Generalized invexity on differentiable manifolds

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Abstract

We extend the notion of invexity, from open subsets of $\mathbf{R}^n$ to differentiable manifolds. As specific tools, we use the behaviour of the differential in pair points and restricted differentiable functions and vector fields along curves on manifolds. Characterizations of local/global minimum points are given. We also argue that the invexity property is relevant only for functions with critical points: every regular function is invex with respect to some properly choosed vector fields family. Examples are given, which show that our "double" invexity is more general than the classical invexity.

A double invex programming is developped, proving a duality and a Kuhn-Tucker-like theorems.

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1 Introduction

Convexity (and arcwise connectedness) for smooth optimization problems gradually evolved on the following scale:

- **the classical setting**: (differentiable) functions on open subsets in $\mathbf{R}^n$; linking made by line segments.

- **the Riemannian setting**: functions on Riemannian manifolds; linking made by geodesic arcs (Udriste [9], Rapcsak [8] for a review)

- **the affine differential setting**: functions on manifolds endowed with linear connections; linking made by arcs of autoparallel curves (Pripoae [6], [7]).

- **the differential setting**: functions on differentiable manifolds; linking made by arcs of differentiable curves (Avriel [1], Preda [5]).

Invexity for smooth optimization problems followed a similar path, but only on the first two steps. The aim of this paper is to complete the next two stages.

In §2, we review different extensions of the original definition of invexity (Pini[4]). We generalize the η-invexity of Mititelu [3] ("simple invexity") to an invexity property which involves two families of vector fields ("double invexity").

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Invexity is also introduced along curves on manifolds, obtaining a geometric interpretation of this property. Examples are given, which show our extensions are effective.

Also in §2, we prove that any regular function is η-invex with respect to some properly choosed family of vector fields, fact which restricts the interest area of this notion to functions with critical points.

We adapt the convex programing on Riemannian manifolds from Udriste [9], to double invex programming (§3) and we prove a duality theorem and a Kuhn-Tucker-like theorem.

In §4 we consider invexity along auto-parallel curves in differentiable affine manifolds and along geodesics in Riemannian manifolds. We show that the "double invexity" is equivalent to the invexity along all the curves, and is equivalent to the invexity along all the geodesics (or along all the auto-parallel curves of a fixed linear connection).

2 Preliminaries.

Consider an n-dimensional differentiable manifold M and $f : M \to \mathbf{R}$ a differentiable function. Denote by T_pM the tangent space at $p \in M$ and by $TM := \{(p, v) \mid p \in M \;\;,\;\; v \in T_pM\}$ the total space of the tangent bundle of M. Let η, $\theta : M \times M \to TM$ be two differentiable functions (with respect with the product differentiable structure on $M \times M$), such that, for every p, $q \in M$ we have $\eta(p, q)$, $\theta(p, q) \in T_qM$. We may view η and θ as two families of vector fields, indexed by all the points of M (the first variable).

Definition 2.1. The function f is $(\eta, \theta) - invex$ if, for every p,$q \in M$

$$f(p) - f(q) \geq df_q(\eta(p, q)) + df_p(\theta(q, p)) \tag{1}$$

If, moreover, there exist two points p, $q \in M$ such that the previous relation holds with strict inequality, we say the function f is *strictly* $(\eta, \theta) - invex$.

Definition 2.2. We say f is $(\eta, \theta)-$ *invex along the curve* $c : [0, 1] \to M$ if the relation (1) holds, for every points $p = c(t)$, $q = c(s)$; t,$s \in [0, 1]$.
(For the sake of simplicity, we suppose the curve to be differentiable, even if most of the subsequent results do not require this property).

Definition 2.3. A (η, θ)-invex function f is said to have positive θ-differential in a point $x_0 \in M$, if $df_y(\theta(x_0, y)) \geq 0$, for every $y \in M$.

Remarks 2.1. (i) Due originally to Pini [4], for a vector fields family defined by the position vector in $\mathbf{R}^n$, the invexity was first generalized by Mititelu ([2], [3]), in the case of an arbitrary vector fields family; we recover its definition, by taking $\theta = 0$ in (1). (For simplicity, the $(\eta, 0)$-invexity will be called η-invexity). Our Definition 2.1. has the advantage to detect the behaviour of the differential of f in both points p and q, in a "skew symmetric vs. symmetric" play.

(ii) A necessary condition for the (η, θ)-invexity is

$$df_p(\eta(p, p)) + df_p(\theta(p, p)) \leq 0 \quad , \quad p \in M$$

(iii) A function f is (η, θ)-invex if and only if it is (η, θ)-invex along every curve.

If, in the definition of the (η, θ)- invexity along a curve c we require the hypothesis η, θ be families of vector fields **along** c (and not global ones), then the ("only if" part of the) previous remark does not hold true.

(iv) If a function $f : \mathbf{R}^n \to \mathbf{R}$ is convex (in the classical sense), then f is η-invex, for $\eta(p, q) = p - q$. So, classical convexity is a special case of invexity.

This fact also holds for some generalized convex functions on manifolds, using geodesics or auto-parallel curves instead of straight lines ([6],[7]).

(v) If a (η, θ)-invex function f has positive θ- differential in all the points, then f is η-invex (the right term in (1) is greater than $df_q(\eta(p, q))$, for every points p and q).

Example 2.1. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x^1, x^2) = -(x^1)^2 - (x^2)^2$ and the two families of vector fields on $\mathbf{R}^2$, given by $\eta(p^1, p^2; q^1, q^2) = \theta(p^1, p^2; q^1, q^2) = = q^1\partial_1 + q^2\partial_2$.

It follows that f is (strictly) (η, θ)-invex. On another hand, f cannot be only α-invex, for no vector fields family α on $\mathbf{R}^2$. In fact, suppose such a family $\alpha = = \alpha^1\partial_1 + \alpha^2\partial_2$ may exist; then, the relation (1) becomes

$$-(p^1)^2 - (p^2)^2 + (q^1)^2 + (q^2)^2 \geq -2q^1\alpha^1 - 2q^2\alpha^2$$

for every p^1, p^2, q^1, $q^2 \in \mathbf{R}$. But, making $q^1 = q^2 = 0$, it follows $(p^1)^2 + (p^2)^2 \leq 0$, for every real p^1 and p^2, which is impossible.

This example shows our "double" invexity is more general than the ("simple") invexity. (Obviously, in this case, the failure of the invexity is due to the fact that the function f is concave -in the classical acception- and, as asserted previously, invexity and convexity are closely related.)

Proposition 2.1. *Let f be a (η, θ)-invex function. If $q_0 \in M$ is a critical point for f, and if f has positive θ-differential in q_0, then q_0 is a (global) minimum point.*

(If, moreover, q_0 is a unique critical point, then it is a global (and unique) minimum point).

Proof. From (1) follows

$$f(p) \geq f(q_0) + df_p(\theta(q_0, p)) \geq f(q_0) \quad , \quad p \in M$$

so q_0 is a global minimum point (but not necessary unique). The second assertion is obvious. □

Corollary 2.1. *A non-constant function on M with a global maximum point q_0 cannot be (η, θ)-invex with positive θ-differential in q_0, for no η and θ.*

Example 2.2. The function in Example 2.1 has the properties from the Corolary 2.1.

Proposition 2.2. *Let f a differentiable function on M, with no critical points. Then:*

(i) there exists a family η of vector fields, such that f be η-invex and

(ii) there exists two families η and θ of vector fields, such that f be strictly (η, θ)-invex.

Proof. Consider g a Riemannian metric on M and remark that $gradf$ is a nowhere vanishing vector field on M. Define

$$\eta(p,q) = \frac{f(p)-f(q)}{\|gradf\ |_q\ \|^2} gradf\ |_q \tag{2}$$

As $df(gradf) = \|gradf\|^2$, it follows that relation (1) holds, for $\theta = 0$. So, the function f is η-invex.

For (ii), we keep the same η and construct the family θ as follows: fix two points $p,\ q \in M$ and define

$$\theta(p,q) = -(\|gradf\ |_q\ \|)^{-2} gradf\ |_q$$

Then, $df_p(\theta(q,p)) = -1$ and the strict inequality holds in (1), so f is a strictly (η, θ)-invex function. □

Remarks 2.2. (i) For regular (i.e. critical points free) functions, the (η, θ)-invexity notion may be irrelevant, as seen from the previous proposition.

(ii) The previous proposition remains also true for strictly η-invexity. (In (2), we decrease the denominator by a positive number).

(iii) The construction (2) provides examples of (regular) η-invex functions which are not strictly η-invex.

3 Duality in (η, θ)- invex programming

In this section, we adapt notions and results from the (generalized) convex programming on Riemannian manifolds ([9]) to the framework of double invexity on differential manifolds.

Consider a differentiable function $f : M \to \mathbf{R}$; $\eta, \theta : M \times M \to TM$ two families of vector fields and the differentiable ("constrained") functions $\psi_i : M \to \mathbf{R}$, $(i = 1, r)$.

Denote $A := \{x \in M \mid \psi_i(x) \geq 0\ , i = \overline{1, r}\}$ the *set of admissible solutions.* We suppose the functions f, $-\psi_1$,..., $-\psi_r$ be (η, θ)-invex and consider the following (η, θ)-*invex program* (*the primal program*)

$$min\{f(x) \mid x \in A\} \tag{3}$$

If $intA \neq \emptyset$, we say the program is *superconsistent* ([1]) (that is there exists a point $y \in A$, such that $\psi_i(y) > 0$, for every $i \in \{1, ..., r\}$). For a point $x \in A$, we denote by

$$I(x) := \{i \mid \psi_i(x) = 0\}$$

Proposition 3.1. *Suppose the program (3) is superconsistent and for a fixed $x_0 \in A$, the functions $(-\psi_i)$ have positive θ-differential in x_0, for every $i \in I(x_0)$.*

Then the set

$$\{(d\psi_i)_{x_0} \mid i \in I(x_0)\}$$

is positively linearly independent in the cotangent space $T^*_{x_0}M$.

Proof. From the hypothesis, there exists a point $y \in A$, such that $\psi_i(y) > 0$, for every $i \in \{1, ..., r\}$. Suppose there exist v^i, $i \in I(x_0)$, some non-negative real numbers such that

$$\sum_{i \in I(x_0)} v^i (d\psi_i)_{x_0} = 0 \tag{4}$$

Due to the (η, θ)-invexity of the functions $(-\psi_i)$, for every $y \in M$ and for every $i \in I(x_0)$ we have

$$0 < \psi_i(y) = \psi_i(y) - \psi_i(x_0) \leq (d\psi_i)_{x_0}(\eta(y, x_0)) + (d\psi_i)_y(\theta(x_0, y))$$

The condition of positively θ-differential ensures that $(d\psi_i)_{x_0}(\eta(y, x_0)) > 0$ for every $i \in I(x_0)$.

Hence, from (4) applied in $\eta(y, x_0))$, we derive $v^i = 0$, for every $i \in I(x_0)$. We deduce the set of differentials $\{(d\psi_i)_{x_0} \mid i \in I(x_0)\}$ is positively linearly independent. □

We adapt now the Lagrangian formalism from convex dual programs ([1]) to (η, θ)-invex programs.

Definition 3.1. We call the Lagrangian of the primal problem (3), the function $L : M \times \mathbf{R}_+^r \to \mathbf{R}$, given by

$$L(x, v) = f(x) - \sum_{i=1}^{r} v^i \psi_i(x)$$

Definition 3.2. We call *the dual problem of the primal problem* (3), the program

$$max\{L(x, v) \mid x \in A \ , \ v \in \mathbf{R}_+^r \ , \ df_x = \sum_{i=1}^{r} v^i (d\psi_i)_x\} \tag{5}$$

Theorem 3.1. (Duality theorem) *Let (3) be a superconsistent program on the differentiable manifold M and $x_0 \in A$ a fixed point. Suppose:*

- the functions f, $-\psi_1$,..., $-\psi_r$ are (η, θ)-invex;

- the functions $(-\psi_i)$ have positive θ-differential in x_0, for every $i \in I(x_0)$;

- the point x_0 is (the) optimal solution of the primal problem (3).

Then there exists a vector $v_0 \in \mathbf{R}_+^r$ such that (x_0, v_0) is the optimal solution of the dual problem (5) and $f(x_0) = L(x_0, v_0)$.

Proof. We apply the Proposition 3.1. for the point x_0 and we deduce the set

$$\{(d\psi_i)_{x_0} \mid i \in I(x_0)\}$$

is positively linearly independent. From the Fritz John theorem, there exists a vector $v_0 \in \mathbf{R}_+^r$ such that

$$v_0^k = 0 \quad , \quad k \notin I(x_0) \tag{6}$$

and

$$df_{x_0} = \sum_{k=1}^{r} v_0^k (d\psi_k)_{x_0} \tag{7}$$

It follows (x_0, v_0) is an admissible solution for the dual problem (5).

We calculate

$$L(x_0, v) = f(x_0) - \sum_{k=1}^{r} v^k \psi_k(x_0) \leq f(x_0) = L(x_0, v_0)$$

If (x_1, v_1) is an admissible solution of the dual problem (5), then x_1 is a critical point for the function $h(x) := L(x, v_1)$. By hypothesis, the function h is (η, θ)-invex so, by Proposition 10, x_1 is a global minimum point for h.

We obtain

$$L(x_1, v_1) \leq L(x_0, v_1) \leq L(x_0, v_0)$$

hence (x_0, v_0) is the optimal solution of the dual problem (5) (i.e. at least as good as any other admissible solution of (5)). □

Theorem 3.2. (Kuhn-Tucker-like theorem) *Let (3) be a superconsistent program on the differentiable manifold M and $x_0 \in A$ a fixed point. Suppose:*

- the functions f, $-\psi_1$,..., $-\psi_r$ are (η, θ)-invex;
- the functions $(-\psi_i)$ have positive θ-differential in x_0, for every $i \in I(x_0)$.

Then the point x_0 is (the) optimal solution of the primal problem (3) if and only if

(i) there exists a vector $v_0 \in \mathbf{R}^r_+$ such that $v_0^j \psi_j(x_0) = 0$ for every $j \in \{1, ..., r\}$ and

(ii) $$L(x_0, v) \leq L(x_0, v_0) \leq L(x, v_0) \quad , \quad x \in M \quad , \quad v \in \mathbf{R}^r_+$$

Proof. Suppose first the point x_0 is the optimal solution of the primal problem (3). Then, x_0 satisfies the conclusion of Theorem 3.1., so there exists a vector $v_0 \in \mathbf{R}^r_+$ such that (6) and (7) hold.

Then, as in the proof of Theorem 3.1, we obtain the condition (ii). From the previous data, condition (i) is also satisfied.

For the converse, suppose the conditions (i) and (ii) hold. From (ii), it follows

$$L(x_0, v) \leq L(x_0, v_0) \quad , \quad v \in \mathbf{R}^r_+$$

and we compute

$$\sum_{k=1}^{r} (v^k - v_0^k) \psi_k(x_0) \geq 0 \quad , \quad v \in \mathbf{R}^r_+$$

For each fixed $i \in \{1, ..., r\}$, we choose successively

$$v^k = v_0^k + \delta_{ki} \quad , \quad k \in \{1, ..., r\}$$

and we deduce

$$\psi_i(x_0) \geq 0 \quad , \quad i \in \{1, ..., r\}$$

We proved $x_0 \in A$.

Again from (ii), we derive

$$L(x_0, v_0) \leq L(x, v_0) \ \ , \ \ x \in M$$

We rewrite this relation as

$$f(x_0) - f(x) + \sum_{k=1}^{r} v_0^k \psi_k(x) \leq 0 \ \ , \ \ x \in M \tag{8}$$

where we used the equality $v_0^k \psi_k(x_0) = 0$ from (i). For $x \in A$, we have $\psi_k(x) \geq 0$, so the sum in (8) is non-negative. It follows $f(x_0) \leq f(x)$, for every $x \in A$; hence, x_0 is an optimal solution of the primal problem (3). Using Proposition 10, we deduce x_0 is a global optimum. □

Remarks 3.1. (i) The previous two results extend similar results in [3], proved for "simply" invex functions on Riemannian manifolds. As it may be easily seen, we did not use any metric notion in our proof; only the differentiable structure of the manifold was considered.

(ii) The line of proof was adapted from [9], where similar results were proven for convex functions on Riemannian manifolds. We stress again the jump we made, from metric to arbitrary differentiable structures on manifolds.

4 Double invexity in the Differential affine and the Riemannian settings

As we asserted previously, the natural context where the ("simple" or "double") invexity may be studied is the differentiable one, not the Riemannian one. So, one may ask if there is any reason to particularize again the framework, and to consider invexity on differential affine manifolds, that is manifolds endowed with a linear (or affine) connection.

The invexity notion is expressed and studied in terms of the differential of the objective function f and (eventually) of the constrained functions ψ_k, so it doesn't contain any germ of some additional (geo)metric structure. As the convexity theory and its generalizations show, geometry is involved as soon as specific classes of curves are used, in order to connect points of the manifold: straight lines (particular geodesics) in Euclidean spaces, arbitrary geodesics in Riemannian manifolds, auto-parallel curves in differential affine spaces. (Connecting points through well choosed families of curves is a key technique in algorithms building for generalized convex optimization.)

By analogy, we may study invexity along the respective classes of curves.

Let ∇ be a linear connection on the differentiable manifold M; that is, ∇ is an operator which associates to every pair of vector fields X and Y, a third one denoted

$\nabla_X Y$. This operator is linear over the ring of germs of functions (in the first variable), **R**-linear in the second variable and

$$\nabla_X(hY) = X(h)Y + h\nabla_X Y$$

for any differentiable function h on M. The pair (M, ∇) is called a differentiable affine manifold.

A differentiable curve $c : [0,1] \to M$ is ∇-auto-parallel (i.e. autoparallel with respect to ∇) if the restriction of ∇ along c parallelizates the "velocity" vector field, that is

$$\nabla_{c'} c' = 0$$

Proposition 4.1. *Let f be a differentiable function on a differentiable affine manifold (M, ∇) and η, θ two families of vector fields on M. Suppose every two points of M may be joined by a ∇-auto-parallel curve. Then, the following assertions are equivalent:*

(i) f is (η, θ)-invex along every curve on M.

(ii) f is (η, θ)-invex along every ∇-auto-parallel curve on M.

(iii) f is (η, θ)-invex.

Proof. (i) is equivalent to (iii) as in the Remark 4,iii. Obviously, (i) implies (ii), so we need to prove only that (ii) implies (iii).

Let p and $q \in M$ and $c : [0,1] \to M$ a ∇-auto-parallel curve, such that $c(0) = p$ and $c(1) = q$. Suppose f is (η, θ)-invex along c, so

$$f(c(t)) - f(c(s)) \geq df_{c(s)}(\eta(c(t), c(s))) + df_{c(t)}(\theta(c(s), c(t))) \quad , \quad s, t \in [0,1]$$

In particular, for $t = 0$ and $s = 1$, we recover the relation (1). We conclude f is (η, θ)-invex. $\square$

Consider now a Riemannian manifold (M, g); the Levi-Civita connection associated to the metric g has two additional properties:

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

$$Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0$$

for every vector fields X,Y,Z. (We denoted by [,] the Poisson bracket on M). A geodesic is an auto-parallel curve with respect to the Levi-Civita connection.

The following result is analogous to the previous proposition, so we skip the proof.

Proposition 4.2. *Let f be a differentiable function on a complete Riemannian manifold (M, g) and η, θ two families of vector fields on M. Then, the following assertions are equivalent:*

(i) f is (η, θ)-invex along every curve in M.

(ii) f is (η, θ)-invex along every geodesic in M.

(iii) f is (η, θ)-invex.

Remarks 4.1. (i) In the hypothesis of Proposition 4.2, it does not appear anymore the condition that any two points may be joined by a geodesic, because this is fulfilled by the completness assumption (consequence of the Hopf-Rinow theorem). On differential affine manifolds, such a result is not true (in general), so we need the respective additional condition.

(ii) The propositions 4.1 and 4.2 are no longer valid if we replace "every curve" by "some curve". For example, consider on $\mathbf{R}^2$ the real valued differentiable function $f(x, y) = x^2$, and the family of vector fields $\eta : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$, $\eta((x, y), (z, w)) = 2x^2\partial_x$. Then, f is η-invex along the curve $c : [0, 1] \to \mathbf{R}^2$, $c(t) = (0, t)$ (the inequality is verified as an identity), but f is not η-invex on the whole $\mathbf{R}^2$; for example, in the points $p = (1, 0)$ and $q = (0, 0)$, the inequality (1) is false.

5 Comments.

Remarks 5.1. (i) The notions and results from the paragraph §2 can be easily generalized for constrained optimization problems: take a subset B of a manifold M and a function $f : B \to \mathbf{R}$; definition 1 works unchanged. For invexity along curves, one must impose some arcwise conectedness property of B (see [9],[5]).

(ii) Following ideas of [8] (recovered in the particular case $\theta = 0$), we say $f : M \to \mathbf{R}$ is:

- (η, θ)-*pseudo-invex* if for every p,$q \in M$

$$df_q(\eta(p, q)) + df_p(\theta(q, p)) \geq 0 \quad \Rightarrow \quad f(p) \geq f(q)$$

- (η, θ)-*quasi-invex* if for every p,$q \in M$

$$f(p) \leq f(q) \quad \Rightarrow \quad df_q(\eta(p, q)) + df_p(\theta(q, p)) \leq 0$$

Obviously, the (η, θ)-invexity implies the (η, θ)-pseudo-invexity and the (η, θ)-quasi-invexity.

Some of the previous results may be adapted for pseudo-invexity and quasi-invexity as well. For example, we have an analogue of the Proposition 2.1., as

Proposition 5.1. *Suppose q_0 is a critical point of the function $f : M \to \mathbf{R}$, and f has positive θ-differential in q_0. If f is (η, θ)-pseudo-invex or (η, θ)-quasi-invex, then q_0 is a minimum point of f.*

Remark 5.2. The (η, θ)-invex programming we developped in §3 may be easily extended for (η, θ)-vector programs, following the theory in [3].

The generalizations suggested previously are quite straightforward. We preferred to treate the (η, θ)-invexity for scalar programs only (and we avoided collateral paths towards pseudo and quasi invexity), in order to illustrate clearer the two extension ideas:

- from "simple" invexity to "double" invexity;
- from Riemannian manifolds to differentiable manifolds.

The first extension needs, usually, additional hypothesis on the behaviour of the differentials. The second extension is allowed as we replace the gradients ("Lagrangian formalism") with the differentials ("Hamiltonian formalism"), in a duality suggested by Analytical Mechanics.

Remark 5.3. Finally, we want to stress again the shift we made ([6], [7]) from **Riemannian optimization** to **differential affine optimization**, which seems to us a more natural setting for generalized (convex) optimization on manifolds.

This paradigm change is similar to Gh. Ţiţeica's viewpoint in the theory of curves and surfaces: some ordinary differential equations, which were leading to apparently metric geometry problems, provided in fact some centro-affine invariants.

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