Strong Morphisms of Groupoids

Gh. Ivan

Abstract

We reffer to the groupoids in the sense of Ehresmann. The aim of this paper is to give some various topics of strong morphisms of groupoids.

Mathematics Subject Classification: 20L13; 20L99: 18B40
Key words: groupoid morphism, normal subgroupoid, strong morphism of groupoids

Introduction

The concept of groupoid in the sense of Ehresmann is a natural generalization of the algebraic notion of groupoid introduced by H. Brandt in the paper: Uber eine Verallgemeinerung der Gruppenbegriffe, Math. Ann., 96 (1926), 360-366.

The notions of topological and differentiable groupoids has been introduced by Ehresmann in 1950 in his paper on connections (cf. [4])

Many authors have investigated the Lie groupoids (in particular, symplectic groupoids) in connection with their applications in differential geometry, symplectic geometry, Poisson geometry, quantum mechanics ergodic theory , geometric quantization and gauge theories (cf. [1], [11] - [15], [17] - [20]).

Recent applications of Lie groupoids endowed with supplementary structures have also contributed to a renewed interest in these studies.

In this paper we study a special case of groupoid morphism, namely: strong morphism of Ehresmann groupoids.

Other special morphisms of groupoids are the following:

- similar morphisms of Brandt groupoids (these morphisms are used in [7] for construct a cohomology theory of Brandt groupoids which extends the usual cohomology theory of groups;
- pullback, fibrewise injective (resp., surjective, bijective) and piecewise injective (resp., surjective, bijective) morphism of groupoids (for various topics concerning these special morphisms see [9]).

Balkan Journal of Geometry and Its Applications, Vol.4, No.1, 1999, pp. 91-102

1 Morphisms of groupoids

In this section we construct the category of Ehresmann groupoids and some important properties concerning the morphisms of groupoids are given.

Definition 1.1. ([15]) A *groupoid* (in the sense of Ehresmann) Γ over Γ_0 or *groupoid* with the base Γ_0, is a pair $(\Gamma;\Gamma_0)$ of sets equipped with:

(i) two surjections $\alpha,\beta:\Gamma\longrightarrow\Gamma_0$, called the *source* and the *target* map;

(ii) a (partial) composition law $\mu:\Gamma_{(2)}\longrightarrow\Gamma, (x,y)\longrightarrow\mu(x,y)=x\cdot y=xy$, with domain $\Gamma_{(2)}=\{(x,y)\in\Gamma\times\Gamma \quad | \quad \beta(x)=\alpha(y)\}$;

(iii) an injection $\epsilon:\Gamma_0\longrightarrow\Gamma, u\longrightarrow\epsilon(u)=\widetilde{u}$, called the *inclusion map*;

(iv) a map $i:\Gamma\longrightarrow\Gamma, \quad x\longrightarrow i(x)=x^{-1}$, called the *inversion map.*

These maps must satysfy the following algebraic axioms generalizing those of groups:

(G1) (*associative law*) For arbitrary $x,y,z\in\Gamma$ the triple product $(xy)z$ is defined iff $x(yz)$ is defined. In case either is defined, we have $\quad (xy)z=x(yz) \quad$; hence, the triple xyz is defined whenever $\beta(x)=\alpha(y)$ and $\beta(y)=\alpha(z)$.

(G2) (*identities*) For each $x\in\Gamma$ we have $(\epsilon(\alpha(x)),x)\in\Gamma_{(2)};(x,\epsilon(\beta(x))\in\Gamma_{(2)}$ and $\epsilon(\alpha(x))\cdot x=x\cdot\epsilon(\beta(x))=x$

(G3) (*inverses*) For each $x\in\Gamma$ we have $(x,i(x))\in\Gamma_{(2)};(i(x),x)\in\Gamma_{(2)}$ and $x\cdot i(x)=\epsilon(\beta(x)), \quad i(x)\cdot x=\epsilon(\alpha(x)) \quad \triangle$

Every group G with e as unity, is a groupoid over $G_0=\{e\}$.

We denote a groupoid Γ over Γ_0 by $(\Gamma,\alpha,\beta,\epsilon,i,\mu;\Gamma_0)$ or $(\Gamma,\alpha,\beta;\Gamma_0)$ or $(\Gamma;\Gamma_0)$.

For each $u\in\Gamma_0$, the set $\Gamma_u=\alpha^{-1}(u)$ (resp. $\Gamma^u=\beta^{-1}(u)$) is called the *α-fibre* (resp. *β-fibre*) of Γ over $u\in\Gamma_0$ and if $u,v\in\Gamma$, we will write $\Gamma_u^v=\Gamma_u\cap\Gamma^v$.

A groupoid Γ over Γ_0 such that Γ_0 is a subset of Γ is called Γ_0-*groupoid* or *Brandt groupoid.*

We summarize some properties of these mappings obtained from definitions.

Proposition 1.1. Let Γ be a groupoid over Γ_0. The following assertions hold:

(i) $\alpha\circ\epsilon=\beta\circ\epsilon=Id_{\Gamma_0}$.

(ii) $\alpha(xy)=\alpha(x) \quad$ and $\quad \beta(xy)=\beta(y) \quad$ for all $\quad (x,y)\in\Gamma_{(2)}$.

(iii) $\epsilon(u)\cdot\epsilon(u)=\epsilon(u) \quad$ for each $\quad u\in\Gamma_0$.

(iv) Let $u,v\in\Gamma_0$. We have:

(a) if $(x,\epsilon(u))\in\Gamma_{(2)}$ such that $x\cdot\epsilon(u)=x$ then $\epsilon(u)=\epsilon(\beta(x))$.

(b) if $(\epsilon(v),x)\in\Gamma_{(2)}$ such that $\epsilon(v)\cdot x=x$ then $\epsilon(v)=\epsilon(\alpha(x))$.

(v) For all $x\in\Gamma \quad$ we have $\quad \beta(x^{-1})=\alpha(x) \quad$ and $\quad \alpha(x^{-1})=\beta(x)$

(vi) For $u\in\Gamma_0 \quad$ we have $\quad (\epsilon(u))^{-1}=\epsilon(u)$.

(vii) $\alpha\circ i=\beta, \quad \beta\circ i=\alpha \quad$ and $\quad i\circ i=Id_\Gamma$.

(viii) For each $u\in\Gamma_0$, the set $\Gamma(u)=\alpha^{-1}(u)\cap\beta^{-1}(u)$ is a group under the restriction of the partial multiplication (this group is be called the *isotropy group at* u of the groupoid Γ).

(ix) In the case $\Gamma_0\subseteq\Gamma$, we have:

(a) $\epsilon(\Gamma_0)=\Gamma_0$.

(b) $\epsilon(u)=u, \quad$ for each $u\in\Gamma_0$. $\triangle$

In view of Proposition 1.1., the element $\epsilon(\alpha(x))$ (resp. $\epsilon(\beta(x))$) is the *left unit* (resp., *right unit*) of $x\in\Gamma$. The subset $\epsilon(\Gamma_0)$ is be called the *unity set* of Γ.

Definition 1.2. (a) A groupoid Γ over Γ_0 is said to be *transitive* if the map $\alpha\times\beta:\Gamma\longrightarrow\Gamma_0\times\Gamma_0$, given by $(\alpha\times\beta)(x)=(\alpha(x),\beta(x)),(\forall)x\in\Gamma$ is surjective.

(b) By *group bundle* we mean a groupoid Γ over Γ_0 such that $\alpha(x) = \beta(x)$ for each $x \in \Gamma$. Moreover, a group bundle is the union of its isotropy groups $\Gamma(u) = \alpha^{-1}(u), u \in \Gamma_0$ (here two elements may be composed iff they lie in the same fiber.) $\triangle$

If $(\Gamma, \alpha, \beta; \Gamma_0)$ is a groupoid over Γ_0, then $Is(\Gamma) = \{x \in \Gamma \mid \alpha(x) = \beta(x)\}$ is a group bundle, called the *isotropy group bundle* associated to Γ. It is easy to see that $\epsilon(\Gamma_0) \subseteq Is(\Gamma)$.

Proposition 1.2. If $(\Gamma, \alpha, \beta; \Gamma_0)$ is a groupoid, then the following assertions hold:

(i) (*cancellation law*) If $x \cdot z_1 = x \cdot z_2$ (resp., $z_1 \cdot x = z_2 \cdot x$) then $z_1 = z_2$.

(ii) If $(x, y) \in \Gamma_{(2)}$ then $(y^{-1}, x^{-1}) \in \Gamma_{(2)}$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

(iii) The isotropy groups $\Gamma(\alpha(x))$ and $\Gamma(\beta(x))$ are isomorphic.

(iv) If Γ is transitive, then the isotropy groups of Γ are groups isomorphes.

Proof. (i) and (ii) These assertions follows from definitions.

(iii) We prove that the map $\varphi : \Gamma(\alpha(x)) \longrightarrow \Gamma(\beta(x)), a \longrightarrow \varphi(a) = x \cdot a \cdot x^{-1}$ is a isomorphism of groups.

(iv) It follows from (iii) and the fact that $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$ is surjective. $\triangle$

Example 1.1. (a) *Nul groupoid.* Any set B is a groupoid on itself with $\Gamma = \Gamma_0 := B$, $\alpha = \beta = \epsilon := id_B$ and every element is a unity, called it *the nul groupoid.* The multiplication is given by $x \cdot x := x$ for all $x \in B$.

(b) *Coarse groupoid.* If B is any non-empty set, then $B \times B$ is a groupoid over B with the rules:

$$\alpha(x, y) := x; \beta(x, y) := y; \epsilon(x) := (x, x), \quad i(x, y) := (y, x)$$

and

$$\mu((x, y), (y', z)) := (x, z) \quad \text{iff} \quad y = y'.$$

The unit set of this groupoid, called it *the coarse groupoid associated* to B, is the diagonal Δ_B of the cartesian product $B \times B$.

(c) *Trivial groupoid.* Let B be any non- empty set and $\mathcal{G}$ be a multiplicative group with e as unity. Construct a transitive groupoid Γ over B, called the *trivial groupoid on B with group* $\mathcal{G}$, in the following way:

$$\Gamma := B \times B \times \mathcal{G}; \ \Gamma_0 := B; \quad \alpha(a, b, x) := a; \quad \beta(a, b, x) := b; \quad \epsilon(b) := (b, b, e);$$

$$i(a, b, x) := (b, a, x^{-1}) \quad \text{and} \quad \mu((a, b, x), (b', c, y)) := (a, c, xy) \quad \text{iff} \quad b = b'.$$

For this groupoid we have

$$\epsilon(\Gamma_0) = \{(b, b, e) \mid b \in B\} \quad \text{and} \quad \Gamma(b) = \{(b, b, x) \mid x \in \mathcal{G}\},$$

which are identified with B resp. $\mathcal{G}$.

If $\mathcal{G} = \{e\}$, then we can identify $B \times B \times \mathcal{G}$ with the coarse groupoid associated to B.

(d) A *vector bundle* $E \xrightarrow{\pi} M$ is a group bundle on M. Here $\Gamma := E$ is the total space, $\Gamma_0 := M$ is the base space, $\alpha = \beta := \pi$ so that $\Gamma_{(2)} := \uplus_{x \in M} E_x \times E_x$ (E_x is the fibre at x) and the composition law is fibrewise addition.$\triangle$

Other examples of groupoids are the following: the *fundamental groupoid of a topological space* (see [6]), the *disjoint union* of a disjoint family of groupoids (see [9]) and the *action groupoid* (see [13]).

Definition 1.3. Let $(\Gamma,\alpha,\beta,\epsilon,i,\mu;\Gamma_0)$ and $(\Gamma',\alpha',\beta',\epsilon',i',\mu';\Gamma'_0)$ be two groupoids. A *morphism of groupoids* or *groupoid morphism* is a pair (f,f_0) of maps $f:\Gamma\longrightarrow\Gamma'$ and $f_0:\Gamma_0\longrightarrow\Gamma'_0$ such that the following two conditions are satisfied:

$$(1)\qquad f(\mu(x,y))=\mu'(f(x),f(y))\quad\text{for every}\quad (x,y)\in\Gamma_{(2)}$$

$$(2)\qquad \alpha'\circ f=f_0\circ\alpha\quad\text{and}\quad \beta'\circ f=f_0\circ\beta.\triangle$$

If $\Gamma_0=\Gamma'_0$ and $f_0=Id_{\Gamma_0}$, we say that f is a Γ_0**- morphism**.$\triangle$

Note that the condition (1) ensure that $(f(x),f(y))\in\Gamma'_{(2)}$, i.e. $\mu'(f(x),f(y))$ is defined whenever $\mu(x,y)$ is defined.

Applying Propositions 1.1 and 1.2 we obtain:

Proposition 1.3. The groupoids morphisms preserve unities and inverses, i.e. $f(\widetilde{u})=\widetilde{f_0(u)}$, $(\forall)u\in\Gamma_0$ and $f(x^{-1})=(f(x))^{-1}$, $(\forall)x\in\Gamma$; in other words, we have: $f\circ\epsilon=\epsilon'\circ f_0$ and $f\circ i=i'\circ f$ $\triangle$

Proposition 1.4. A pair $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is a groupoid morphism iff the following condition holds:

$$(3)\quad (\forall)(x,y)\in\Gamma_{(2)}\implies (f(x),f(y))\in\Gamma'_{(2)}\quad\text{and}\quad f(\mu(x,y))=\mu'(f(x),f(y))$$

Proof. The condition (3) is a consequence of Definition 1.4. and Prop.1.3.

Conversely, let $f:\Gamma\longrightarrow\Gamma'$ which satisfy (3) and we define the map $f_0:\Gamma_0\longrightarrow\Gamma'_0$ by $f_0(u)=\alpha'(f(\epsilon(u)))$, $(\forall)u\in\Gamma_0$. We prove that $\alpha'\circ f=f_0\circ\alpha$ and $\beta'\circ f=f_0\circ\beta$.

Indeed, since $(x,\epsilon(\beta(x)))\in\Gamma_{(2)}$ it follows that $(f(x),f(\epsilon(\beta(x))))\in\Gamma'_{(2)}$ and

$$f(x)\cdot f(\epsilon(\beta(x))))=f(x\cdot\epsilon(\beta(x)))=f(x);\ \text{but}\ f(x)\cdot\epsilon'(\beta'(f(x)))=f(x);$$

$$\implies\epsilon'(\beta'(f(x)))=f(\epsilon(\beta(x)))\implies\alpha'(\epsilon'(\beta'(f(x))))=\alpha'(f(\epsilon(\beta(x))))$$

and applying Prop.1.1. we obtain succesivelly

$$\beta'(f(x))=(f_0\circ\alpha)(\epsilon(\beta(x)))\implies\beta'(f(x))=f_0(\beta(x))$$

i.e. $\beta'\circ f=f_0\circ\beta$. Similarly we prove that $\alpha'\circ f=f_o\circ\alpha.\triangle$

Example 1.2. (a) If $(\Gamma,\alpha,\beta,\epsilon;\Gamma_0)$ is a groupoid, then $(Id_\Gamma,Id_{\Gamma_0})$ is a groupoid morphism.

(b) If $(f,f_0):(\Gamma,\Gamma_0)\to(\Gamma';\Gamma'_0)$ and $(g,g_0):(\Gamma',\Gamma'_0)\to(\Gamma'',\Gamma''_0)$ are groupoid morphisms, then the composition $(g,g_0)o(f,f_0):(\Gamma,\Gamma_0)\to(\Gamma'',\Gamma''_0)$ defined by $(g,g_0)\circ(f,f_0)=(g\circ f,g_0\circ f_0)$ is a groupoid morphism.$\triangle$

If $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is a groupoid morphism, then for every $u,v\in\Gamma_0$ we have:

$$f(\Gamma_u)\subseteq\Gamma'_{f_0(u)};\quad f(\Gamma^v)\subseteq(\Gamma')^{f_0(v)}\ \text{and}\ f(\Gamma_u^v)\subseteq(\Gamma')_{f_0(u)}^{f_0(v)}.$$

Then the restriction of f to $\Gamma_u,\Gamma^v,\Gamma_u^v$ respectively, defines the groupoid morphisms

$$\Gamma_u\longrightarrow\Gamma'_{f_0(u)};\ \Gamma^v\longrightarrow(\Gamma')^{f_0(v)},\Gamma_u^v\longrightarrow(\Gamma')_{f_0(u)}^{f_0(v)},$$

denoted by f_u,f^v and f_u^v.

Definition 1.4. A groupoid morphism $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is said to be *isomorphism* of groupoids if there exists a groupoid morphism $(g,g_0):(\Gamma';\Gamma'_0)\to(\Gamma,\Gamma_0)$

with $(g,g_0)o(f,f_0)=(id_\Gamma,id_{\Gamma_0})$ and $(f,f_0)o(g,g_0)=(id_{\Gamma'},id_{\Gamma'_0})$. Two groupoids $(\Gamma;\Gamma_0)$ and $(\Gamma';\Gamma'_0)$ are said to be **isomorphic** if there exists an isomorphism $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$.△

Proposition 1.5. Let $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ be a groupoid morphism. Then the following assertions hold:

(i) If f is injective (resp., surjective), then also is f_0.

(ii) (f,f_0) is an isomorphism iff the map f is bijective.

(iii) $f(Is(\Gamma))\subseteq Is(\Gamma')$.

(iv) A groupoid morphism (f,f_0) such that f is surjective and f_0 is injective (in particular, every surjective Γ_0- morphism of groupoids) preserve the isotropy group bundles, i.e. $f(Is(\Gamma))=Is(\Gamma')$.

Proof. (i) This follows imediately from Definition 1.3. and Proposition 1.4.

(ii) It is a consequence of Definition 1.4. and of the assertion (i).

(iii) Let $x'\in f(Is(\Gamma))$. Then $x'=f(x)$ with $x\in Is(\Gamma)$ and we have

$$\alpha'(x')=\alpha'(f(x))=f_0(\alpha(x))=f_0(\beta(x))=\beta'(f(x))=\beta'(x'),$$

since $\alpha(x)=\beta(x)$; hence $x'\in Is(\Gamma')$. Therefore, $f(Is(\Gamma))\subseteq Is(\Gamma')$.

(iv) It suffices to prove that $Is(\Gamma')\subseteq f(Is(\Gamma))$. Let $x'\in Is(\Gamma')$ i.e. $x'\in\Gamma'$ such that $\alpha(x')=\beta'(x')$. For $x'\in\Gamma'$ there exists $x\in\Gamma$ such that $x'=f(x)$, since f is surjective. Then $\alpha'(f(x))=\beta'(f(x))$ and we obtain that $f_0(\alpha(x))=f_0(\beta(x))$. Hence, $\alpha(x)=\beta(x)$, since f_0 is injective. Thus, $x\in Is(\Gamma)$ and $x'\in f(Is(\Gamma))$. Therefore, $Is(\Gamma')\subseteq f(Is(\Gamma))$.△

Example 1.3. (a) Let $(\Gamma,\alpha,\beta,\epsilon;\Gamma_0)$ be a groupoid and $(\Gamma_0\times\Gamma_0,\alpha',\beta',\epsilon';\Gamma_0)$ the coarse groupoid associated to Γ_0.Then $\alpha\times\beta:\Gamma\longrightarrow\Gamma_0\times\Gamma_0$, $(\alpha\times\beta)(x)=(\alpha(x),\beta(x))$ is a Γ_0- morphism of the groupoid Γ into the coarse groupoid $\Gamma_0\times\Gamma_0$.

b) Let $(\Gamma,\alpha,\beta,\epsilon,i,\mu;\Gamma_0)$ be a groupoid over Γ_0 and X a set with the same cardinal as Γ_0, i.e. there exists a bijection φ from Γ_0 to X. Then Γ has a canonical structure of a groupoid over X, that is $(\Gamma,\alpha',\beta',\epsilon',i',\mu';X)$ is a groupoid over X where $\alpha':=\varphi o\alpha;\beta':=\varphi o\beta;\epsilon':=\epsilon o\varphi^{-1};i':=\varphi oi;\mu':=\mu$.Moreover, $(id_\Gamma,\varphi):(\Gamma;\Gamma_0)\to(\Gamma;X)$ is an isomorphism of groupoids.△

Example 1.4. (*the induced groupoid*) Let $(\Gamma,\alpha,\beta,\epsilon;\Gamma_0)$ be a groupoid, X an abstract set and $f:X\longrightarrow\Gamma_0$ a map from X to Γ_0. Then the set:

$$f^*(\Gamma)=\{(x,y,a)\in X\times X\times\Gamma\quad|\quad f(x)=\alpha(a),f(y)=\beta(a)\}$$

has a canonical structure of groupoid over X with respect to the following rules:

$$\alpha^*(x,y,a):=x;\beta^*(x,y,a):=y;\epsilon^*(x):=(x,x,\epsilon(f(x)));i^*(x,y,a):=(y,x,i(a)),$$

and

$$\mu^*((x,y,a),(y',z,b)):=(x,z,\mu(a,b))$$

iff $y=y'$ and $(a,b)\in\Gamma_{(2)}$.

The groupoid $(f^*(\Gamma),\alpha^*,\beta^*,\epsilon^*,\mu^*;\Gamma_0)$ is called the *induced groupoid* or the *inverse image* of Γ *under* f; it is denoted sometimes by $f^*(\Gamma)$.

If $f^*(\Gamma)$ is the induced groupoid of Γ under $f:X\longrightarrow\Gamma_0$ then $f^*_\Gamma:f^*(\Gamma)\longrightarrow\Gamma$ defined by $f^*_\Gamma(x,y,a)=a$ together with f define a groupoid morphism $(f^*_\Gamma,f):(f^*(\Gamma);X)\longrightarrow(\Gamma;\Gamma_0)$ and it is called the *canonical morphism of an induced groupoid*.△

2 Strong morphisms of groupoids

This section is dedicated to study of a particular type of groupoid morphisms, namely: the strong morphisms of groupoids. One of the most important results of strong morphisms is the correspondence theorem for subgroupoids (resp., for normal subgroupoids).

Definition 2.1. A *subgroupoid* of a groupoid $(\Gamma;\Gamma_0)$ is a pair $(\Gamma';\Gamma'_0)$ of subsets, where $\Gamma' \subseteq \Gamma$, $\Gamma'_0 \subseteq \Gamma_0$ such that the following conditions are verified:

(i) $\alpha(\Gamma') \subseteq \Gamma'_0$; $\beta(\Gamma') \subseteq \Gamma'_0$

(ii) for every $x, y \in \Gamma'$ such that the product $x \cdot y$ is defined implies that $x \cdot y \in \Gamma'$, i.e. Γ' is closed under the partial multiplication.

(iii) $(\forall) u \in \Gamma'_0 \implies \epsilon(u) \in \Gamma'$

(iv) $(\forall) x \in \Gamma' \implies x^{-1} \in \Gamma'$.

A subgroupoid $(\Gamma';\Gamma'_0)$ of $(\Gamma;\Gamma_0)$ is **wide** if $\Gamma'_0 = \Gamma_0$. $\triangle$

Definition 2.2. A *normal subgroupoid* of a groupoid $(\Gamma;\Gamma_0)$ is a wide subgroupoid N of Γ such that: for any $\lambda \in N$ and any $x \in \Gamma$ such that $\beta(x) = \alpha(\lambda) = \beta(\lambda)$ we have $x \cdot \lambda \cdot x^{-1} \in N$.$\triangle$

Example 2.1. (a) If $(\Gamma;\Gamma_0)$ is a groupoid, then $\epsilon(\Gamma_0) = \{\widetilde{u} \quad | \quad u \in \Gamma_0\}$ is a normal subgroupoid of Γ over Γ_0, called the *nul subgroupoid* of Γ.

(b) If $(\Gamma;\Gamma_0)$ is a groupoid, then $Is(\Gamma) = \bigcup_{u\in\Gamma_0} \Gamma_u^u$ is a normal subgroupoid of Γ over Γ_0, called the *inner subgroupoid* of Γ.

(c) The *kernel* of a groupoid morphism $(f.f_0) : (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$ defined by: $Kerf = \{x \in \Gamma \quad | \quad f(x) \in \epsilon'(\Gamma'_0)\}$ is a normal subgroupoid of Γ over Γ_0.$\triangle$

Proposition 2.1. Let $(f, f_0) : (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$ be a groupoid morphism. Then the following assertions hold:

(i) If $(\Omega';\Omega'_0)$ is a subgroupoid of $(\Gamma';\Gamma'_0)$, then $(f^{-1}(\Omega'); f_0^{-1}(\Omega'_0))$ is a subgroupoid of $(\Gamma;\Gamma_0)$.

(ii) If Ω' is a normal subgroupoid of Γ', then $f^{-1}(\Omega')$ is a normal subgroupoid of Γ such that $Kerf \subseteq f^{-1}(\Omega)$.

Proof. (i) We prove that $(f^{-1}(\Omega'); f_0^{-1}(\Omega'_0))$ satisfies the conditions of Definition 2.1.

- $\alpha(f^{-1}(\Omega')) \subseteq f_0^{-1}(\Omega'_0)$. Indeed, if $u \in \alpha(f^{-1}(\Omega'))$ it follows that $u = \alpha(x)$ with $x \in f^{-1}(\Omega')$. Then $f_0(u) = f_0(\alpha(x)) = \alpha'(f(x)) \in \Omega'_0$, since $f(x) \in \Omega'$ and $\alpha'(\Omega') \subseteq \Omega'_0$. Hence, $u \in f_0^{-1}(\Omega'_0)$. Similarly, $\beta(f^{-1}(\Omega')) \subseteq f_0^{-1}(\Omega'_0)$.

- Let $x, y \in f^{-1}(\Omega')$ such that $x \cdot y$ is defined, i.e. $\beta(x) = \alpha(y)$. It follows that $f(x), f(y) \in \Omega'$ and $\beta'(f(x)) = f_0(\beta(x)) = f_0(\alpha(y)) = \alpha'(f(y))$; hence, $f(x) \cdot f(y)$ is defined in Γ'. Then $f(x) \cdot f(y) \in \Omega'$, since Ω' is subgroupoid. Then $f(x \cdot y) \in \Omega'$, i.e. $x \cdot y \in f^{-1}(\Omega')$.Therefore, the condition (ii) of Definition 2.1. is verified.

- For every $u \in f_0^{-1}(\Omega'_0)$ we have $\epsilon(u) \in f^{-1}(\Omega')$. Indeed, $f_0(u) \in \Omega'_0$ and we have $\epsilon'(f_0(u)) \in \Omega'$, since Ω' is subgroupoid. Then $f(\epsilon(u)) \in \Omega'$, i.e. $\epsilon(u) \in f^{-1}(\Omega')$.

- For every $x \in f^{-1}(\Omega')$, we have $x^{-1} \in f^{-1}(\Omega')$. Indeed, from $f(x) \in \Omega'$ follows $(f(x))^{-1} \in \Omega'$, since Ω' is subgroupoid. Then $f(x^{-1}) \in \Omega'$, i.e. $x^{-1} \in f^{-1}(\Omega')$.

(ii) In view of (i) follows that $f^{-1}(\Omega';\Gamma_0)$ is a subgroupoid of $(\Gamma;\Gamma_0)$.

Let $\lambda \in f^{-1}(\Omega')$ and $x \in \Gamma$ such that $\beta(x) = \alpha(\lambda) = \beta(\lambda)$ and we prove that $x \cdot \lambda \cdot x^{-1} \in f^{-1}(\Omega')$.

Indeed, we have $f(\lambda) \in \Omega'$ and $\beta'(f(x)) = f_0(\beta(x)) = f_0(\alpha(\lambda)) = \alpha'(f(\lambda))$ and $\beta'(f(x)) = f_0(\beta(x)) = f_0(\beta(\lambda)) = \beta'(f(\lambda))$. From $f(\lambda) \in \Omega'$, $\beta'(f(x)) = \alpha'(f(\lambda)) =$

$\beta'(f(\lambda))$ and the fact that Ω' is normal in Γ' follows $f(x)\cdot f(\lambda)\cdot(f(x))^{-1}\in\Omega'$. Hence, $f(x\cdot\lambda\cdot x^{-1})\in\Omega'$, i.e. $x\cdot\lambda\cdot x^{-1}\in f^{-1}(\Omega')$. Therefore, $f^{-1}(\Omega')$ is normal.

- We have $Kerf\subseteq f^{-1}(\Omega')$. Indeed, for $x\in Kerf$, we have $f(x)=\epsilon'(u')$ with $u'\in\Gamma_0'$ and by the condition (iii) of Definition 2.1. follows $\epsilon'(u')\in\Omega'$. Then $f(x)\in\Omega'$, i.e. $x\in f^{-1}(\Omega')$. $\triangle$

Corollary 2.1. *Let $f:\Gamma\longrightarrow\Gamma'$ be a Γ_0- groupoid morphism. Then the following assertions hold:*

(i) If $(\Omega';\Omega_0')$ is a subgroupoid of $(\Gamma';\Gamma_0)$, then $(f^{-1}(\Omega');f_0^{-1}(\Omega_0'))$ is a subgroupoid of $(\Gamma;\Gamma_0)$.

(ii) If Ω' is a normal subgroupoid of Γ', then $f^{-1}(\Omega')$ is a normal subgroupoid of Γ such that $Kerf\subseteq f^{-1}(\Omega)$.

Proof. We apply the Proposition 2.1.$\triangle$

Remark 2.1. If $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma_0')$ is a groupoid, then not always, $Imf=\{f(x)\quad|\quad x\in\Gamma\}$ is a subgroupoid of Γ'. For example, let

$$\Gamma=\{(0,0);(0,1);(1,0);(1,1)\}=B\times B$$

the coarse groupoid associated to set $B=\{0,1\}$ and let the map $f:\Gamma\longrightarrow Z$ defined by $f(0,0)=0;f(0,1)=1;f(1,0)=-1;f(1,1)=0$. W e denote by $f_0:B\longrightarrow\{0\}$ the map defined by $f_0(0)=0$ and $f_0(1)=0$. We can prove easily the conditions of Definition 1.3. are satisfied for the pair (f,f_0) of the coarse groupoid Γ over B into the group additiv $\mathbf{Z}$ of entiers numbers over $\{0\}$, having $Imf=\{0,-1,1\}$ which is not a subgroup of $\mathbf{Z}$. Hence Imf is not a subgroupoid. $\triangle$

Definition 2.3. A *strong morphism of groupoids* or *groupoid strong morphism* is a groupoid morphism $(f,f_0):(\Gamma;\Gamma')\longrightarrow(\Gamma';\Gamma_0')$ such that the following condition holds:

$$\text{for every}\quad(f(x),f(y))\in\Gamma'_{(2)}\quad\text{we have}\quad(x,y)\in\Gamma_{(2)}.\triangle\tag{4}$$

Remark 2.2. The concept of strong morphism has considered by A. Ramsay (cf. [18]) in the case of Brandt groupoids, called it *true morphism* of groupoids. $\triangle$

Remark 2.3. If (f,f_0) is a strong morphism of groupoids, then

$$f_u:\Gamma_u\longrightarrow\Gamma'_{f_0(u)};f^v:\Gamma^v\longrightarrow(\Gamma')^{f_0(v)}\text{ and }f_u^v:\Gamma_u^v\longrightarrow(\Gamma')^{f_0(v)}_{f_0(u)}$$

are also strong morphisms of groupoids. $\triangle$

Theorem 2.1. *(i) If $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma_0')$ is a groupoid morphism such that the map f_0 is injective, then (f,f_0) is a groupoid strong morphism.*

(ii) Every Γ_0- morphism of groupoids $f:\Gamma\longrightarrow\Gamma'$ is a groupoid strong morphism.

Proof. (i) We suppose that $(f(x),f(y))\in\Gamma'_{(2)}$, with $x,y\in\Gamma$. Then

$$\beta'(f(x))=\alpha'(f(y))\Longrightarrow(\beta'\circ f)(x)=(\alpha'\circ f)(y)\Longrightarrow(f_0\circ\beta)(x)=(f_0\circ\alpha)(y)\Longrightarrow$$

$$\Longrightarrow f_0(\beta(x))=f_0(\alpha(y))\Longrightarrow\beta(x)=\alpha(y)(\text{since } f_0 \text{ is injective})\Longrightarrow(x,y)\in\Gamma_{(2)}.$$

Hence (f,f_0) is a groupoid strong morphism.

(ii) This is a consequence of (ii), since $f_0=Id_{\Gamma_0}$. $\triangle$

Example 2.2. (i) The morphism $\alpha\times\beta:\Gamma\longrightarrow\Gamma_0\times\Gamma_0$, given in Definition 1.2., is a groupoid strong morphism.

(ii) The canonical morphism (f^*_Γ, f) of induced groupoid $f^*(\Gamma)$ of Γ by $f: X \longrightarrow \Gamma_0$ is not a groupoid strong morphism. $\triangle$

Proposition 2.2. Let $(f, f_0): (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$ be a groupoid strong morphism. Then the following assertions hold:

(i) If $(\Omega;\Omega_0)$ is a subgroupoid of $(\Gamma;\Gamma_0)$, then $(f(\Omega); f_0(\Omega_0))$ is a subgroupoid of $(\Gamma';\Gamma'_0)$. In particular, Imf is a subgroupoid of Γ' over Imf_0.

(ii) If f is surjective and Ω is a normal subgroupoid of Γ, then $f(\Omega)$ is a normal subgroupoid of Γ'.

Proof. (i) We have $(\alpha'(f(\Omega)) \subseteq f_0(\Omega_0)$. Indeed, for any $u' \in \alpha'(f(\Omega))$ exists $y' \in f(\Omega)$ such that $u' = \alpha'(y')$. For $y' \in f(\Omega)$ exists $y \in \Omega$ such that $f(y) = y'$. Then $u' = \alpha'(y') = \alpha'(f(y)) = f_0(\alpha(y))$; hence $u' \in f_0(\Omega)$, since $\alpha(y) \in \Omega_0$. Similarly, $\beta'(f(\Omega)) \subseteq f_0(\Omega_0)$.

- We have $\epsilon'(u') \subseteq f(\Omega)$, for all $u' \in f_0(\Omega_0)$.Indeed, for $u' \in f_0(\Omega_0)$ exists $u \in \Omega_0$ such that $u' = f_0(u) \implies \epsilon'(u') = \epsilon'(f_0(u)) = f(\epsilon(u)) \in f(\Omega)$, since $\epsilon(u) \in \Omega$.

- Let $x', y' \in f(\Omega)$ such that $x' \cdot y'$ is defined.We prove that $x' \cdot y' \in f(\Omega)$. Indeed, $x' = f(x), y' = f(y)$ with $x, y \in \Omega$. Since, $x' \cdot y'$ is defined it implies that $(f(x), f(y)) \in \Gamma'_{(2)}$, and we have $(x, y) \in \Gamma_{(2)}$, since f is a groupoid strong morphism. Hence $x \cdot y$ is defined. We have $x \cdot y \in \Omega$, since Ω is subgroupoid of Γ,and therefore $x' \cdot y' = f(x) \cdot f(y) = f(x \cdot y) \in f(\Omega)$.

- For any $x' \in f(\Omega)$ we have $(x')^{-1} \in f(\Omega)$. Indeed, $x' = f(x)$, with $x \in \Omega \implies (x')^{-1} = (f(x))^{-1} = f(x^{-1}) \in f(\Omega)$, since $x^{-1} \in \Omega$.

Therefore $(f(\Omega); f_0(\Omega_0))$ is a subgroupoid of $(\Gamma';\Gamma'_0)$.

(ii) By (i) $(f(\Omega);\Gamma'_0)$ is a subgroupoid of $(\Gamma';\Gamma'_0)$, since f_0 is surjective.

Let $\lambda' \in f(\Omega)$ and $x' \in \Gamma'$ such that $\beta'(x') = \alpha'(\lambda) = \beta'(\lambda)$. We prove that $x' \cdot \lambda' \cdot (x')^{-1} \in f(\Omega)$.

Indeed, $\lambda' = f(\lambda)$ with $\lambda \in \Omega$ and $x' = f(x)$ with $x \in \Gamma$, since f is surjective. From $(f(x), f(\lambda)), (f(\lambda), (f(x))^{-1}) \in \Gamma'_{(2)}$, it follows that $(x, \lambda), (\lambda, x^{-1}) \in \Gamma_{(2)}$, since f is a groupoid strong morphism. It follows that $x \cdot \lambda \cdot x^{-1}$ is defined and $x \cdot \lambda \cdot x^{-1} \in \Omega$, since Ω is normal in Γ. Hence, $f(x \cdot \lambda \cdot x^{-1}) \in f(\Omega)$ and

$$f(x) \cdot f(\lambda) \cdot f(x^{-1}) = f(x) \cdot f(\lambda) \cdot (f(x))^{-1} = x' \cdot \lambda' \cdot (x')^{-1} \in f(\Omega).$$

Thus, $f(\Omega)$ is a normal subgroupoid of Γ'. $\triangle$

Corollary 2.2. *Let $f: \Gamma \longrightarrow \Gamma'$ be a Γ_0- morphism of groupoids. Then the following assertions hold:*

(i) If $(\Omega;\Omega_0)$ is a subgroupoid of $(\Gamma;\Gamma_0)$, then $(f(\Omega); f_0(\Omega_0))$ is a subgroupoid of $(\Gamma';\Gamma_0)$. In particular, Imf is a subgroupoid of Γ' over Imf_0.

(ii) If f is surjective and Ω is a normal subgroupoid of Γ, then $f(\Omega)$ is a normal subgroupoid of Γ'.

Proof. We apply Theorem 2.1.(ii) and Proposition 2.2.$\triangle$

If $(\Gamma;\Gamma_0)$ is a groupoid, we denote by $\mathcal{S}(\Gamma;\Gamma_0)$ (resp., $\mathcal{N}(\Gamma)$) the set of the subgroupoids (resp., the normal subgroupoids) of $(\Gamma;\Gamma_0)$.

If $(f, f_0): (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$ is a groupoid morphism, we denote by $\widetilde{\mathcal{S}}(\Gamma;\Gamma_0)$ (resp., $\widetilde{\mathcal{N}}(\Gamma)$) the set of the subgroupoids (resp., the normal subgroupoids) of $(\Gamma;\Gamma_0)$, which contains the kernel of f, i.e.:

$\mathcal{S}(\Gamma;\Gamma_0) = \{\Omega \mid \Omega$ *is a subgroupoid of* $(\Gamma;\Gamma_0)$ *such that* $Kerf \subseteq \Omega\}$

$\mathcal{N}(\Gamma) = \{\Omega \mid \Omega$ *is a normal subgroupoid of* Γ *such that* $Kerf \subseteq \Omega\}$.

In view of Example 2.1.(a),(b),(c) we have that $\mathcal{S}(\Gamma;\Gamma_0)$, $\mathcal{N}(\Gamma)$), $\widetilde{\mathcal{S}}(\Gamma;\Gamma_0)$ and $\widetilde{\mathcal{N}}(\Gamma)$) are nonempty sets.

Theorem 2.2. (**the correspondence theorem for subgroupoids**) *For any surjective strong morphism of groupoids* $(f, f_0) : (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$*, there exists a bijection from the set* $\mathcal{S}(\Gamma';\Gamma'_0)$ *of the subgroupoids of* $(\Gamma';\Gamma'_0)$ *to the set* $\mathcal{S}(\Gamma;\Gamma_0)$ *of the subgroupoids of* $(\Gamma;\Gamma_0)$.

Proof. We take the maps

$$\varphi : \widetilde{\mathcal{S}}(\Gamma;\Gamma_0) \longrightarrow \mathcal{S}(\Gamma';\Gamma'_0)$$

and

$$\psi : \mathcal{S}(\Gamma';\Gamma'_0) \longrightarrow \widetilde{\mathcal{S}}(\Gamma;\Gamma_0),$$

given by:

$$\varphi(\Omega) = f(\Omega), \quad (\forall)\Omega \in \widetilde{\mathcal{S}}(\Gamma); \tag{5}$$

$$\psi(\Omega') = f^{-1}(\Omega'), \quad (\forall)\Omega' \in \mathcal{S}(\Gamma'). \tag{6}$$

By Proposition 2.2.(i), it follows that $f(\Omega)$ is a subgroupoid of Γ', for all $\Omega \in \widetilde{\mathcal{S}}(\Gamma)$. Hence, φ is well-defined. Also, by Proposition 2.1.(i),it follows that $f^{-1}(\Omega')$ is a subgroupoid of Γ, for all $\Omega' \in \mathcal{S}(\Gamma')$. Hence, ψ is well-defined.

The maps φ and ψ given by (5) and (6) have the following properties:

$$\psi \circ \varphi = Id_{\widetilde{\mathcal{S}}(\Gamma)} \quad \text{and} \quad \varphi \circ \psi = Id_{\mathcal{S}(\Gamma')}. \tag{7}$$

The equalities (7) are equivalently with:

$$f^{-1}(f(\Omega)) = \Omega, \quad (\forall)\Omega \in \widetilde{\mathcal{S}}(\Gamma) \quad \text{and} \quad f(f^{-1}(\Omega')) = \Omega', \quad (\forall)\Omega' \in \mathcal{S}(\Gamma').$$

- (a) If $x \in \Omega$, then $f(x) \in f(\Omega)$ and we have $x \in f^{-1}(f(\Omega))$. Hence, $\Omega \subseteq f^{-1}(f(\Omega))$.

- (b) If $x \in f^{-1}(f(\Omega))$, then $f(x) \in f(\Omega)$ and exists $y \in \Omega$ such that $f(x) = f(y)$. We have $f(x) \cdot (f(y))^{-1} = \epsilon'(f(y))$. Therefore, $f(x \cdot y^{-1}) = \epsilon'(f(y))$ and we obtain that $x \cdot y^{-1} \in Kerf$. Thus, $x \cdot y^{-1} = z$, with $z \in Kerf \subseteq \Omega$. Hence, $x = z \cdot y$ with $y, z \in \Omega$ and we have $x \in \Omega$. Therefore, $f^{-1}(f(\Omega)) \subseteq \Omega$.

From (a) and (b), it follows the first equality of $(7')$.

- (c) If $x' \in f(f^{-1}(\Omega'))$, then $x' = f(x) \in f(\Omega)$ with $x \in f^{-1}(\Omega')$ and follows $f(x) \in \Omega'$. Hence $x' \in \Omega'$. Therefore, $f(f^{-1}(\Omega')) \subseteq \Omega'$.

- (d) If $x' \in \Omega'$, exists $x \in \Gamma$ such that $x' = f(x)$, since f is surjective. Then $x \in f^{-1}(\Omega')$, since $f(x) \in \Omega'$.Therefore, $x' \in f(f^{-1}(\Omega'))$. Hence, $\Omega' \subseteq f(f^{-1}(\Omega'))$.

From (c) and (d), it follows the second equality of $(7')$.

From (7), it follows that ψ is invertible. Hence, ψ is a bijection.$\triangle$

Corollary 2.3. (**the correspondence theorem for subgroupoids via a Γ_0-morphism**) *For any surjective* Γ_0*-morphism of groupoids* $f : \Gamma \longrightarrow \Gamma'$*, there exists a bijection from the set* $\mathcal{S}(\Gamma';\Gamma_0)$ *of the subgroupoids of* $(\Gamma';\Gamma_0)$ *to set* $\mathcal{S}(\Gamma;\Gamma_0)$ *of the subgroupoids of* $(\Gamma;\Gamma_0)$.

Proof. It is a consequence of Theorems 2.1.(ii) and 2.2. $\triangle$

Applying the Propositions 2.1.(ii) and 2.2.(ii) we can prove similarly the following theorem.

Theorem 2.3. (**the correspondence theorem for normal subgroupoids**) *For any surjective strong morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma_0')$, *there exists a bijection from the set* $\mathcal{N}(\Gamma')$ *of the normal subgroupoids of* $(\Gamma'; \Gamma_0')$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$.$\triangle$

Corollary 2.4. (**the correspondence theorem for normal subgroupoids via a Γ_0-morphism**) *For any surjective* Γ_0*-morphism of groupoids* $f : \Gamma \longrightarrow \Gamma'$, *there exists a bijection from the set* $\mathcal{N}(\Gamma')$ *of the normal subgroupoids of* $(\Gamma'; \Gamma_0)$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$.

Proof. It is a consequence of Theorems 2.1.(ii) and 2.3. $\triangle$

Remark 2.3. (i) The Theorems 2.2 and 2.3. generalise the correspondence theorems for subgroups and normal subgroups by a surjective morphism of groups.

(ii) The Theorems 2.2 and 2.3. are not true for arbitrary surjective morphisms of groupoids.

(iii) If $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma_0')$ is a groupoid strong morphism, then $(\widetilde{f}, \widetilde{f_0}) : (\Gamma; \Gamma_0) \longrightarrow (Imf; Imf_0)$ is a surjective strong morphism of groupoids, where $\widetilde{f}, \widetilde{f_0}$ are given by $\widetilde{f}(x) = f(x), \quad (\forall) x \in \Gamma \quad$ and $\quad \widetilde{f_0}(u) = f_0(u), \quad (\forall) u \in \Gamma_0$.$\triangle$

Theorem 2.4. *For any strong morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma_0')$, *there exists a bijection from the set* $\mathcal{S}(Imf; Imf_0)$ *of the subgroupoids of* $(Imf; Imf_0)$ *to the set* $\mathcal{S}(\Gamma; \Gamma_0)$ *of the subgroupoids of* $(\Gamma; \Gamma_0)$.

Proof. We apply the Theorem 2.2. of the strong morphism of groupoids $(\widetilde{f}, \widetilde{f_0}) : (\Gamma; \Gamma_0) \longrightarrow (Imf; Imf_0)$ associated to (f, f_0). $\triangle$

Similarly, we can prove the following theorem.

Theorem 2.5. *For any strong morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma_0')$, *there exists a bijection from the set* $\mathcal{N}(\Gamma')$ *of the normal subgroupoids of* $(\Gamma'; \Gamma_0')$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$. $\triangle$

Corollary 2.5. *Let* $f : \Gamma \longrightarrow \Gamma'$ *a* Γ_0*- morphism of groupoids. Then the following assertions hold:*

(i) There exists a bijection from the set $\mathcal{S}(Imf; Imf_0)$ *of the subgroupoids of* $(Imf; Imf_0)$ *to the set* $\mathcal{S}(\Gamma; \Gamma_0)$ *of the subgroupoids of* $(\Gamma; \Gamma_0)$.

(ii) There exists a bijection from the set $\mathcal{N}(Imf)$ *of the normal subgroupoids of* $(Imf; Imf_0)$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$.

Proof. This is a consequence of Theorems 2.1.(ii), 2.4 and 2.5. $\triangle$

Remark 2.4. We conclude that the strong morphisms of groupoids have the same properties as the morphisms of groups. $\triangle$

References

[1] C. Albert & P. Dazord, *Groupoides de Lie et groupoides symplé ctiques.* Séminaire Sud- Rhodasien, Publ. Dept. Math. Lyon (1-partie 1989, 2- partie 1990).

[2] A. Coste, P. Dazord & A. Weinstein, *Groupoides symplectiques.* Publ. Dept. Math. Lyon, 2/A(1987),1-62.

[3] B. Dumons & Gh. Ivan, *Introduction à la théorie des groupoides.* Dept.Math. Univ. Poitiers(France), URA, C.N.R.S. D1322, nr.86(1994), 86 p.

[4] C. Ehresmann, *Oeuvres complétes. Parties I.1, I.2. Topologie algébrique et géométrie différentielle.*1950.

[5] P. J. Higgins, *Notes on categories and groupoids*, Von Nostrand Reinhold, London, 1971.

[6] S. T. Hu, *Elements of general topology*, Holden-Day, San Francisco, 1968.

[7] Gh. Ivan, *On cohomology of Brandt groupoids*, Proceed. of International Conference on Group Theory, Timişoara, 17-20 september 1992, Anal. Şt. Univ. Timişoara, Seria St. Mat.Vol.1993,p. 109-116.

[8] Gh. Ivan, *Cayley theorem for monoidoids*, Glasnik Matematicki, Vol. 31(51), 1996, 73-82.

[9] Gh. Ivan, *Special morphisms of Ehresmann groupoids*, Novi Sad J. Math.,Vol.29,1999, to appear.

[10] M. V. Karasev, *Analogues of objects of Lie groups theory for nonlinear Poisson brackets.* Math. U.S.S.R. Izv., 26(1987), 497-527.

[11] P. Liebermann, *On symplectic and contact Lie groupoids.* Diff. Geometry and its appl. Proc. Conf. Opava (Czechoslovakia), august 24-28, 1992, Silezian University Opava, 1993, 29-45.

[12] P. Liebermann, *Lie algebroids and mechanics*, Archivum Math. (BRNO), Tomus 32 (1996), 142-162.

[13] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc., Lectures Notes Series, 124, Cambridge Univ.Press., 1987.

[14] G. W. Mackey, *Ergodic theory, groups theory and differential geometry*, Proc. Nat. Acad. Sci., USA, 50(1963), 1184-1191.

[15] K. Mikami & A. Weinstein, *Moments and reduction for symplectic groupoid actions*, Publ. RIMS Kyoto Univ., 42(1988), 121-140.

[16] J. Pradines, *Théorie de Lie pour les groupoides différentiables*, C.R. Acad. Sci. Paris, 263, 1966, p.907-910; 264, 1967, p.245-248.

[17] M. Puta, *Hamiltonian mechanics and geometric quantization*, Kluwer,1993.

[18] A. Ramsey, *Virtual groups and group actions*, Advances in Math., 6(1971), 253-322.

[19] J. Renault, *A groupoid approach to C^*-algebras*, Lectures Notes Series, 793, Springer, 1980.

[20] G. Virsik, *Total connections in Lie groupoids*, Archivum Math. (BRNO), Tomus 31 (1995), 183-200.

[21] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc., 16(1987), 101-104.

University of Timişoara
Department of Mathematics
4, Bd. V. Pârvan
1900, Timişoara, Romania