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Maia type fixed point theorems for Ćirić-Prešić operators

Margareta-Eliza Balazs
Technical University of Cluj-Napoca,
North University Center of Baia Mare, Romania
email: margareta_eliza_b@yahoo.com

Abstract. The main aim of this paper is to obtain Maia type fixed point results for Ćirić-Prešić contraction condition, following Ćirić L. B. and Prešić S. B. result proved in [Ćirić L. B.; Prešić S. B. On Prešić type generalization of the Banach contraction mapping principle, Acta Math. Univ. Comenian. (N.S.), 2007, v 76, no. 2, 143–147] and Luong N. V. and Thuan N. X. result in [Luong, N. V., Thuan, N. X., Some fixed point theorems of Prešić-Ćirić type, Acta Univ. Apulensis Math. Inform., No. 30, (2012), 237–249]. We unified these theorems with Maia's fixed point theorem proved in [Maia, Maria Grazia. Un'osservazione sulle contrazioni metriche. (Italian) Rend. Sem. Mat. Univ. Padova 40 1968 139–143] and the obtained results are proved is the present paper. An example is also provided.

1 Introduction and preliminaries

Prešić S. B. [11] extended the famous Banach contraction principle [2] to the case of product spaces in 1965. Recently, in 2007, Ćirić and Prešić [10], generalized the Prešić's theorem introducing Ćirić-Prešić contraction condition. Other important Prešić fixed point theorem generalizations and some related results can be found in Păcurar's papers [7], [8].

The following result was given by M. G. Maia [4] in 1968 and is also a generalization of Banach contraction mapping principle for sets endowed with two

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comparable metrics. Maia type fixed point results for singlevalued or multivalued operators have been studied in [9], [12], [13], [14].

Theorem 1 [14], [4]

Let X *be a nonempty set,* d *and* ρ *two metrics on* X *and* $f : X \to X$ *an operator. We suppose that:*

(i) $d(x, y) \le \rho(x, y)$ *for all* $x, y \in X$;
(ii) (X, d) *is a complete metric space;*
(iii) $f : (X, d) \to (X, d)$ *is continuous;*
(iv) $f : (X, \rho) \to (X, \rho)$ *is an* α*-contraction.*

Then:

(a) $F_f = \{x^*\}$;
(b) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$, *for all* $x \in X$;
(c) $f^n(x) \xrightarrow{\rho} x^*$ *as* $n \to \infty$, *for all* $x \in X$;
(d) $\rho(x, x^*) \le \frac{1}{1-\alpha}\rho(x, f(x))$, *for each* $x \in X$.

In 2007, Ćirić L. B. and Prešić S. B. generalized the Prešić's theorem introducing Ćirić-Prešić contraction condition. Their fixed point result can be stated as follows:

Theorem 2 [10] *Let* (X, d) *be a complete metric space,* k *a positive integer and* $T : X^k \to X$ *a mapping satisfying the following contractive type condition:*

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\}, \quad (1)$$

where $\lambda \in (0, 1)$ *is constant and* $x_1, \dots, x_{k+1} \in X$.

Then there exists a point $x^* \in X$ *such that* $T(x^*, \dots, x^*) = x^*$. *Moreover, if* $x_1, x_2, x_3, \dots, x_{k+1}$ *are arbitrary points in X and for* $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ *is convergent and*

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n).$$

If, in addition, we suppose that on diagonal $\Delta \subset X^k$,

$$d(T(u, \dots, u), T(v, \dots, v)) < d(u, v) \quad (2)$$

holds for all $u, v \in X$, *with* $u \ne v$, *then* x^* *is the unique fixed point of* T *in* X *with* $T(x^*, \dots, x^*) = x^*$.

Remark 1 [10] *Theorem 2 is a generalization of Prešić fixed point theorem (see [11]), as the Prešić's contraction condition implies the conditions 1 and 2.*

$$\begin{aligned}
&d(T(x_1,x_2,\dots,x_k),T(x_2,x_3,\dots,x_{k+1}))\le\\
&\le \alpha_1 d(x_1,x_2)+\alpha_2 d(x_2,x_3)+\dots+\alpha_k d(x_k,x_{k+1})\le\\
&\le(\alpha_1+\alpha_2+\dots+\alpha_k)\max\{d(x_1,x_2),d(x_2,x_3),\dots,d(x_k,x_{k+1})\}\le\\
&\le\lambda\max\{d(x_i,x_{i+1}):1\le i\le k\}
\end{aligned}$$

and

$$\begin{aligned}
&d(T(u,u,\dots,u),T(v,v,\dots,v))\le\\
&\le d(T(u,u,\dots,u),T(u,\dots,u,v))+d(T(u,\dots,u,v),T(u,\dots,u,v,v))+\\
&\quad+\dots+d(T(u,v,\dots,v),T(v,v,\dots,v))\le\\
&\le\alpha_k d(u,v)+\alpha_{k-1}d(u,v)+\dots+\alpha_1 d(u,v)=\\
&=(\alpha_k+\alpha_{k-1}+\dots+\alpha_1)d(u,v)<d(u,v).
\end{aligned}$$

Following the above result, the next lemma is a generalization of the Prešić's lemma in [11].

Lemma 1 *Let $k\in\mathbb{N}, k\neq 0$ and $\lambda\in(0,1)$. If $\{\Delta_n\}_{n\ge1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k}\le\lambda\max\{\Delta_n,\Delta_{n+1},\dots,\Delta_{n+k-1}\}, n\ge1, \tag{3}$$

then there exist $L>0$ and $\theta\in(0,1)$ such that

$$\Delta_n\le L\cdot\theta^n,\ \text{for all } n\ge1. \tag{4}$$

Proof. Similarly with the proof of the result [10], we have:

Let $\Delta_1,\Delta_2,\dots,\Delta_k$ be k positive elements of the sequence $\{\Delta_n\}_{n\ge1}$ satisfying (3).

Denoting $L=\max\{\Delta_1,\Delta_2,\dots,\Delta_k\}$, we obtain

$$\Delta_k\le\lambda\max\{\Delta_1,\Delta_2,\dots,\Delta_k\}=\lambda L$$

We assume that (4) holds for $n,n+1,\dots,n+k-1$ and we prove that it takes place for $n+k$.

$$\Delta_i\le L\theta^i,\ i=n,n+1,\dots,n+k-1$$

where $\theta = \lambda^{\frac{1}{k}}$, $L = \max\{\frac{\Delta_1}{\theta}, \frac{\Delta_2}{\theta^2}, \ldots, \frac{\Delta_k}{\theta^k}\}$.

$$\begin{aligned}\Delta_{n+k} &\leq \lambda \max\{\Delta_n, \Delta_{n+1}, \ldots, \Delta_{n+k-1}\} \\ &\leq \lambda \max\{L\theta^n, L\theta^{n+1}, \ldots, L\theta^{n+k-1}\} \\ &= L\lambda \max\{\theta^n, \theta^{n+1}, \ldots, \theta^{n+k-1}\}.\end{aligned}$$

As $\theta \in (0,1)$, $\theta^{n+1} < \theta^n$, we have

$$\begin{aligned}&\Delta_{n+k} \leq L\lambda\theta^n \ (0 < \theta < 1) \\ &\Delta_{n+k} \leq L\theta^{n+k}.\end{aligned}$$

□

Remark 2 [12] *For any operator* $f : X^k \to X$, k *a positive integer, we can define its associate operator* $F : X \to X$ *by*

$$F(x) = f(x, \ldots, x), \ x \in X.$$

$x \in X$ *is a fixed point of* $f : X^k \to X$ *if and only if* x *is a fixed point of its associate operator* F.

Remark 3 *Particular cases:*
1. From Maia's fixed point theorem when $d \equiv \rho$, *we get Banach's fixed point theorem.*
2. A Maia type fixed point theorem for Prešić-Kannan operators has been obtained by Balazs M. [1].

Starting from these results, the aim of this paper is to extend Theorem 2 and Theorem 2.2, Theorem 2.5 from [5], to the case of a set endowed with two comparable metrics.

2 The main results

Theorem 3 *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X, k *a positive integer,* $\lambda \in (0,1)$ *a constant and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{\rho(x_i, x_{i+1}) : 1 \leq i \leq k\} \quad (5)$$

for any $x_1, x_2, \ldots, x_{k+1} \in X$.

We suppose that:
(i) $d(x, y) \leq \rho(x, y)$ *for all* $x, y \in X$;
(ii) (X, d) *is a complete metric space;*
(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous;*
(iv) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y) \tag{6}$$

holds for all $x, y \in X$, *with* $x \neq y$.
Then:
(a) f *has a unique fixed point* x^*, $F_f = \{x^*\}$, $f(x^*, x^*, \dots, x^*) = x^*$;
(b) *the sequence* $\{x_n\}_{n\geq 1}$ *with* $x_1, x_2, \dots, x_k \in X$, *and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \geq 1$$

converges to x^* *w.r.t.* d.

Proof. Let $\{x_n\}_{n\geq 1}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, $n \geq 1$, with $x_1, x_2, \dots, x_k$ arbitrary elements in X.

$$\begin{aligned} \rho(x_{n+k}, x_{n+k+1}) &= \rho(f(x_n, x_{n+1}, \dots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \dots, x_{n+k})) \\ &\leq \lambda \max\{\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2}), \dots, \rho(x_{n+k-1}, x_{n+k})\} \end{aligned}$$

Denoting $\rho(x_n, x_{n+1}) = \Delta_n$ we have

$$\Delta_{n+k} \leq \lambda \max\{\Delta_n, \Delta_{n+1}, \dots, \Delta_{n+k-1}\},\ n \geq 1$$

The conditions in Lemma 1 are fulfilled and there exist $L > 0$ and $\theta \in (0, 1)$ such that

$$\begin{aligned} \Delta_n &\leq L\theta^n,\ n \geq 1 \\ \rho(x_{n+k}, x_{n+k+1}) &\leq \lambda \max\{L\theta^n, L\theta^{n+1}, \dots, L\theta^{n+k-1}\} \\ &\leq \lambda L\theta^n \end{aligned}$$

From [10], $\lambda = \theta^k$, so $\rho(x_{n+k}, x_{n+k+1}) \leq L\theta^{n+k}$.
For $n, p \in \mathbb{N}^*$ with $p > n$, we have

$$\begin{aligned} \rho(x_n, x_{n+p}) &= \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+p-1}, x_{n+p}) \leq \\ &\leq L\theta^n + L\theta^{n+1} + \cdots + L\theta^{n+k-1} = \\ &= L\theta^n(1 + \theta + \cdots + \theta^{p-1}) = \\ &= L\theta^n \frac{1 - \theta^p}{1 - \theta},\ n \geq 1,\ p \geq 1. \end{aligned}$$

Since $\theta \in (0, 1)$, it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, ρ). From (i) it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete metric space (X, d), so $\{x_n\}_{n\geq 1}$ is also convergent: there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_{n+k}, x^*) = 0$.

By the continuity of f and the associate operator $F : X \to X$, $F(x) = f(x, x, \ldots, x)$, for any $x \in X$, we have:

$$\begin{aligned} d(F(x^*), x^*) &= d(f(x^*, \ldots, x^*), x^*) = \\ &= d(f(\lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_{n+k-1}), x^*) = \\ &= \lim_{n\to\infty} (d(f(x_n, \ldots, x_{n+k-1})), x^*) = \lim_{n\to\infty} d(x_{n+k}, x^*) = 0. \end{aligned}$$

Therefore $x^* = f(x^*, \ldots, x^*) = F(x^*)$ is a fixed point of f.

We suppose there exists another fixed point of f, $y^* = f(y^*, \ldots, y^*)$,

$$d(x^*, y^*) = d(f(x^*, \ldots, x^*), f(y^*, \ldots, y^*))$$

from (iv) we have

$$d(x^*, y^*) < d(x^*, y^*)$$

which is a contradiction. The uniqueness of the fixed point is proved. □

Remark 4 *We have the following important particular cases of Theorem 3:*
1. If $k = 1$, by Theorem 3 we get Maia fixed point theorem.
2. If $d = \rho$, by Theorem 3 we get Ćirić and Prešić fixed point theorem [10].

Following the results in [3], we extend them to the case of a set endowed with two comparable metrics.

Remark 5 [3]

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying
(i) φ *is continuous and non-decreasing;*
(ii) $\sum_{i=1}^{\infty} \varphi^i(t) < \infty$, *for all $t \in (0, \infty)$.*

Lemma 2 [5] *Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is non-decreasing. Then for every $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$, implies $\varphi(t) < t$.*

Remark 6 [3]

Property (ii) from Remark 5 implies $\lim_{n\to\infty} \varphi^n(t) = 0$ for every $t > 0$. Therefore, by Lemma 2, if $\varphi \in \Phi$ then $\varphi(t) < t$, for every $t > 0$.

Theorem 4 *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X*,* k *a positive integer,* $\varphi \in \Phi$ *and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \varphi(\max\{\rho(x_i, x_{i+1}) : 1 \le i \le k\}) \quad (7)$$

for any $x_1, x_2, \dots, x_{k+1} \in X$*.*

We suppose that:

(i) $d(x, y) \le \rho(x, y)$ *for all* $x, y \in X$*;*

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous;*

(iv) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y) \quad (8)$$

holds for all $x, y \in X$*, with* $x \ne y$*.*

Then:

(a) f *has a unique fixed point* x^**,* $F_f = \{x^*\}$*,* $f(x^*, x^*, \dots, x^*) = x^*$*;*

(b) *the sequence* $\{x_n\}_{n\ge1}$ *with* $x_1, x_2, \dots, x_k \in X$*, and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \ n \ge 1$$

converges to x^* *w.r.t.* d*.*

Proof. Let $\{x_n\}_{n\ge1}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, $n \ge 1$, with $x_1, x_2, \dots, x_k \in X$. For simplicity, we set

$$\theta = \max\{\rho(x_1, x_2), \rho(x_2, x_3), \dots, \rho(x_k, x_{k+1})\}.$$

If $x_1 = x_2 = \dots = x_{k+1} = x^*$, then x^* is a fixed point of f, therefore we assume they are not all equal, i.e., $\theta > 0$.

We have

$$\begin{aligned}
\rho(x_{k+1}, x_{k+2}) &= \rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \\
&\le \varphi(\max\{\rho(x_1, x_2), \rho(x_2, x_3), \dots, \rho(x_k, x_{k+1}\}) \le \\
&\le \varphi(\theta) < \theta. \\
\rho(x_{k+2}, x_{k+3}) &= \rho(f(x_2, x_3, \dots, x_{k+1}), f(x_3, x_4, \dots, x_{k+2})) \le \\
&\le \varphi(\max\{\rho(x_2, x_3), \rho(x_3, x_4), \dots, \rho(x_{k+1}, x_{k+2}\}) \le \\
&\le \varphi(\max\{\theta, \varphi(\theta)\}) = \varphi(\theta) < \theta. \\
&\dots \\
\rho(x_2k, x_{2k+1}) &= \rho(f(x_k, x_{k+1}, \dots, x_{2k-1}), f(x_{k+1}, x_{k+2}, \dots, x_{2k})) \le \\
&\le \varphi(\max\{\rho(x_k, x_{k+1}), \rho(x_{k+1}, x_{k+2}), \dots, \rho(x_{2k-1}, x_{2k}\}) \le \\
&\le \varphi(\max\{\theta, \varphi(\theta), \dots, \varphi(\theta)\}) = \varphi(\theta) < \theta.
\end{aligned}$$

$$\begin{aligned}\rho(x_{2k+1}, x_{2k+2}) &= \rho(f(x_{k+1}, x_{k+2}, \dots, x_{2k}), f(x_{k+2}, x_{k+3}, \dots, x_{2k+1})) \leq \\ &\leq \varphi(\max\{\rho(x_{k+1}, x_{k+2}), \rho(x_{k+2}, x_{k+3}), \dots, \rho(x_{2k}, x_{2k+1}\}) \leq \\ &\leq \varphi(\max\{\varphi(\theta), \varphi(\theta), \dots, \varphi(\theta)\}) = \varphi^2(\theta) < \varphi(\theta).\end{aligned}$$

By induction, we get

$$\rho(x_{nk+1}, x_{nk+2}) \leq \varphi^n(\theta),\ n \geq 1$$

or

$$\rho(x_{n+1}, x_{n+2}) \leq \varphi^{\left[\frac{n}{k}\right]}(\theta),\ n \geq k.$$

By property (ii) from Remark 5, we have

$$\lim_{n\to\infty} \rho(x_{n+1}, x_{n+2}) = 0 \tag{9}$$

For $n, p \in \mathbb{N}$, $n > k$, we have

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+p-1}, x_{n+p}) \leq \\ &\leq \varphi^{\left[\frac{n-1}{k}\right]}(\theta) + \varphi^{\left[\frac{n}{k}\right]}(\theta) + \cdots + \varphi^{\left[\frac{n+p-2}{k}\right]}(\theta)\end{aligned} \tag{10}$$

Denoting $l = \left[\frac{n-1}{k}\right]$ and $m = \left[\frac{n+p-2}{k}\right]$, $l \leq m$.

From inequality 10, we have

$$\begin{aligned}\rho(x_n, x_{n+p}) \leq\ & \underbrace{\varphi^l(\theta) + \varphi^l(\theta) + \cdots + \varphi^l(\theta)}_{k \text{ times}} + \\ & + \underbrace{\varphi^{l+1}(\theta) + \varphi^{l+1}(\theta) + \cdots + \varphi^{l+1}(\theta)}_{k \text{ times}} + \\ & + \cdots + \\ & + \underbrace{\varphi^m(\theta) + \varphi^m(\theta) + \cdots + \varphi^m(\theta)}_{k \text{ times}}\end{aligned}$$

and that is

$$\rho(x_n, x_{n+p}) \leq k \sum_{i=l}^{m} \varphi^i(\theta). \tag{11}$$

By property (ii)

$$\lim_{l\to\infty} \sum_{i=l}^{m} \varphi^i(\theta) = 0.$$

So we have that $\rho(x_n, x_{n+p}) \to 0$, when $n \to \infty$. The sequence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, ρ). From (i) it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete metric space (X, d), so $\{x_n\}_{n\geq 1}$ is also convergent: there exists $x^* \in X$ such that $d(x_{n+k}, x^*) = 0$.

By the continuity of f and the associate operator $F : X \to X$, $F(x) = f(x, x, \dots, x)$, for any $x \in X$, we have

$$\begin{aligned} d(F(x^*), x^*) &= d(f(x^*, \dots, x^*), x^*) = \\ &= d(f(\lim_{n\to\infty} x_n, \dots, \lim_{n\to\infty} x_{n+k-1}), x^*) = \\ &= \lim_{n\to\infty}(d(f(x_n, \dots, x_{n+k-1})), x^*) = \lim_{n\to\infty} d(x_{n+k}, x^*) = 0. \end{aligned}$$

Therefore $x^* = f(x^*, \dots, x^*) = F(x^*)$ is a fixed point of f.

We suppose there exists another fixed point of f, $y^* = f(y^*, \dots, y^*)$, $x^* \neq y^*$,

$$d(x^*, y^*) = d(f(x^*, \dots, x^*), f(y^*, \dots, y^*))$$

from (iv) we have

$$d(x^*, y^*) = d(f(x^*, \dots, x^*), f(y^*, \dots, y^*)) < d(x^*, y^*)$$

which is a contradiction. The uniqueness of the fixed point is proved. □

Remark 7 *We have the following particular cases of Theorem 4:*

1. If $\varphi(t) = \lambda t$, for all $t \in [0, \infty)$ and $\lambda \in (0, 1)$, by Theorem 4 we get Theorem 3.

2. If $d = \rho$, by Theorem 4 we get Theorem 2.2 in [3].

The next theorem is an extension of Theorem 4 to monotone nondecreasing mappings in ordered metric spaces. First we recall some useful notions [3]:

Let $(X, \preceq)$ be a partially ordered set and we consider the following partial order on X^k

for $x, y \in X^k$, $x = (x_1, x_2, \dots, x_k)$, $y = (y_1, y_2, \dots, y_k)$

$$x \sqsubseteq y \Leftrightarrow x_1 \preceq y_1, x_2 \preceq y_2, \dots, x_k \preceq y_k.$$

Definition 1 [3] *Let $(X, \preceq)$ be a partially ordered set and $f : X^k \to X$ a mapping.*

f is said to be monotone non-decreasing if for all $x, y \in X^k$,

$$x \sqsubseteq y \Rightarrow f(x) \preceq f(y),$$

where $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k)$.

Theorem 5 *Let* X *be a nonempty set,* $(X, \preceq)$ *a partially ordered set,* d *and* ρ *two metrics on* X*,* k *a positive integer,* $\varphi \in \Phi$ *and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \varphi(\max\{\rho(x_i, x_{i+1}) : 1 \leq i \leq k\}) \quad (12)$$

for any $x_1, x_2, \dots, x_{k+1} \in X$ *and* $x_1 \preceq x_2 \preceq \cdots \preceq x_{k+1}$.

We suppose that:

(i) $d(x, y) \leq \rho(x, y)$ *for all* $x, y \in X$;

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous*

or

X *has the property: if* $\{x_n\}_{n\geq 1}$ *is a monotone non-decreasing sequence,* $x_n \to x$ *then* $x_n \preceq x$*, for any* $n \geq 1$*;*

(iv) *there exists* k *elements* $x_1, x_2, \dots, x_k \in X$ *such that*

$$x_1 \preceq x_2 \preceq \cdots \preceq x_k \text{ and } x_k \preceq f(x_1, x_2, \dots, x_k);$$

(v) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y) \quad (13)$$

holds for all $x, y \in X$*, with* $x \neq y$.

Then:

(a) f *has a unique fixed point* x^**,* $F_f = \{x^*\}$*,* $f(x^*, x^*, \dots, x^*) = x^*$;

(b) *the sequence* $\{x_n\}_{n\geq 1}$ *with* $x_1, x_2, \dots, x_k \in X$*, and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \geq 1,\ x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \dots$$

converges to x^* *w.r.t.* d.

Proof. From (iv), if we denote $x_{k+1} = f(x_1, x_2, \dots, x_k) \succeq x_k$, $x_{k+2} = f(x_2, x_3, \dots, x_{k+1}) \succeq x_{k+1}$ and so on, we obtain the sequence $\{x_n\}_{n\geq 1}$,

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \geq 1,\ x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \dots$$

The alternative assumption (iii) is usual in fixed point theory in ordered metric spaces. The first paper that first considered this assumption is due to Nieto, Juan J.; Rodríguez-López, Rosana. Existence and uniqueness results for fuzzy differential equations subject to boundary value conditions. Mathematical models in engineering, biology and medicine, 264–273, AIP Conf. Proc., 1124, Amer. Inst. Phys., Melville, NY, 2009.

For the next part of the proof, see the proof of Theorem 4. □

Corollary 1 *Let X be a nonempty set, $(X, \preceq)$ a partially ordered set, d and ρ two metrics on X, k a positive integer, $\lambda \in (0,1)$ a constant and $f : X^k \to X$ a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \lambda \max\{\rho(x_i, x_{i+1}) : 1 \leq i \leq k\} \quad (14)$$

for any $x_1, x_2, \dots, x_{k+1} \in X$ and $x_1 \preceq x_2 \preceq \dots \preceq x_{k+1}$.

We suppose that:

(i) $d(x,y) \leq \rho(x,y)$ *for all* $x, y \in X$;

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous*

or

X *has the property: if* $\{x_n\}_{n\geq 1}$ *is a monotone non-decreasing sequence,* $x_n \to x$ *then* $x_n \preceq x$, *for any* $n \geq 1$;

(iv) *there exists* k *elements* $x_1, x_2, \dots, x_k \in X$ *such that*

$$x_1 \preceq x_2 \preceq \dots \preceq x_k \text{ and } x_k \preceq f(x_1, x_2, \dots, x_k);$$

(v) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y) \quad (15)$$

holds for all $x, y \in X$, *with* $x \neq y$.

Then:

(a) f *has a unique fixed point* x^*, $F_f = \{x^*\}$, $f(x^*, x^*, \dots, x^*) = x^*$;

(b) *the sequence* $\{x_n\}_{n\geq 1}$ *with* $x_1, x_2, \dots, x_k \in X$, *and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \geq 1,\ x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq \dots$$

converges to x^* *w.r.t.* d.

Remark 8 *We have the following particular cases of Theorem 5:*

1. If $\varphi(t) = \lambda t$, *for all* $t \in [0, \infty)$ *and* $\lambda \in (0,1)$, *by Theorem 5 we get Theorem 3 for ordered metric space, see Corollary 1.*

2. If $d = \rho$, *by Theorem 5 we get Theorem 2.5 in [3].*

The following example, adapted after Example 1 in [10], illustrates the result in this paper.

Example 1 *Let d be the euclidean distance and ρ be the sum-distance, metrics on $X = [0,1] \cup [2,3]$. For $k = 2$, let $f : X^2 \to X$ be a mapping defined by*

$$f(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{4}; & (x_1, x_2) \in [0,1] \times [0,1] \\ \frac{x_1+x_2+4}{4}; & (x_1, x_2) \in [2,3] \times [2,3] \\ \frac{x_1+x_2-2}{4}; & (x_1, x_2) \in [0,1] \times [2,3] \text{ or } (x,y) \in [2,3] \times [0,1]. \end{cases}$$

satisfying the condition 14.

$$d(f(x_1,x_2),f(x_2,x_3)) = \sqrt{(f(x_1,x_2)-f(x_2,x_3))^2} = |f(x_1,x_2)-f(x_2,x_3)|$$
$$= \rho(f(x_1,x_2),f(x_2,x_3))$$

$f : (X^2, d) \to (X, d)$ *is continuous. Hence, the conditions* $(i) - (iv)$ *from Theorem 3 are satisfied.*

Let $\{x_n\}_{n\geq 1}$, *defined by* $x_{n+2} = f(x_n, x_{n+1})$.

For $n = 1$, *we have* $x_3 = f(x_1, x_2)$.

Then,

for any $x_1, x_2 \in [0, 1]$ *we have* $f(x_1, x_2) = x_3 \in [0, 1]$, *and*

for any $x_1, x_2 \in [2, 3]$ *we have* $f(x_1, x_2) = x_3 \in [2, 3]$.

For $x_1, x_2 \in [0, 1]$ *or* $x_1, x_2 \in [2, 3]$ *we have*

$$\rho(f(x_1,x_2),f(x_2,x_3)) = |\frac{x_1+x_2}{4} - \frac{x_2+x_3}{4}| = |\frac{x_1-x_2}{4} + \frac{x_2-x_3}{4}| \leq$$
$$\leq |\frac{x_1-x_2}{4}| + |\frac{x_2-x_3}{4}| \leq \frac{1}{4}\cdot \max\{\rho(x_1,x_2),\rho(x_2,x_3)\}.$$

For $(x_1, x_2) \in [0, 1] \times [2, 3]$ *or* $(x_1, x_2) \in [2, 3] \times [0, 1]$ *we have* $f(x_1, x_2) = x_3 \in [0, 1]$.

Therefore,

if $x_2 \in [2, 3]$, *then*

$$\rho(f(x_1,x_2),f(x_2,x_3)) = |\frac{x_1+x_2}{4} - \frac{x_2+x_3}{4}| \leq \frac{1}{4}\cdot \max\{\rho(x_1,x_2),\rho(x_2,x_3)\}.$$

if $x_2 \in [0, 1]$, *then*

$$\rho(f(x_1,x_2),f(x_2,x_3)) = |\frac{x_1+x_2-2}{4} - \frac{x_2+x_3}{4}| = |\frac{x_1-x_2}{4} - \frac{1}{2} + \frac{x_2-x_3}{4}| \leq$$
$$\leq |\frac{x_1-x_2}{4} - \frac{1}{2}| + |\frac{x_2-x_3}{4}| < |\frac{x_1-x_2}{4}| + |\frac{x_2-x_3}{4}| \leq$$
$$\leq \frac{1}{4}\cdot \max\{\rho(x_1,x_2),\rho(x_2,x_3)\}$$

So f *is a Ciric-Prešić operator, with* $\lambda = \frac{1}{4} \in (0, 1)$.

Since $\lambda = \frac{1}{4} \in (0, 1)$, *it follows that* $\{x_n\}_{n\geq 1}$ *is a Cauchy sequence in* (X, ρ). *From* (i) *we have that* $\{x_n\}_{n\geq 1}$ *is a Cauchy sequence in* (X, d), *which in the complete metric space* (X, d), *is also convergent. So there exists* $x^* \in [0, 1] \cup [2, 3]$ *such that*

$$\lim_{n\to\infty} x_n = x^*,\ \ x^* = f(x^*, x^*).$$

$$d(x_3, 0) = d(f(x_1, x_2), f(0, 0)) = \sqrt{(f(x_1, x_2) - f(0, 0))^2} = |f(x_1, x_2) - f(0, 0)| = 0$$

$$d(x_3, 2) = d(f(x_1, x_2), f(2, 2)) = \sqrt{(f(x_1, x_2) - f(2, 2))^2} = |f(x_1, x_2) - f(2, 2)| = 0$$

From the continuity of f *in* (X, d)*, we have*

$$\lim_{n\to\infty} f(x_1, x_2) = f(0, 0),$$

and

$$\lim_{n\to\infty} f(x_1, x_2) = f(2, 2),$$

so $f(0, 0) = 0$ *and* $f(2, 2) = 2$*,* $F_f = \{0; 2\}$.

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