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A survey of the alternating sum-of-divisors function

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Dedicated to the memory of my friend and colleague, Professor Antal Bege

Abstract. We survey arithmetic and asymptotic properties of the alternating sum-of-divisors function β defined by $\beta(p^a) = p^a - p^{a-1} + p^{a-2} - \cdots + (-1)^a$ for every prime power p^a ($a \geq 1$), and extended by multiplicativity. Certain open problems are also stated.

1 Introduction

Let β denote the multiplicative arithmetic function defined by $\beta(1) = 1$ and

$$\beta(p^a) = p^a - p^{a-1} + p^{a-2} - \cdots + (-1)^a \tag{1}$$

for every prime power p^a ($a \geq 1$). That is,

$$\beta(n) = \sum_{d \mid n} d\, \lambda(n/d) \tag{2}$$

for every integer $n \geq 1$, where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function, $\Omega(n)$ denoting the number of prime power divisors of n.

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The function β, as a variation of the sum-of-divisors function σ, was considered by Martin [19], Iannucci [16], Zhou and Zhu [32] regarding the following problem. In analogy with the perfect numbers, n is said to be imperfect if $2\beta(n) = n$. More generally, n is said to be k-imperfect if $k\beta(n) = n$ for some integer $k \geq 2$. The only known imperfect numbers are $2, 12, 40, 252, 880$, $10\,880, 75\,852$, $715\,816\,960$ and $3\,074\,457\,344\,902\,430\,720$ (sequence A127725 in [33]). Examples of 3-imperfect numbers are $6, 120, 126, 2520$. No k-imperfect numbers are known for $k > 3$. See also the book of Guy [9, p. 72].

This function occurs in the literature also in another context. Let

$$b(n) = \#\{k : 1 \leq k \leq n \text{ and } \gcd(k, n) \text{ is a square}\}.$$

Then $b(n) = \beta(n)$ $(n \geq 1)$, see Cohen [5, Cor. 4.2], Sivaramakrishnan [23], [24, p. 201], McCarthy [20, Sect. 6], [22, p. 25], Bege [3, p. 39], Iannucci [16, p. 12]. A modality to show the identity $b(n) = \beta(n)$ is to apply a familiar property of the Liouville function, namely,

$$\sum_{d|n} \lambda(d) = \chi(n) \quad (n \geq 1), \tag{3}$$

where χ is the characteristic function of the set of squares. Using (3),

$$b(n) = \sum_{k=1}^{n} \chi(\gcd(k, n)) = \sum_{k=1}^{n} \sum_{d|\gcd(k,n)} \lambda(d) = \sum_{d|n} \lambda(d) \sum_{1 \leq k \leq n,\ d|k} 1$$
$$= \sum_{d|n} \lambda(d) \frac{n}{d} = \beta(n).$$

In this paper we survey certain known properties of the function β, and give also other ones (without references), which may be known, but we could not locate them in the literature.

We point out that the function β has a double character. On the one hand, certain properties of this function are similar to those of the sum-of-divisors function σ, due to the fact that both β and σ are the Dirichlet convolution of two completely multiplicative functions. Such functions are called in the literature specially multiplicative functions or quadratic functions. Their study in connection to the Busche-Ramanujan identities goes back to the work of Vaidyanathaswamy [30]. See also [15, 21, 22, 24].

On the other hand, further properties of this function are analogous to those of the Euler's totient function φ, as a consequence of the representation of β given above.

We call the function β the *alternating sum-of-divisors function* or *alternating sigma function*. Sivaramakrishnan [24, p. 201] remarked that it may be termed the square totient function.

It is possible, of course, to define other alternating sums of the positive divisors of n. For example, let $\theta(n) = \sum_{d|n} d\,\lambda(d)$ $(n \geq 1)$. Then $\theta(n) = \lambda(n)\beta(n)$ $(n \geq 1)$. This is sequence A061020 in [33]. Another example: let $n = d_1 > d_2 > \cdots > d_{\tau(n)} = 1$ be the divisors of n, in decreasing order, and let $A(n) = \sum_{j=1}^{\tau(n)} (-1)^{j-1} d_j$, cf. [2]. Note that the function A is not multiplicative.

We do not give detailed proofs, excepting the proofs of formulae (10), (16), (17) and of the Proposition in Section 7, which are included in Section 8. We leave to the interested reader to compare the corresponding properties of the functions β, σ and φ. See, for example, the books [1, 10, 22, 24, 27].

In Section 7 we pose certain open problems. One of them is concerning super-imperfect numbers n, defined by $2\beta(\beta(n)) = n$. This notion seems not to appear in the literature. The super-imperfect numbers up to 10^7 are $n = 2, 4, 8, 128, 32\,768$. The number $2\,147\,483\,648$ is also super-imperfect.

The corresponding concept for the sigma function is the following: A number n is called superperfect if $\sigma(\sigma(n)) = 2n$. The even superperfect numbers are 2^{p-1}, where $2^p - 1$ is a Mersenne prime, cf. [26] (sequence A019279 in [33]). No odd superperfect numbers are known.

2 Basic properties

It is clear from (1) that for every prime power p^a $(a \geq 1)$,

$$\beta(p^a) = \frac{p^{a+1} + (-1)^a}{p+1} = \begin{cases} \frac{p^{a+1}-1}{p+1}, & \text{if } a \geq 1 \text{ is odd,} \\ \frac{p^{a+1}+1}{p+1}, & \text{if } a \geq 2 \text{ is even.} \end{cases} \tag{4}$$

We obtain from (2),

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s)}{\zeta(s)} \quad (\Re(s) > 2),$$

where ζ is the Riemann zeta function, leading to another convolution representation of β, namely

$$\beta(n) = \sum_{d^2 k = n} \varphi(k) \quad (n \geq 1), \tag{5}$$

cf. McCarthy [20, Sect. 6], [22, p. 25], Bege [3, p. 39].

We have $\varphi(n) \le \beta(n) \le n$ ($n \ge 1$). More exactly, it follows from (5) that for every $n \ge 1$,

$$\beta(n) = \varphi(n) + \sum_{d^2 k = n,\, d>1} \varphi(k) \ge \varphi(n),$$

with equality for the squarefree values of n. Also,

$$\beta(n) \le \sum_{dk=n} \varphi(k) = n,$$

with equality only for $n = 1$.

Moreover, $\beta(n) \le \varphi^*(n)$ for every $n \ge 1$, with equality if and only if n is squarefree or twice a squarefree number. This follows easily from (4). Here φ^* is the unitary Euler function, which is multiplicative and given by $\varphi^*(p^a) = p^a - 1$ for every prime power p^a ($a \ge 1$), cf. [22, 24]. Also, $\beta(n) \ge \sqrt{n}$ ($n \ge 1$, $n \ne 2$, $n \ne 6$).

Similar to the corresponding property of the function σ, $\beta(n)$ is odd if and only if n is a square or twice a square.

The function β appears in certain elementary identities regarding the set of squares, for example in

$$\sum_{\substack{k=1 \\ \gcd(k,n) \text{ a square}}}^{n} k = \frac{n(\beta(n) + \chi(n))}{2} \quad (n \ge 1),$$

$$\prod_{\substack{k=1 \\ \gcd(k,n) \text{ a square}}}^{n} k = n^{\beta(n)} \prod_{d \mid n} (d!/d^d)^{\lambda(n/d)} \quad (n \ge 1),$$

which can be deduced from (3).

3 Generalizations

An obvious generalization of β is the function β_a ($a \in \mathbb{C}$) defined by

$$\beta_a(n) = \sum_{d \mid n} d^a \lambda(n/d) \quad (n \ge 1). \tag{6}$$

If $a = m$ is a positive integer, then the following representation can be given: $\beta_m(n) = \#\{k : 1 \le k \le n^m, (k, n^m)_m \text{ is a } 2m\text{-th power}\}$, where $(a, b)_m$

stands for the largest common m-th power divisor of a and b. See McCarthy [20, Sect. 6], [22, p. 51].

Note that if $a = 0$, then $\beta_0 = \chi$, the characteristic function of the set of squares, used above.

For an arbitrary nonempty set S of positive integers let $\varphi_S(n) = \#\{k : 1 \leq k \leq n, \gcd(k,n) \in S\}$. For $S = \{1\}$ and S the set of squares this reduces to Euler's function φ and to the function β, respectively. The function φ_S was investigated by Cohen [6]. For every set S one has

$$\varphi_S(n) = \sum_{d \mid n} d\mu_S(n/d) \quad (n \geq 1),$$

where the function μ_S is defined by $\sum_{d \mid n} \mu_S(d) = \chi_S(n)$ $(n \geq 1)$, i.e., $\mu_S = \mu * \chi_S$ in terms of the Dirichlet convolution $*$, χ_S and μ denoting the characteristic function of S and the Möbius function, respectively.

Also, let

$$B(r,n) = \sum_{\substack{k=1 \\ \gcd(k,n) \text{ a square}}}^{n} \exp(2\pi ikr/n),$$

which is an analog of the Ramanujan sum to be considered in Section 5. Then

$$B(r,n) = \sum_{d \mid \gcd(r,n)} d\,\lambda(n/d) \quad (r, n \geq 1),$$

see Sivaramakrishnan [23], [24, p. 202], Haukkanen [13]. For $r = n$ one has $B(n,n) = \beta(n)$.

These generalizations can also be combined. See also Haukkanen [11, 12]. Many of the results given in the present paper can be extended for these generalizations.

We consider in what follows only the functions β_a defined by (6) and do not deal with other generalizations.

4 Further properties

For every $n, m \geq 1$,

$$\beta(n)\beta(m) = \sum_{d \mid \gcd(n,m)} \beta(nm/d^2)\,d\,\lambda(d), \tag{7}$$

and equivalently,

$$\beta(nm) = \sum_{d|\gcd(n,m)} \beta(n/d)\beta(m/d)d\mu^2(d), \tag{8}$$

cf. [23], [22, p. 26]. Here (7) and (8) are special cases of the Busche-Ramanujan identities, valid for specially multiplicative functions. See [15, 21, 22, 24, 30] for their discussions and proofs.

Direct proofs of (7) and (8) can be given by showing that both sides of these identities are multiplicative, viewed as functions of two variables and then computing their values for prime powers. Recall that an arithmetic function f of two variables is called multiplicative if it is nonzero and $f(n_1m_1, n_2m_2) = f(n_1, n_2)f(m_1, m_2)$ holds for any $n_1, n_2, m_1, m_2 \geq 1$ such that gcd$(n_1n_2, m_1m_2) = 1$. See [30], [29], [24, Ch. VII].

The proof of the equivalence of identities of type (7) and (8) is outlined in [14], referring to the work of Vaidyanathaswamy [30].

It follows at once from (8) that $\beta(nm) \geq \beta(n)\beta(m)$ for every $n, m \geq 1$, i.e., β is super-multiplicative. Formula (8) leads also to the double Dirichlet series

$$\sum_{n,m=1}^{\infty} \frac{\beta(nm)}{n^s m^t} = \frac{\zeta(s-1)\zeta(2s)\zeta(t-1)\zeta(2t)\zeta(s+t-1)}{\zeta(s)\zeta(t)\zeta(2(s+t-1))},$$

valid for $s, t \in \mathbb{C}$ with $\Re(s) > 2, \Re(t) > 2$.

The generating power series of β is

$$\sum_{n=1}^{\infty} \beta(n)x^n = \sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{(1-x^n)^2} \quad (|x| < 1),$$

which is a direct consequence of (2).

Consider the functions β_a defined by (6). One has

$$\sum_{n=1}^{\infty} \frac{\beta_a(n)\beta_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a-b)\zeta(2s-2a)\zeta(2s-2b)}{\zeta(s-a)\zeta(s-b)\zeta(2s-a-b)}, \tag{9}$$

valid for every $s, a, b \in \mathbb{C}$ with $\Re(s) > 1 + \max(0, \Re(a), \Re(b), \Re(a+b))$.

This formula is similar to Ramanujan's well-known result for the product $\sigma_a(n)\sigma_b(n)$, where $\sigma_a(n) = \sum_{d|n} d^a$. Formula (9) is due to Chowla [4], in an equivalent form for the product $\theta_a(n)\theta_b(n)$, where $\theta_a(n) = \sum_{d|n} d^a\lambda(d)$.

Formula (9) and that of Ramanujan follow from the next more general result concerning the product of two arbitrary specially multiplicative functions.

If f, g, h, k are completely multiplicative functions, then

$$(f * g)(h * k) = fh * fk * gh * gk * w, \tag{10}$$

where $w(n) = \mu(m)f(m)g(m)h(m)k(m)$ if $n = m^2$ is a square and $w(n) = 0$ otherwise.

This result is given by Vaidyanathaswamy [30, p. 621], Lambek [17], Subbarao [25]. See also [24, p. 50]. The proof of (10) can be carried out using Euler products. This is well-known in the case of Ramanujan's result regarding $\sigma_a\sigma_b$, and is presented in several texts, cf., e.g., [10, Th. 305], [22, Prop. 5.4]. An alternative proof is given by Lambek [17].

In Section 8 we offer another less known short proof of (10).

In the case of the functions $f(n) = n^a$, $h(n) = n^b$, $g(n) = k(n) = \lambda(n)$ we obtain (9) by using the known formulae for the Dirichlet series corresponding to the right hand side of (10).

If $f(n) = n^a$, $h(n) = n^b$, $g(n) = \lambda(n)$, $k(n) = 1$, then we deduce

$$\sum_{n=1}^{\infty} \frac{\beta_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s-a)\zeta(s-a-b)\zeta(2s)\zeta(2s-2b)\zeta(2s-a-b)}{\zeta(s)\zeta(s-b)\zeta(4s-2a-2b)},$$

valid for the same region as (9).

Remark that we obtain, as direct corollaries, the next formulae:

$$\sum_{n=1}^{\infty} \frac{\beta^2(n)}{n^s} = \frac{\zeta(s)\zeta(s-2)\zeta(2s-2)}{\zeta^2(s-1)},$$

$$\sum_{n=1}^{\infty} \frac{\beta(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-2)}{\zeta(s-1)}, \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{\beta(n)\sigma(n)}{n^s} = \frac{\zeta(s-2)\zeta(2s)\zeta^2(2s-2)}{\zeta(s)\zeta(4s-4)},$$

all valid for $\Re(s) > 3$. Here (11) is obtained from (9) by choosing $a = 1$ and $b = 0$.

From these Dirichlet series representations we can deduce the following convolutional identities:

$$\beta^2(n) = \sum_{dk=n} d\, 2^{\omega(d)}\lambda(d)\sigma_2(k) \quad (n \geq 1), \tag{12}$$

$$\beta(n^2) = \sum_{dk=n} d\mu(d)\sigma_2(k) \quad (n \geq 1), \tag{13}$$

$$\beta(n)\sigma(n) = \sum_{d^2k=n} d^2 2^{\omega(d)} \beta_2(k) \quad (n \geq 1), \tag{14}$$

where $\omega(n)$ denotes the number of distinct prime factors of n.

5 Asymptotic behavior

The average order of $\beta(n)$ is $(\pi^2/15)n$, more exactly,

$$\sum_{n\leq x} \beta(n) = \frac{\pi^2}{30}x^2 + \mathcal{O}\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{15}$$

Formula (15) follows from the convolution representation (5) and from the known estimate of Walfisz regarding $\sum_{n\leq x}\varphi(n)$ with the same error term as above.

There are also other asymptotic properties of the φ function, which can be transposed to β by using that $\beta(n) \geq \varphi(n)$, with equality for n squarefree. For example,

$$\liminf_{n\to\infty} \frac{\beta(n)\log\log n}{n} = e^{-\gamma},$$

where γ is Euler's constant (cf. [10, Th. 328] concerning φ). Another example: the set $\{\beta(n)/n : n \geq 1\}$ is dense in the interval $[0,1]$.

Let $c_r(n)$ denote the Ramanujan sum, defined as the sum of n-th powers of the primitive r-th roots of unity. Then

$$\frac{\beta(n)}{n} = \frac{\pi^2}{15}\sum_{r=1}^{\infty}\frac{\lambda(r)}{r^2}c_r(n) \tag{16}$$

$$= \frac{\pi^2}{15}\left(1 - \frac{(-1)^n}{2^2} - \frac{2\cos(2\pi n/3)}{3^2} + \frac{2\cos(\pi n/2)}{4^2} + \ldots\right),$$

showing how the values of $\beta(n)/n$ fluctuate harmonically about their mean value $\pi^2/15$, cf. [7], [22, p. 245].

A quick direct proof of formula (16) is given in Section 8. We refer to [18] for a recent survey of expansions of functions with respect to Ramanujan sums.

From the identities (12), (13) and (14) we deduce the following asymptotic formulae:

$$\sum_{n\leq x}\beta^2(n) = \frac{2\zeta(3)}{15}x^3 + \mathcal{O}\left(x^2(\log x)^2\right),$$

$$\sum_{n\le x} \beta(n^2) = \frac{2\zeta(3)}{\pi^2}x^3 + \mathcal{O}\left(x^2\log x\right),$$

$$\sum_{n\le x} \beta(n)\sigma(n) = \frac{\pi^6}{2430\zeta(3)}x^3 + \mathcal{O}\left(x^2\right).$$

We also have

$$\sum_{n\le x} \frac{1}{\beta(n)} = K_1 \log x + K_2 + \mathcal{O}\left(x^{-1+\varepsilon}\right), \qquad (17)$$

for every $\varepsilon > 0$, where K_1 and K_2 are constants,

$$K_1 = \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{a=1}^{\infty}\frac{1}{\beta(p^a)}\right).$$

For the proof of (17) see Section 8.

6 Unitary analog

Consider the function β^* defined by

$$\beta^*(n) = \sum_{d\|n} d\,\lambda(n/d) \quad (n\ge 1),$$

where the sum is over the unitary divisors d of n. Recall that d is a unitary divisor of n if $d \mid n$ and $\gcd(d, n/d) = 1$. Here $\beta^*(p^a) = p^a + (-1)^a$ for every prime power p^a ($a \ge 1$) and

$$\sum_{n=1}^{\infty} \frac{\beta^*(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s)\zeta(2s-1)}{\zeta(s)\zeta(4s-2)} \quad (\Re(s) > 2). \qquad (18)$$

The formula (18) can be derived using Euler products or by establishing the convolutional identity

$$\beta^*(n) = \sum_{dk^2=n} \beta(d)k\,q(k),$$

q standing for the characteristic function of the squarefree numbers. This leads also to the asymptotic formula

$$\sum_{n\le x} \beta^*(n) = \frac{63\zeta(3)}{2\pi^4}x^2 + \mathcal{O}\left(x(\log x)^{5/3}(\log\log x)^{4/3}\right).$$

Note the following interpretation: $\beta^*(n) = \#\{k : 1 \leq k \leq n$ and $\gcd(k,n)_*$ is a square$\}$, where $(a,b)_*$ is the largest divisor of a which is a unitary divisor of b.

7 Super-imperfect numbers, open problems

A number n is super-imperfect if $2\beta(\beta(n)) = n$, cf. Introduction. Observe that, excepting 4, all the other examples of super-imperfect numbers are of the form $n = 2^{2^k-1}$ with $k \in \{1,2,3,4,5\}$. The proof of the next statement is given in Section 8.

Proposition. For $k \geq 1$ the number $n_k = 2^{2^k-1}$ is super-imperfect if and only if $k \in \{1,2,3,4,5\}$.

Problem 1. Is there any other super-imperfect number?

More generally, we define n to be (m,k)-imperfect if $k\beta^{(m)}(n) = n$, where $\beta^{(m)}$ is the m-fold iterate of β. For example, 3, 15, 18, 36, 72, 255 are $(2,3)$-imperfect, 6, 12, 24, 30, 60, 120 are $(2,6)$-imperfect, 6, 36, 144 are $(3,6)$-imperfect numbers.

We refer to [8] regarding (m,k)-perfect numbers, defined by $\sigma^{(m)}(n) = kn$.

Problem 2. Investigate the (m,k)-imperfect numbers.

The numbers $n = 1, 20, 116, 135, 171, 194, 740, \ldots$ are solutions of the equation $\beta(n) = \beta(n+1)$.

Problem 3. Are there infinitely many numbers n such that $\beta(n) = \beta(n+1)$?

Remark that it is not known if there are infinitely many numbers n such that $\sigma(n) = \sigma(n+1)$ (sequence A002961 in [33]). See also Weingartner [31].

The next problem is the analog of Lehmer's open problem concerning the φ function.

Problem 4. Is there any composite number $n \neq 4$ such that $\beta(n)$ divides $n - 1$?

Up to 10^6 there are no such composite numbers.

The computations were performed using Maple. The function $\beta(n)$ was generated by the following procedure:

```
beta:= proc(n) local x, i: x:= 1:
for i from 1 to nops(ifactors(n)[2])
do p_i:= ifactors(n)[2][i][1]: a_i:= ifactors(n)[2][i][2];
x:= x*((p_i^(a_i+1)+(-1)^(a_i))/(p_i+1)): od: RETURN(x) end;
# alternating sum-of-divisors function
```

8 Proofs

Proof of formula (10): Write

$$(f*g)(n)(h*k)(n) = \sum_{\substack{d|n \\ e|n}} f(d)g(n/d)h(e)k(n/e),$$

where $d \mid n, e \mid n \iff \text{lcm}[d,e] \mid n$. Write $d = mu$, $e = mv$ with $\gcd(u,v) = 1$. Then $\text{lcm}[d,e] = muv$ and obtain that this sum is

$$\sum_{\substack{muv|n \\ \gcd(u,v)=1}} f(mu)g(n/(mu))h(mv)k(n/(mv))$$

$$= \sum_{muv|n} f(mu)g(n/(mu))h(mv)k(n/(mv)) \sum_{\delta|(\gcd(u,v)} \mu(\delta).$$

Putting now $u = \delta x$, $v = \delta y$ and using that the considered functions are all completely multiplicative the latter sum is

$$\sum_{\delta^2 xymt=n} (\mu fghk)(\delta)(fk)(x)(gh)(y)(fh)(m)(gk)(t),$$

finishing the proof (cf. [30, p. 621] and [27, p. 161]).

Proof of formula (16): Let $\eta_r(n) = r$ if $r \mid n$ and $\eta_r(n) = 0$ otherwise. Applying that $\sum_{d|r} c_d(n) = \eta_r(n)$ we deduce

$$\frac{\beta(n)}{n} = \sum_{d|n} \frac{\lambda(d)}{d} = \sum_{d=1}^{\infty} \frac{\lambda(d)}{d^2}\eta_d(n) = \sum_{d=1}^{\infty} \frac{\lambda(d)}{d^2} \sum_{r|d} c_r(n)$$

$$= \sum_{r=1}^{\infty} \frac{\lambda(r)}{r^2} c_r(n) \sum_{k=1}^{\infty} \frac{\lambda(k)}{k^2} = \frac{\zeta(4)}{\zeta(2)} \sum_{r=1}^{\infty} \frac{\lambda(r)}{r^2} c_r(n),$$

using that λ is completely multiplicative and its Dirichlet series is $\sum_{n=1}^{\infty} \lambda(n)/n^s = \zeta(2s)/\zeta(s)$. The rearranging of the terms is justified by the absolute convergence.

Proof of formula (17): Write

$$\frac{1}{\beta(n)} = \sum_{\substack{dk=n \\ \gcd(d,k)=1}} \frac{h(d)}{\varphi(k)}$$

as the unitary convolution of the functions h and $1/\varphi$, where h is multiplicative and for every prime power p^a ($a \ge 1$),

$$\frac{1}{\beta(p^a)} = h(p^a) + \frac{1}{\varphi(p^a)}, \quad h(p^a) = -\frac{p^{a-1} + (-1)^a}{p^{a-1}(p-1)(p^{a+1} + (-1)^a)}.$$

Here

$$|h(p^a)| \le \frac{1}{p^a(p-1)^2}, \quad |h(n)| \le \frac{f(n)}{\varphi(n)} \ (n \ge 1),$$

with $f(n) = \prod_{p|n}(p(p-1))^{-1}$. We deduce

$$\sum_{n \le x} \frac{1}{\beta(n)} = \sum_{d \le x} h(d) \sum_{\substack{k \le x/d \\ \gcd(d,k)=1}} \frac{1}{\varphi(k)},$$

and use the known estimates for the inner sum. The same arguments were applied in the proof of [28, Th. 2].

Proof of the Proposition of Section 7: The fact that the numbers n_k with $1 \le k \le 5$ are super-imperfect follow also by direct computations, but the following arguments reveal a connection to the Fermat numbers $F_m = 2^{2^m} + 1$.

For $n_k = 2^{2^k - 1}$ with $k \ge 1$ we have

$$\beta(n_k) = \frac{2^{2^k} - 1}{3} = F_1 F_2 \cdots F_{k-1}$$

(for $k = 1$ this is 1, the empty product). Since the numbers F_m are pairwise relatively prime,

$$\beta(\beta(n_k)) = \beta(F_1)\beta(F_2) \cdots \beta(F_{k-1}).$$

Now for $2 \le k \le 5$, using that F_1, F_2, F_3, F_4 are primes,

$$\beta(\beta(n_k)) = 2^{2^1} \cdot 2^{2^2} \cdot \ldots \cdot 2^{2^{k-1}} = 2^{2^k - 2} = \frac{n_k}{2},$$

showing that n_k is super-imperfect.

Now let $k \ge 6$. We use that F_5 is composite and that $\beta(n) \lneqq n - 1$ for every $n \ne 4$ composite. Hence $\beta(F_5) \lneqq 2^{2^5}$ and

$$\beta(\beta(n_k)) \lneqq \beta(F_1)\beta(F_2) \cdots \beta(F_{k-1}) = 2^{2^k - 2} = \frac{n_k}{2},$$

ending the proof.

Note that for $k \ge 2$ the number $m_k = 2^{2^k - 1} F_1 F_2 \cdots F_{k-1}$ is imperfect if and only if $k \in \{2, 3, 4, 5\}$. This follows by similar arguments. The imperfect numbers of this form are 40, 10 880, 715 816 960 and 3 074 457 344 902 430 720, given in the Introduction.

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