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HYERS–ULAM–RASSIAS STABILITY OF A GENERALIZED PEXIDER FUNCTIONAL EQUATION

AHMED CHARIFI[1], BELAID BOUIKHALENE[2]
AND ELHOUCIEN ELQORACHI[3]*

This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by C. Park

ABSTRACT. In this paper, we obtain the Hyers–Ulam–Rassias stability of the generalized Pexider functional equation

$$\sum_{k\in K} f(x+k\cdot y) = |K|g(x) + |K|h(y),\ x, y \in G,$$

where G is an abelian group, K is a finite abelian subgroup of the group of automorphism of G.
The concept of Hyers–Ulam–Rassias stability originated from Th.M. Rassias' Stability Theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72(1978), 297-300.

1. INTRODUCTION

The stability problem of functional equations was posed for the first time by S.M. Ulam [28] in the year 1940. Ulam stated the problem as follows:

> Given a group G_1, a metric group (G_2, d), a number $\varepsilon > 0$ and a mapping $f : G_1 \to G_2$ which satisfies $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G_1$, does there exist an homomorphism $h : G_1 \to G_2$

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and a constand $k > 0$, depending only on G_1 and G_2 such that $d(f(x), h(x)) \leq k\varepsilon$ for all x in G_1?

The first affirmative answer was given by D.H. Hyers [8], under the assumption that G_1 and G_2 are Banach spaces.
In 1978, Th.M. Rassias [19] gave a remarkable generalization of the Hyers' result which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.1. [19] *Let $f : V \to X$ be a mapping between Banach spaces and let $p < 1$ be fixed. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|x\|^p)$$

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).
If, in addition, $f(tx)$ is continuous in t for each fixed x, then T is linear.

In 1990, Th.M. Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for values of p greater or equal to one. Z. Gajda [6] following the same approach as in [19] provided an affirmative solution to Rassias' question for p strictly greater than one. However, it was shown independently by J. Gajda [6] and Th.M. Rassias and P. Šemrl [22] that a similar result for the case of value of p equal to one can not be obtained.
The concept of the linear mapping, that was introduced for the first time in 1978 by Th.M. Rassias and followed later by several other mathematicians is known today as Hyers–Ulam–Rassias stability. Several papers have been published in this subject and some interesting variants of Ulam's problem have been also investigated by a number of mathematicians. We refer the reader to the following references [5, 6, 7, 9, 10, 12, 16, 17, 18], [20]– [24].
The Hyers–Ulam–Rassias stability for the functional equations of the forms

$$f(x+y) = g(x) + h(y) \quad (x, y \in G)$$

$$f(x+y) + f(x-y) = g(x) + h(y), \quad (x, y \in G),$$

where G is an abelian group, has been obtained in several works. In particular, for Cauchy's equation, Jensen's equations, the quadratic equation and the Pexider equation. We can see for example [2, 3, 4, 11, 13, 14, 15].

The main purpose of this paper is to investigate the Hyers–Ulam–Rassias stability of the functional equation

$$\Sigma_{k\in K} f(x + k \cdot y) = |K|g(x) + |K|h(y),\ x, y \in G.$$

Indeed we prove the stability theorem of the generalized Jensen's functional equation

$$\Sigma_{k\in K} f(x + k \cdot y) = |K|f(x),\ x, y \in G, \tag{1.1}$$

and the generalized Pexider functional equation

$$\Sigma_{k\in K} g(x+k\cdot y) = |K|g(x) + |K|h(y),\ x,y\in G,$$

$$\Sigma_{k\in K} f(x+k\cdot y) = |K|g(x) + |K|h(y),\ x,y\in G,$$

Where G is an abelian group, K is a finite abelian subgroup of $Aut(G)$ (the group of automorphism of G), and $|K|$ denotes the order of K.

These functional equations have been studied by H. Stetkær [25, 26, 27].

Recently, M. Ait Sibaha, B. Bouikhalene, and E. Elqorachi [1] proved the Hyers–Ulam–Rassias stability of the K-quadratic functional equation

$$\Sigma_{k\in K} f(x+k\cdot y) = |K|f(x) + |K|f(y),\ x,y\in G. \tag{1.2}$$

It is convenient to state the stability theorem obtained in [1], the result will be used later.

Theorem 1.2. [1] *Let G be an abelian group. Let K be a finite abelian subgroup of the group of automorphism of G, let $\delta > 0$, and let B be a Banach space. Suppose that $f\colon G \to B$ satisfies the inequality*

$$\|\sum_{k\in K} f(x+k\cdot y) - |K|f(x) - |K|f(y)\| \le \delta$$

for all $x,y\in G$. Then the limit $q(x) = lim_{n\to\infty}\frac{f_n(x)}{2^n}$, with

$$f_0(x) = f(x) \ \text{ and } \ f_n(x) = \frac{1}{|K|}\sum_{k\in K} f_{n-1}(x+k\cdot x) \ \text{ for } \ n\ge 1$$

exists for all $x\in G$, and $q\colon G\to B$ is a unique K-quadratic mapping which satisfies

$$\|f(x) - q(x)\| \le \frac{\delta}{|K|} \ \text{ for all } x\in G.$$

The present paper is a continuation of the previous work By M. Ait Sibaha, B. Bouikhalene and E. Elqorachi [1].

2. Hyers–Ulam–Rassias stability of the generalized Jensen's functional equation (1.1)

In this section we study the Hyers–Ulam–Rassias stability of the generalized Jensen's functional equation

$$\Sigma_{k\in K} f(x+k\cdot y) = |K|f(x),\ x,y\in G. \tag{2.1}$$

Theorem 2.1. *Let $\delta > 0$. Suppose that the mapping $f\colon G\to B$ satisfies the inequality*

$$\|\Sigma_{k\in K} f(x+k\cdot y) - |K|f(x)\| < \delta,\ x,y\in G, \tag{2.2}$$

where B is a Banach space. Then, there exists a unique mapping $J: G\to B$ solution of (2.1) such that $f - J$ is bounded ($\|f(x) - J(x)\| \le \delta$) and $J(e) = f(e)$.

Proof. Assume that $f\colon G \to B$ satisfies the inequality (2.2). We use induction on n to prove that the sequence functions

$$f_0(x) = f(x) \text{ and}$$

$$f_n(x) = \Sigma_{k\in K} f_{n-1}(x - k\cdot x),\ x \in G,\ \ n \in \mathbb{N} \tag{2.3}$$

satisfy the following statements

$$f_n(e) = |K|^n f(e) \tag{2.4}$$

$$\|f_n(x) - |K| f_{n-1}(x)\| \le (|K|-1)^{n-1}\delta \tag{2.5}$$

$$\|f_n(x) - |K|^n f(x)\| \le [|K|^n - (|K|-1)^n]\delta. \tag{2.6}$$

and

$$\|\Sigma_{k\in K} f_n(x + k\cdot y) - |K| f_n(x)\| \le (|K|-1)^n\delta. \tag{2.7}$$

By using the definition of f_n and inequality (2.2), we get

$$f_1(e) = |K| f_0(e) = |K| f(e),$$

$$\|f_1(x) - |K| f_0(x)\| = \|\Sigma_{k\in K} f(x - k\cdot x) - |K| f(x)\| < (|K|-1)^{1-1}\delta,$$

$$\|f_1(x) - |K|^1 f(x)\| \le [|K|^1 - (|K|-1)^1]\delta.$$

It follows from (2.3) that

$$\begin{aligned}
\Sigma_{t\in K} f_1(x + t\cdot y) &= \Sigma_{t\in K}\Sigma_{k\in K} f(x + t\cdot y - k\cdot(x + t\cdot y)) \\
&= |K| f(e) + \Sigma_{k\in K\setminus\{I\}}\Sigma_{t\in K} f(x + t\cdot y - k\cdot(x + t\cdot y)) \\
&= f_1(e) + \Sigma_{k\in K\setminus\{I\}}\Sigma_{t\in K} f(x - k\cdot x + t\cdot(y - k\cdot y)).
\end{aligned}$$

On the other hand we have

$$|K| f_1(x) = |K|\Sigma_{k\in K} f(x - k\cdot x) = f_1(e) + |K|\Sigma_{k\in K\setminus\{I\}} f(x - k\cdot x),$$

hence, we deduce that

$$\begin{aligned}
\|\Sigma_{t\in K} f_1(x + t\cdot y) - |K| f_1(x)\| &= \|\Sigma_{k\in K\setminus\{I\}}\Sigma_{t\in K} f(x - k\cdot x + t\cdot(y - k\cdot y)) \\
&\quad - |K|\Sigma_{k\in K\setminus\{I\}} f(x - k\cdot x)\| \\
&\le \Sigma_{k\in K\setminus\{I\}}\|\Sigma_{t\in K} f(x - k\cdot x + t\cdot(y - k\cdot y)) \\
&\quad - |K| f(x - k\cdot x)\| \\
&\le (|K|-1)^1\delta.
\end{aligned}$$

Consequently, the assertions (2.4), (2.5), (2.6) and (2.7) are true for $n = 1$. Assuming that the assertions are true for all integers i, $1 \le i \le n$. It follows from (2.3), (2.2) and the induction assumptions that

$$f_{n+1}(e) = \Sigma_{k\in K} f_n(e - k\cdot e) = |K| f_n(e) = |K|^{n+1} f(e),$$

$$\begin{aligned}\|f_{n+1}(x)-|K|f_n(x)\| &= \|\Sigma_{k\in K}f_n(x-k\cdot x)-|K|\Sigma_{k\in K}f_{n-1}(x-k\cdot x)\|\\ &= \|f_n(e)+\Sigma_{k\in K\backslash\{I\}}f_n(x-k\cdot x)\\ &\quad -|K|f_{n-1}(e)-|K|\Sigma_{k\in K\backslash\{I\}}f_{n-1}(x-k\cdot x)\|\\ &\leq \Sigma_{k\in K\backslash\{I\}}\|f_n(x-k\cdot x)-|K|f_{n-1}(x-k\cdot x)\|\\ &\leq (|K|-1)(|K|-1)^{n-1}\delta\\ &= (|K|-1)^n\delta.\end{aligned}$$

$$\begin{aligned}\|f_{n+1}(x)-|K|^{n+1}f(x)\| &\leq \|f_{n+1}(x)-|K|f_n(x)|\\ &\quad +|K|\|f_n(x)-|K|^nf(x)\|\\ &\leq (|K|-1)^n\delta+|K|[|K|^n-(|K|-1)^n]\delta\\ &= [|K|^{n+1}-(|K|-1)^{n+1}]\delta.\end{aligned}$$

Now, for all $x,y\in G$ we get

$$\begin{aligned}&\|\Sigma_{t\in K}f_{n+1}(x+t\cdot y)-|K|f_{n+1}(x)\|\\ &\qquad = \||K|f_n(e)+\Sigma_{t\in K}\Sigma_{k\in K\backslash\{I\}}f_n(x+t\cdot y-k\cdot(x+ty))\\ &\qquad\quad -|K|f_n(e)-|K|\Sigma_{k\in K\backslash\{I\}}f_n(x-k\cdot x)\|\\ &\qquad \leq \Sigma_{k\in K\backslash\{I\}}\|\Sigma_{t\in K}f_n(x-k\cdot x+t\cdot(y-k\cdot y))\\ &\qquad\quad -|K|f_n(x-k\cdot x)\|\\ &\qquad \leq (|K|-1)(|K|-1)^n\delta\\ &\qquad = (|K|-1)^{n+1}\delta,\end{aligned}$$

which gives the desired results.
From (2.5), it follows that the sequence functions

$$g_n(x)=\frac{f_n(x)}{|K|^n}$$

is a Cauchy sequence. Since B is complete, the above sequence has a limit and we denote it by $J(x)$. Thus

$$J(x)=\lim_{n\to+\infty}\frac{f_n(x)}{|K|^n}.$$

From (2.4), it follows that $J(e)=f(e)$.
From inequality (2.6), we get that $f-J$ is bounded: $\|f(x)-J(x)\|\leq\delta$ for all $x\in G$. From (2.7), we deduce that J: $G\to B$ satisfies the generalized Jensen's functional equation (2.1).
Now, let H: $G\to B$ be another solution of the generalized Jensen's functional equation

$$\Sigma_{k\in K}H(x+k\cdot y)=|K|H(x)\ \text{ for all } x,y\in G \tag{2.8}$$

which satisfies $H(e)=f(e)$ and $\|f(x)-H(x)\|\leq\delta$.
We will prove by induction that

$$\|f_n(x)-|K|^nH(x)\|<(|K|-1)^n\delta. \tag{2.9}$$

By using (2.8), (2.3) and the condition $H(e) = f(e)$, we get

$$\begin{aligned}
\|f_1(x) - |K|H(x)\| &= \|\Sigma_{k\in K} f(x - k\cdot x) - \Sigma_{k\in K} H(x - k\cdot x)\| \\
&= \|f(e) + \Sigma_{k\in K\setminus\{I\}} f(x - k\cdot x) - H(e) - \\
&\quad \Sigma_{k\in K\setminus\{I\}} H(x - k\cdot x)\| \\
&\leq \Sigma_{k\in K\setminus\{I\}} \|f(x - k\cdot x) - H(x - k\cdot x)\| \\
&\leq (|K| - 1)\delta.
\end{aligned}$$

Assuming that (2.9) is true for all integers $i \leq n$, hence we have

$$\begin{aligned}
\|f_{n+1}(x) - |K|^{n+1}H(x)\| &= \|\Sigma_{k\in K} f_n(x - k\cdot x) - |K|^n \Sigma_{k\in K} H(x - k\cdot x)\| \\
&= \||K|^n f(e) + \Sigma_{k\in K\setminus\{I\}} f_n(x - k\cdot x) \\
&\quad -|K|^n H(e) - |K|^n \Sigma_{k\in K\setminus\{I\}} H(x - k\cdot x)\| \\
&\leq \Sigma_{k\in K\setminus\{I\}} \|f_n(x - k\cdot x) - |K|^n H(x - k\cdot x)\| \\
&\leq (|K| - 1)(|K| - 1)^n \delta \\
&= (|K| - 1)^{n+1}\delta.
\end{aligned}$$

By letting $n \to +\infty$ we get from inequality $\|\frac{f_n(x)}{|K|^n} - H(x)\| \leq (\frac{|K|-1}{|K|})^n \delta$ that $J = H$. This completes the proof of Theorem 2.1. □

In the following corollary, we give the Hyers–Ulam–Rassias stability of the functional equation

$$\Sigma_{k\in K} g(x + k\cdot y) = |K|g(x) + |K|h(y),\ x, y \in G.$$

Corollary 2.2. *Let $\delta > 0$ and let $g, h : G \to B$ satisfy the inequality*

$$\|\Sigma_{k\in K} g(x + k\cdot y) - |K|g(x) - |K|h(y)\| < \delta,\ x, y \in G. \tag{2.10}$$

Then, there exist a K-quadratic mapping $q : G \to B$ and a mapping $J : G \to B$ a solution of equation (1.1) such that $J(e) = g(e) - h(e)$,

$$\|h(x) - q(x)\| \leq \frac{3\delta}{|K|}$$

and

$$\|g(x) - q(x) - J(x)\| \leq 4\delta + \frac{3\delta}{|K|}$$

for all $x \in G$.

Proof. Let $D(y, z)$ be the function defined on $G \times G$ by the following relation

$$D(y, z) = \||K|^2 h(z) + |K|^2 h(y) - |K|\Sigma_{k\in K} h(y + k\cdot z)\|.$$

The function $D(y,z)$ can also be written as follows

$$\begin{aligned}
& D(y,z) \\
= \;& \|[|K|^2 g(x) + |K|^2 h(z) - |K|\Sigma_{k\in K} g(x+k\cdot z)] \\
& +[|K|^2 h(y) + |K|\Sigma_{k\in K} g(x+k\cdot z) - \Sigma_{k\in K}\Sigma_{t\in K} g(x+t\cdot(y+k\cdot z))] \\
& +[\Sigma_{k\in K}\Sigma_{t\in K} g(x+t\cdot(y+k\cdot z) - |K|^2 g(x) - |K|\Sigma_{k\in K} h(y+k\cdot z))]\| \\
= \;& \|[|K|^2 g(x) + |K|^2 h(z) - |K|\Sigma_{k\in K} g(x+k\cdot z)] \\
& +[|K|^2 h(y) + |K|\Sigma_{k\in K} g(x+k\cdot z) - \Sigma_{k\in K}\Sigma_{t\in K} g((x+k\cdot z)+t\cdot y))] \\
& +[\Sigma_{k\in K}\Sigma_{t\in K} g(x+t\cdot(y+k\cdot z) - |K|^2 g(x) - |K|\Sigma_{k\in K} h(y+k\cdot z))]\| \\
\leq \;& |K|\|\Sigma_{k\in K} g(x+k\cdot z) - |K|g(x) - |K|h(z)\| \\
& +\Sigma_{k\in K}\|\Sigma_{t\in K} g((x+k\cdot z)+t\cdot y)) - |K|g(x+k\cdot z) - |K|h(y)\| \\
& +\Sigma_{k\in K}\|\Sigma_{t\in K} g(x+t\cdot(y+k\cdot z) - |K|g(x) - |K|h(y+k\cdot z))\| \\
\leq \;& 3|K|\delta.
\end{aligned}$$

Hence, for all $x,y\in G$ we have

$$\|\Sigma_{k\in K} h(x+k\cdot y) - |K|h(x) - |K|h(y)\| \leq 3\delta. \tag{2.11}$$

According to Theorem 1.2, there exists a unique function q: $G\to B$ which is solution of the K-quadratic functional equation (1.2) such that $\|h(x)-q(x)\| \leq \frac{3\delta}{|K|}$ for all $x\in G$.
Now, if we put $l=g-h$, by using (2.10) and (2.11) we get

$$\begin{aligned}
\|\Sigma_{k\in K} l(x+k\cdot y) - |K|l(x)\| \;&\leq\; \|\Sigma_{k\in K} g(x+k\cdot y) - |K|g(x) - |K|h(y)\| \\
&\quad +\|\Sigma_{k\in K} h(x+k\cdot y) - |K|h(x) - |K|h(y)\| \\
&\leq\; 4\delta \text{ for all } x,y\in G.
\end{aligned}$$

In view of Theorem 2.1, there exists a mapping J: $G\to B$ solution of the generalized Jensen's functional equation (1.1) such that $g-q-J$ is bounded and $J(e)=g(e)-h(e)$. This completes the proof. $\square$

In the following corollary we obtain the Hyers–Ulam–Rassias stability of the functional equation

$$\Sigma_{k\in K} f(x+k\cdot y) = |K|g(x) + |K|h(y),\ x,y\in G.$$

Corollary 2.3. *Let $\delta>0$ and let $f,g,h: G\to B$ satisfy the inequality*

$$\|\Sigma_{k\in K} f(x+k\cdot y) - |K|g(x) - |K|h(y)\| < \delta \text{ for all } x,y\in G. \tag{2.12}$$

Then, there exists a K-quadratic mapping $q: G\to B$, there exists a mapping $J: G\to B$ solution of equation (1.1) such that $J(e)=f(e)$,

$$\|f(x)-q(x)-J(x)\| \leq 8\delta + \frac{6\delta}{|K|}$$

$$\|g(x)-q(x)-J(x)+h(e)\| \leq 8\delta + \frac{7\delta}{|K|}$$

and

$$\|h(x)-q(x)-h(e)\| \leq \frac{6\delta}{|K|}$$

for all $x \in G$.

Proof. Putting $y = e$ in (2.12), we see that

$$\||K|f(x) - |K|g(x) - |K|h(e)\| \leq \delta,$$

so inequality (2.12) can be written as follows

$$\|\Sigma_{k\in K} f(x + k \cdot y) - |K|f(x) - |K|(h(y) - h(e))\| \leq 2\delta \text{ for all } x, y \in G.$$

Hence, from Corollary 2.2, we get the rest of the proof. □

In the following result, we obtain the Hyers–Ulam–Rassias stability of the Jensen's functional equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) \quad (x, y \in G) \tag{2.13}$$

in the spirit of Th.M. Rassias and P. Găvruta, where σ: $G \to G$ is a homomorphism of the abelian group G such that $\sigma \circ \sigma = I$.

Theorem 2.4. *Let G be an abelian group, let $(E, \|.\|)$ be a normed space, and let φ: $G \times G \to [0, \infty)$ be a mapping such that*

$$\psi(x, y) = \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \tag{2.14}$$

for all $x, y \in G$.
Assume that the map f: $G \to E$ satisfies the inequality

$$\|f(x + y) + f(x + \sigma(y)) - 2f(x)\| \leq \varphi(x, y) \tag{2.15}$$

for all $x, y \in G$. Then, there exists a unique mapping J: $G \to E$, solution of equation (2.13)*, such that $J(e) = f(e)$ and*

$$\|f(x) - J(x)\| \leq \frac{1}{2}\varphi(x, -x) + \frac{1}{4}\psi(x - \sigma(x), \sigma(x) - x) \tag{2.16}$$

for all $x \in G$.

Proof. Assume that f: $G \to E$ satisfies (2.15). We claim that for any positive integer $n \geq 2$, the following statements hold

$$\|f_n(x) - 2f_{n-1}(x)\| \leq \varphi(2^{n-2}x - 2^{n-2}\sigma(x), 2^{n-2}\sigma(x) - 2^{n-2}x), \tag{2.17}$$

$$\begin{aligned} &\|f_n(x) - 2^n f(x)\| \\ &\qquad \leq 2^{n-1}\varphi(x, -x) + \sum_{i=0}^{n-2} 2^{n-i-2}\varphi(2^i x - 2^i\sigma(x), 2^i\sigma(x) - 2^i x) \end{aligned} \tag{2.18}$$

and

$$\|f_n(x + y) + f_n(x + \sigma(y)) - 2f_n(x)\| \leq \varphi(2^{n-2}x - 2^{n-2}\sigma(x), 2^{n-2}y - 2^{n-2}\sigma(y)),$$

where

$$f_0(x) = f(x) \quad \text{and}$$
$$f_n(x) = f_{n-1}(e) + f_{n-1}(x - \sigma(x)) \quad (\text{for } n \geq 1).$$

By using some computations of the proof of Theorem 2.1, we can show by induction on n the above statements. From (2.14) and (2.17) it follows that for any fixed $x \in G$ the sequence

$$\{\frac{f_n(x)}{2^n}\}$$

is a Cauchy sequence and by the completeness of E this sequence has a limit. We denote it by $J(x)$. Hence,

$$J(x) = \lim_{n \to +\infty} \frac{f_n(x)}{2^n}.$$

Indeed, From (2.18), we obtain the inequality (2.16).
Finally, for the uniqueness of J, we use the following induction relation

$$2^n J(x) = J(2^{n-1}x - 2^{n-1}\sigma(x)) + (2^n - 1)J(e)$$

and inequality (2.16). □

Corollary 2.5. *Let G be an abelian group, and $(E, \|.\|)$ a normed space. If a mapping $f : G \to E$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \le \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \ge 0$, $0 \le p < 1$ and for all $x, y \in G$, then there exists a unique mapping $J : G \to E$, solution of the functional equation (2.13) such that $J(e) = f(e)$ and

$$\|f(x) - J(x)\| \le \theta\{\|x\|^p + \frac{1}{2-2^p}\|x - \sigma(x)\|^p\}$$

for all $x \in G$.

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[1,2] Bouikhalene Belaid, Laboratory LAMA, Department of Mathematics, University of Ibn Tofail, Faculty of Sciences,
BP 133, Kénitra 14000, Morocco.
E-mail address: charifi2000@yahoo.fr and bbouikhalene@yahoo.fr

[3] Elqorachi Elhoucien, Laboratory LAMA, Harmonic Analysis and functional equations Team, Department of Mathematics, Faculty of Sciences, University Ibn Zohr, Agadir, Morocco.
E-mail address: elqorachi@hotamail.com