wish. We have proved that for every nonnegative and nonzero $f \in C(\mathfrak{h}(\mu))$ there exist ε and n such that

(a) $$T^n f(x) \geq \varepsilon > 0 \quad \text{for all} \quad x \in \mathfrak{h}(\mu).$$

For arbitrary $f \in C(\mathfrak{h}(\mu))$ we denote

$$O(f) = \max_{x\in\mathfrak{h}(\mu)} f(x) - \min_{y\in\mathfrak{h}(\mu)} f(y) \geq 0.$$

Clearly, $O(T^n f)$ is nonincreasing. By (a)

$$O(T^n f) < O(f) \quad \text{for some} \quad n \geq 1$$

whenever f is nonconstant. If g is any limit function of the sequence $T^n f$ (it exists by compactness of trajectories), then $O(Tg) = O(g)$, what follows from monotonicity of $O(T^n f)$. Therefore all limit functions g are constant. Since T is markovian $(T\mathbf{1} = \mathbf{1})$ this implies that $T^n f \to \Lambda(f)$ uniformly, where $\Lambda(f)$ is a constant function. From the general theory of Markov operators the functional $\Lambda(f)$ has the form $\int f\,dm$, where m is the unique T^*-invariant probability such that $S(m) = \mathfrak{h}(\mu)$ (see [6] for all details). In particular,

$$\check{\mu}^{*n} * \mu^{*n} = T^{*n}\delta_e$$

converges weakly to m. Clearly

$$m = \varrho = \lim_{N_l\to\infty} \frac{1}{N_l} \sum_{n=0}^{N_l-1} \check{\mu}^{*n} * \mu^{*n}.$$

To prove that m is the Haar measure ω on $\mathfrak{h}(\mu)$ it is sufficient to show that

$$\int f_h(g)\,dm(g) = \int f(g)\,dm(g)$$

for all $f \in C(\mathfrak{h}(\mu))$ and $h \in \mathfrak{h}(\mu)$, where $f_h(g) = f(gh)$.
For this note that

$$\lim_{n\to\infty} \check{\mu}^{*n} * \omega * \mu^{*n} = m \quad \text{and} \quad \delta_{x^{-1}} * \omega * \delta_y$$

do not depend on $x, y \in S(\mu^{*n})$ (thus they coincide with $\check{\mu}^{*n} * \omega * \mu^{*n}$). Given $\varepsilon > 0$ there exists n such that

$$\left|\int f(g)\,dm(g) - \int f(x^{-1}gy)\,d\omega(g)\right| < \varepsilon$$

and

$$\left|\int f_h(g)\,dm(g) - \int f_h(x^{-1}gy)\,d\omega(g)\right| < \varepsilon$$

for all $x, y \in S(\mu^{*n})$. Since $\mathfrak{h}(\mu)$ is a normal subgroup of $G(\mu)$ we get $yh = \tilde{h}y$ for some $\tilde{h} \in \mathfrak{h}(\mu)$. Hence

$$\int f_h(x^{-1}gy)\, d\omega(g) = \int f(x^{-1}gyh)\, d\omega(g) = \int f(x^{-1}g\tilde{h}y)\, d\omega(g) = \int f(x^{-1}gy)\, d\omega(g),$$

and we get

$$\left| \int f(g)\, dm(g) - \int f_h(g)\, dm(g) \right| < 2\varepsilon.$$

Since ε is arbitrary the invariance of m follows. We conclude $m = \omega$.

Note that $\mathfrak{h}(\mu) = \mathfrak{h}(\check{\mu})$. In particular, $\check{\mu}$ is concentrated as well. Therefore $\lim_{n\to\infty} \mu^{*n} * \check{\mu}^{*n} = \omega$ and the proof is complete. □

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