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A Hájek–Rényi type inequality and its applications

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Abstract

A general method is presented to obtain strong laws of large numbers. Then it is applied for certain dependent random variables to obtain some strong laws.

1. Introduction

It is well-known that the Hájek–Rényi inequality (see [7]) is a generalization of the Kolmogorov inequality. In this paper we show (Theorem 2.1) that Kolmogorov's inequality implies a certain Hájek–Rényi type inequality. Using this fact we give a general method to obtain strong laws of large numbers (Theorem 2.4). Actually our method is the same as the one applied in Fazekas and Klesov [5] and Fazekas et al. [6] but here we use probabilities instead of moments. In the proof we follow the lines of [5].

Our theorem offers a general tool: if a maximal inequality is known for a certain sequence of random variables then one can easily obtain a strong law of large numbers. Our scheme helps to find the conditions and the normalizing constants.

In section 3 we apply our theorem to give alternative proofs for some known strong laws of large numbers. We deal with associated, negatively associated random variables and demimartingales.

2. Results

Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{R}$ the set of real numbers. If $a_1, a_2, \ldots \in \mathbb{R}$ then in case $A = \emptyset$ let $\max_{k\in A} a_k = 0$ and $\sum_{k\in A} a_k = 0$. Let $\{X_k,\ k \in \mathbb{N}\}$ be a sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and $S_k = \sum_{i=1}^k X_i$ for all $k \in \mathbb{N}$.

Theorem 2.1. *Let $\{\alpha_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers and $r > 0$. Then the following two statements are equivalent.*

(i) *There exists $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$*

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k.$$

(ii) *There exists $c > 0$ such that for any nondecreasing sequence $\{\beta_k,\ k \in \mathbb{N}\}$ of positive real numbers, any $n \in \mathbb{N}$ and any $\varepsilon > 0$*

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k|\, \beta_k^{-1} \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k \beta_k^{-r}.$$

Proof. The proof is based on the idea of the proof of Theorem 1.1 in Fazekas and Klesov [5]. It is clear that (ii) implies (i). Now we turn to (i) $\Rightarrow$ (ii). Let $0 < \beta_1 \leqslant \beta_2 \leqslant \ldots$, $n \in \mathbb{N}$ and $\varepsilon > 0$ are fixed. Without loss of generality we can assume that $\beta_1 = 1$. Introduce the following notation

$$\begin{aligned}
A_i &= \{m \ :\ 1 \leqslant m \leqslant n \text{ and } 2^i \leqslant \beta_m^r < 2^{i+1}\}, \quad i = 0, 1, 2, \ldots, \\
I &= \max\{i \ :\ A_i \neq \emptyset\}, \\
m_i &= \begin{cases} \max A_i, & \text{if } A_i \neq \emptyset, \\ m_{i-1}, & \text{if } A_i = \emptyset, \end{cases} \qquad i = 0, 1, 2, \ldots \quad \text{and} \quad m_{-1} = 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{P}\Big(\max_{k\leqslant n} |S_k|\, \beta_k^{-1} \geqslant \varepsilon\Big) &\leqslant \sum_{i=0}^{I} \mathrm{P}\Big(\max_{k\in A_i} |S_k| \geqslant \varepsilon 2^{i/r}\Big) \\
&\leqslant \sum_{i=0}^{I} \mathrm{P}\Big(\max_{k\leqslant m_i} |S_k| \geqslant \varepsilon 2^{i/r}\Big) \leqslant \sum_{i=0}^{I} c\varepsilon^{-r} 2^{-i} \sum_{k=1}^{m_i} \alpha_k \\
&= c\varepsilon^{-r} \sum_{k=0}^{I} \sum_{j\in A_k} \alpha_j \sum_{i=k}^{I} 2^{-i} \leqslant 2c\varepsilon^{-r} \sum_{k=0}^{I} 2^{-k} \sum_{j\in A_k} \alpha_j \\
&\leqslant 2c\varepsilon^{-r} \sum_{k=0}^{I} \sum_{j\in A_k} \alpha_j 2\beta_j^{-r} = 4c\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k \beta_k^{-r}.
\end{aligned}$$

Thus the theorem is proved. $\square$

The following two lemmas are due to Fazekas and Klesov (see [4, Lemma 2.1 and Lemma 2.2]).

Lemma 2.2. *Let $\{\lambda_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers. Assume that $\sum_{k=1}^{\infty} \lambda_k 2^{-k} < \infty$. Then there exists a nondecreasing unbounded sequence $\{\gamma_k,\ k \in \mathbb{N}\}$ of positive real numbers such that*

$$\sum_{k=1}^{\infty} \lambda_k \gamma_k^{-1} < \infty \quad \textit{and} \quad \lim_{k\to\infty} \gamma_k 2^{-k} = 0. \tag{2.1}$$

Proof. If finitely many λ_k are positive then the statements are obvious. Suppose that there are infinitely many positive λ_k. Let $z = \sum_{k=1}^{\infty} \lambda_k 2^{-k}$ and let n_i be the smallest integer such that

$$\sum_{k=n_i}^{\infty} \lambda_k 2^{-k} \leqslant z 2^{-i}, \quad i = 0, 1, \ldots.$$

Let $q_{-1} = 0$, $q_i = \min\{n_j \ :\ j = 0, 1, \ldots \text{ and } n_j > q_{i-1}\}$ $(i = 0, 1, \ldots)$,

$$B_i = \{k \in \mathbb{N} \ :\ q_i \leqslant k < q_{i+1}\} \quad (i = 0, 1, \ldots)$$

and $\gamma_k = 2^{k-i/2}$ for $k \in B_i$. Property $\gamma_k \leqslant \gamma_{k+1}$ has to be verified only for $k = q_{i+1} - 1$, $i = 0, 1, \ldots$. In this case $\gamma_{k+1}/\gamma_k = \sqrt{2}$ so $\{\gamma_k,\ k \in \mathbb{N}\}$ is nondecreasing. This equality implies $\lim_{i\to\infty} \gamma_{q_i} = \infty$, so $\{\gamma_k,\ k \in \mathbb{N}\}$ is unbounded. Now we turn to (2.1).

$$\sum_{k=1}^{\infty} \lambda_k \gamma_k^{-1} = \sum_{i=0}^{\infty} \sum_{k\in B_i} \lambda_k \gamma_k^{-1} \leqslant \sum_{i=0}^{\infty} 2^{i/2} \sum_{k=n_i}^{\infty} \lambda_k 2^{-k} \leqslant z \sum_{i=0}^{\infty} 2^{-i/2} < \infty.$$

The last statement follows from the definition of γ_k. $\square$

Lemma 2.3. *Let $\{\alpha_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers, $\{b_k,\ k \in \mathbb{N}\}$ a nondecreasing unbounded sequence of positive real numbers and $r > 0$. Assume that $\sum_{k=1}^{\infty} \alpha_k b_k^{-r} < \infty$. Then there exists a nondecreasing unbounded sequence $\{\beta_k,\ k \in \mathbb{N}\}$ of positive real numbers such that*

$$\sum_{k=1}^{\infty} \alpha_k \beta_k^{-r} < \infty \quad \textit{and} \quad \lim_{k\to\infty} \beta_k b_k^{-1} = 0. \tag{2.2}$$

Proof. Let $w_0 = 0$, $w_i = \max\{k \in \mathbb{N} \ :\ b_k^r \leqslant 2^i\}$ $(i \in \mathbb{N})$,

$$C_i = \{k \in \mathbb{N} \ :\ w_{i-1} + 1 \leqslant k \leqslant w_i\} \quad (i \in \mathbb{N})$$

and $\lambda_i = \sum_{k\in C_i} \alpha_k$. Since

$$\sum_{k=1}^{\infty} \alpha_k b_k^{-r} = \sum_{i=1}^{\infty} \sum_{k\in C_i} \alpha_k b_k^{-r} \geqslant \sum_{i=1}^{\infty} \lambda_i 2^{-i}$$

we get that $\sum_{i=1}^{\infty} \lambda_i 2^{-i} < \infty$. So all conditions of Lemma 2.2 are satisfied. Let $\{\gamma_k,\ k \in \mathbb{N}\}$ be fixed by Lemma 2.2. Now we put

$$\beta_k = \gamma_i^{1/r} \quad \text{for} \quad k \in C_i.$$

Then

$$\infty > \sum_{i=1}^{\infty} \lambda_i \gamma_i^{-1} \sum_{i=1}^{\infty} \sum_{k \in C_i} \alpha_k \gamma_i^{-1} = \sum_{k=1}^{\infty} \alpha_k \beta_k^{-r}.$$

The other statements are obvious. □

Theorem 2.4. *Let $\{\alpha_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers, $r > 0$ and $\{b_k,\ k \in \mathbb{N}\}$ a nondecreasing unbounded sequence of positive real numbers. Assume that*

$$\sum_{k=1}^{\infty} \alpha_k b_k^{-r} < \infty$$

and there exists $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$

$$\mathrm{P}\Big(\max_{k \leqslant n} |S_k| \geqslant \varepsilon \Big) \leqslant c\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k. \tag{2.3}$$

Then

$$\lim_{n\to\infty} S_n b_n^{-1} = 0 \quad \textit{almost surely } (a.s.).$$

Proof. The proof is based on the idea of the proof of Theorem 2.1 in Fazekas and Klesov [4]. Let $\{\beta_k,\ k \in \mathbb{N}\}$ be fixed by Lemma 2.3. Then (2.3) and Theorem 2.1 imply that there exists $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$

$$\mathrm{P}\Big(\max_{k \leqslant n} |S_k|\, \beta_k^{-1} \geqslant \varepsilon \Big) \leqslant c\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k \beta_k^{-r}.$$

By this fact we get for any fixed $m \in \mathbb{N}$

$$\mathrm{P}\Big(\sup_{k} |S_k|\, \beta_k^{-1} > \varepsilon_m \Big) \leqslant \lim_{n\to\infty} \mathrm{P}\Big(\max_{k \leqslant n} |S_k|\, \beta_k^{-1} \geqslant \varepsilon_m \Big) \leqslant c\varepsilon_m^{-r} \sum_{k=1}^{\infty} \alpha_k \beta_k^{-r},$$

where $\{\varepsilon_m,\ m \in \mathbb{N}\}$ a nondecreasing unbounded sequence of positive real numbers. So we have by (2.2)

$$\lim_{m\to\infty} \mathrm{P}\Big(\sup_{k} |S_k|\, \beta_k^{-1} > \varepsilon_m \Big) = 0.$$

Hence, using continuity of probability, we have

$$\mathrm{P}\Big(\sup_{k} |S_k|\, \beta_k^{-1} > \varepsilon_m \text{ for all } m \in \mathbb{N} \Big) = 0.$$

Consequently $\sup_k |S_k|\, \beta_k^{-1} < \infty$ a.s. Thus by (2.2) we get

$$\lim_{k\to\infty} |S_k(\omega)|\, b_k^{-1} = \lim_{k\to\infty} \left(|S_k(\omega)|\, \beta_k^{-1} \right) \left(\beta_k b_k^{-1} \right) = 0$$

for almost every $\omega \in \Omega$. Thus the theorem is proved. □

3. Some applications

We shall prove that some known results (i.e. Theorem 3.3, Theorem 3.4, Theorem 3.7, Theorem 3.8 and Theorem 3.12) are special cases of Theorem 2.4.

Associated random variables

Definition 3.1 (Esary et al. [3]). A finite family $\{X_1, \dots, X_n\}$ of random variables is called *associated* if

$$\operatorname{cov}\big(f(X_1, \dots, X_n), g(X_1, \dots, X_n)\big) \geqslant 0$$

for any real coordinatewise nondecreasing functions f, g on $\mathbb{R}^n$ such that the above covariance exists. An infinite family of random variables is associated if its every finite subfamily is associated.

Lemma 3.2 (Matuła [11], Lemma 1). *Assume that $X_1, \dots, X_n$ are associated zero mean random variables with finite second moments. Then for every $\varepsilon > 0$*

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant 8\varepsilon^{-2}\,\mathrm{E}\,S_n^2.$$

Theorem 3.3 (Matuła [11], Theorem 1). *Let $\{X_k,\ k\in\mathbb{N}\}$ be a sequence of associated random variables with finite second moments and $\{a_k,\ k\in\mathbb{N}\}$ a sequence of positive real numbers satisfying $\sum_{k=1}^{\infty} a_k = \infty$. Let $b_n = \sum_{i=1}^{n} a_i$. Assume that*

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} a_i a_j \operatorname{cov}(X_i, X_j) b_j^{-2} < \infty.$$

Then

$$\lim_{n\to\infty} (S_n^* - \mathrm{E}\,S_n^*) b_n^{-1} = 0 \quad a.s.,$$

where $S_n^* = \sum_{i=1}^{n} a_i X_i$.

Proof. Without loss of generality we can assume that $\mathrm{E}\,X_k = 0$ for all $k\in\mathbb{N}$. Let $\alpha_k = \mathrm{E}\,S_k^{*2} - \mathrm{E}\,S_{k-1}^{*2}$, where $S_0^* = 0$. Then for all $k\in\mathbb{N}$

$$0 \leqslant \alpha_k \leqslant 2\sum_{i=1}^{k} a_i a_k \operatorname{cov}(X_i, X_k),$$

so we have

$$\sum_{k=1}^{\infty} \alpha_k b_k^{-2} \leqslant \sum_{k=1}^{\infty}\sum_{i=1}^{k} 2a_i a_k \operatorname{cov}(X_i, X_k) b_k^{-2} < \infty.$$

It is easy to see that $\{a_k X_k,\ k\in\mathbb{N}\}$ is associated thus, by Lemma 3.2,

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k^*| \geqslant \varepsilon\Big) \leqslant 8\varepsilon^{-2}\,\mathrm{E}\,S_n^{*2} = 8\varepsilon^{-2}\sum_{k=1}^{n}\alpha_k$$

for any $\varepsilon > 0$. Consequently, by Theorem 2.4, we get $\lim_{n\to\infty} S_n^* b_n^{-1} = 0$ a.s. $\square$

Theorem 3.4 (Birkel [1], Theorem 2 and Christofides [2], Corollary 2.2). *Let $\{X_k,\ k \in \mathbb{N}\}$ be a sequence of associated random variables with finite second moments. If*

$$\sum_{k=1}^{\infty} k^{-2} \operatorname{cov}(X_k, S_k) < \infty$$

then

$$\lim_{n\to\infty} (S_n - \mathrm{E}\, S_n) n^{-1} = 0 \quad a.s.$$

Proof. Without loss of generality we can assume that $\mathrm{E}\, X_k = 0$ for all $k \in \mathbb{N}$. Let $\alpha_k = \operatorname{cov}(X_k, S_k)$, $b_k = k$ and $S_0 = 0$. Then, by Lemma 3.2, we have

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant 8\varepsilon^{-2}\, \mathrm{E}\, S_n^2 = 8\varepsilon^{-2} \sum_{k=1}^{n} \big(\mathrm{E}\, S_k^2 - \mathrm{E}\, S_{k-1}^2\big) \leqslant 16\varepsilon^{-2} \sum_{k=1}^{n} \alpha_k.$$

Thus Theorem 2.4 implies the statement. □

Negatively associated random variables

Definition 3.5 (Joag-Dev and Proschan [8]). A finite family $\{X_1, \dots, X_n\}$ of random variables is called *negatively associated* if for any disjoint nonempty subsets $A, B \subset \{1, \dots, n\}$, $A = \{i_1, \dots, i_l\}$, $B = \{i_{l+1}, \dots, i_n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^l$ and g on $\mathbb{R}^{n-l}$

$$\operatorname{cov}\big(f(X_{i_1}, \dots, X_{i_l}), g(X_{i_{l+1}}, \dots, X_{i_n})\big) \leqslant 0.$$

An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

The following lemma is a special case of Theorem 2.1 of Liu et al. [9]. (See Lemma 1 of Matuła [10], too.)

Lemma 3.6. *Assume that $X_1, \dots, X_n$ are negatively associated zero mean random variables with finite second moments. Then for every $\varepsilon > 0$*

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant 32\varepsilon^{-2} \sum_{k=1}^{n} \mathrm{E}\, X_k^2.$$

Theorem 3.7 (Matuła [11], Theorem 2). *Let $\{X_k,\ k \in \mathbb{N}\}$ be a sequence of negatively associated random variables with finite second moments and $\{a_k,\ k \in \mathbb{N}\}$ a sequence of positive real numbers satisfying $\sum_{k=1}^{\infty} a_k = \infty$. Let $b_n = \sum_{i=1}^{n} a_i$. Assume that*

$$\sum_{k=1}^{\infty} a_k^2 b_k^{-2}\, \mathrm{D}^2\, X_k < \infty.$$

Then

$$\lim_{n\to\infty} (S_n^* - \mathrm{E}\, S_n^*) b_n^{-1} = 0 \quad a.s.,$$

where $S_n^* = \sum_{i=1}^{n} a_i X_i$.

Proof. Without loss of generality we can assume that $\mathrm{E}\,X_k = 0$ for all $k \in \mathbb{N}$. Let $\alpha_k = a_k^2\,\mathrm{E}\,X_k^2$. It is clear that $\{a_k X_k,\ k \in \mathbb{N}\}$ is negatively associated, so by Lemma 3.6 we have

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k^*| \geqslant \varepsilon\Big) \leqslant 32\varepsilon^{-2}\sum_{k=1}^{n}\alpha_k$$

for any $\varepsilon > 0$. Thus Theorem 2.4 implies the statement. □

Theorem 3.8 (Liu et al. [9], Theorem 3.1). *Let $\{X_k,\ k \in \mathbb{N}\}$ be a sequence of negatively associated random variables with finite second moments and $\{b_k,\ k \in \mathbb{N}\}$ a nondecreasing and unbounded sequence of positive real numbers. Assume that*

$$\sum_{k=1}^{\infty} b_k^{-2}\,\mathrm{D}^2\,X_k < \infty.$$

Then

$$\lim_{n\to\infty} (S_n - \mathrm{E}\,S_n)b_n^{-1} = 0 \quad a.s.$$

Proof. Without loss of generality we can assume that $\mathrm{E}\,X_k = 0$ for all $k \in \mathbb{N}$. Let $\alpha_k = \mathrm{E}\,X_k^2$. Then Lemma 3.6 and Theorem 2.4 imply the statement. □

Demimartingales

We shall use the following notations:

$$X^+ = \max\{0, X\} \quad \text{and} \quad X^- = -\min\{0, X\}.$$

Definition 3.9 (Newman and Wright [12]). Let $\{S_k,\ k \in \mathbb{N}\}$ be an L^1 sequence of random variables. Assume that for $j \in \mathbb{N}$

$$\mathrm{E}\big((S_{j+1} - S_j)\, f\,(S_1, \ldots, S_j)\big) \geqslant 0$$

for all coordinatewise nondecreasing functions f on $\mathbb{R}^j$ such that the expectation is defined. Then $\{S_k,\ k \in \mathbb{N}\}$ is called a *demimartingale*. If in addition the function f is assumed to be nonnegative, the sequence $\{S_k,\ k \in \mathbb{N}\}$ is called a *demisubmartingale*.

Lemma 3.10 (Christofides [2], Theorem 2.1). *Let $\{S_k,\ k \in \mathbb{N}\cup\{0\}\}$ be a demisubmartingale with $S_0 = 0$. Let $\{b_k,\ k \in \mathbb{N}\}$ be a nondecreasing sequence of positive real numbers. Then for all $\varepsilon > 0$*

$$\mathrm{P}\Big(\max_{k\leqslant n} S_k b_k^{-1} \geqslant \varepsilon\Big) \leqslant \varepsilon^{-1}\sum_{k=1}^{n} b_k^{-1}\,\mathrm{E}\left(S_k^+ - S_{k-1}^+\right).$$

The following lemma is a corollary of Lemma 2.1 and Corollary 2.1 of Christofides [2].

Lemma 3.11. *If $\{S_k,\ k \in \mathbb{N}\}$ is demimartingale then $\{(S_k^+)^r,\ k \in \mathbb{N}\}$ and $\{(S_k^-)^r,\ k \in \mathbb{N}\}$ are demisubmartingales for all $r \geqslant 1$.*

Theorem 3.12 (Christofides [2], Theorem 2.2). *Let $\{S_k,\ k \in \mathbb{N} \cup \{0\}\}$ be a demimartingale with $S_0 = 0$. Let $\{b_k,\ k \in \mathbb{N}\}$ be a nondecreasing and unbounded sequence of positive real numbers. Let $r \geqslant 1$ and $\mathrm{E}\,|S_k|^r < \infty$ for each $k \in \mathbb{N}$. Assume that*

$$\sum_{k=1}^{\infty} b_k^{-r}\,\mathrm{E}\left(|S_k|^r - |S_{k-1}|^r\right) < \infty.$$

Then

$$\lim_{n\to\infty} S_n b_n^{-1} = 0 \quad a.s.$$

Proof. Let $\alpha_k = \mathrm{E}\left(|S_k|^r - |S_{k-1}|^r\right)$ for all $k \in \mathbb{N}$ and $\varepsilon > 0$. By Lemma 3.11 and 3.10

$$\begin{aligned} &\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant \mathrm{P}\Big(\max_{k\leqslant n} (S_k^+)^r \geqslant \varepsilon^r/2\Big) + \mathrm{P}\Big(\max_{k\leqslant n} (S_k^-)^r \geqslant \varepsilon^r/2\Big) \\ &\leqslant 2\varepsilon^{-r} \sum_{k=1}^{n} \mathrm{E}\left((S_k^+)^r + (S_k^-)^r - (S_{k-1}^+)^r - (S_{k-1}^-)^r\right) = 2\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k. \end{aligned}$$

Thus Theorem 2.4 implies the statement. □

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