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Some special cases of a general convergence rate theorem in the law of large numbers

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Abstract

Tómács in [6] proved a general convergence rate theorem in the law of large numbers for arrays of Banach space valued random elements. We shall study this theorem in case Banach space of type φ and for two special arrays.

Key Words: Convergence rates; Arrays of Banach space valued random variables; Banach space of type φ

1. Introduction and notation

Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{R}$ the set of real numbers. Let Φ_0 denote the set of functions $f\colon [0,\infty) \to [0,\infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to satisfy the Δ_2-condition ($f \sim \Delta_2$) if there exists a constant $c > 0$ such that $f(2t) \leqslant cf(t)$ for all $t > 0$.

Let B be a real separable Banach space with norm $\|.\|$ and zero element $\mathbf{0}$. If X is a B-valued random variable (r.v.) and $\boldsymbol{E}\|X\| < \infty$ then $\boldsymbol{E}X$ stands for the Bochner integral of X.

Throughout the paper let $\{k_n, n \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers. Let $\{X_{nk}, n \in \mathbb{N}, k = 1,\dots,k_n\}$ be an array of B-valued r.v.'s. It is rowwise independent, if $X_{n1},\dots,X_{nk_n}$ are independent r.v.'s for any fixed $n \in \mathbb{N}$. Let $S_{k_n} = \sum_{k=1}^{k_n} X_{nk}$. If $k_n = n$ for all n, then we denote S_{k_n} by S_n. This corresponds to the case of ordinary sequences.

The array $\{X_{nk}, n \in \mathbb{N}, k = 1,\dots,k_n\}$ is said to be *bounded in probability* if for all $\varepsilon > 0$ there exists $A > 0$ such that $\boldsymbol{P}(\|X_{nk}\| \geqslant A) < \varepsilon$ for all $n \in \mathbb{N}$ and $k = 1,\dots,k_n$.

The following remark give a sufficient condition for the boundedness in probability.

Remark 1.1. If there exists a constant $M > 0$ such that $\boldsymbol{E}\,\|X_{nk}\| \leqslant M$ for every $n \in \mathbb{N}, k = 1, \dots, k_n$, then the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is bounded in probability. (The reader can readily verify this statement.)

Definition 1.2 (Gut [2]). We say that the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is *weakly mean dominated* (w.m.d.) by the r.v. X, if for some $\gamma > 0$,

$$\frac{1}{k_n}\sum_{k=1}^{k_n} \boldsymbol{P}(\|X_{nk}\| > t) \leqslant \gamma \boldsymbol{P}(|X| > t) \quad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad n \in \mathbb{N}.$$

The following theorem a general convergence rate theorem, which is proved in [6].

Theorem 1.3 (Tómács [6], Theorem 3.1). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that there exists a sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$ is bounded in probability. Let $\alpha, \vartheta, \varphi \in \Phi_0$, and assume that α is not bounded, $\vartheta, \varphi \sim \Delta_2$, $\vartheta \not\equiv 0$. Let $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$, $n = 0, 1, 2, \dots$. Assume that $\boldsymbol{E}\,\varphi(|X|) < \infty$, $\boldsymbol{E}\,\vartheta(|X|) < \infty$ and $\lim_{n\to\infty} \alpha(n)/\gamma_n = \infty$.*

Let either $\mu(n) = \beta(n-1)$ for all $n \in \mathbb{N}$ or $\mu(n) = \beta(n)$ for all $n \in \mathbb{N}$. In second case assume that there exists a constant $c > 0$ such that for $n \in \mathbb{N}$ large enough $c\beta(n) \leqslant \beta(n-1)$.

Let $n_0 \in \mathbb{N}$ be such that $\vartheta(\alpha(n)) > 0$ for all $n \geqslant n_0$. If there exist $j \in \mathbb{N}$ and $r > 0$ such that

$$\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n}\left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j} < \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \boldsymbol{P}\big(\|S_n\| > \varepsilon\alpha(n)\big) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

In the following two corollaries of Theorem 1.3 we use some special notations: Following Gut [1], introduce the functions ψ and M_r with

$$\psi(t) = \operatorname{Card}\{n \in \mathbb{N} : k_n \leqslant t\} \quad \text{for} \quad t \geqslant 0,$$

and

$$M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if} \quad t \geqslant 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if} \quad 0 \leqslant t < 1,$$

where $r \in \mathbb{R}$, $\operatorname{Card}A$ is the cardinality of the set A and $[.]$ denotes the integer function. Let $M = M_2$. Let $f \circ g$ be the composite function of functions f and g.

Remark 1.4. $M_r \circ \psi \in \Phi_0$ and

$$(M_r \circ \psi)(t) = M_r(\psi(t)) = \begin{cases} \sum_{i=1}^{n} k_i^{r-1} = M_r(n), & \text{if} \quad k_n \leqslant t < k_{n+1}, \\ k_1^{r-1} = M_r(1), & \text{if} \quad 0 \leqslant t < k_1. \end{cases}$$

The following corollary is a generalization of Theorem 6.2 of Fazekas [5].

Corollary 1.5 (Tómács [6], Corollary 3.2). *Let* $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ *be an array of rowwise independent* B*-valued r.v.'s which is w.m.d. by the r.v.* X*. Let* $M \circ \psi \sim \Delta_2$, $r, s, t > 0$, $rs > t$*. Assume that* $\{\|S_{k_n}\| / k_n^{1/s}, n \in \mathbb{N}\}$ *is bounded in probability. Furthermore, if* $r > 2$ *we assume that* $\{M(n)/M(n-1), n \in \mathbb{N}\}$ *is bounded. If* $\boldsymbol{E} M^{r/2}(\psi(|X|^{t/r})) < \infty$ *and* $\boldsymbol{E}|X|^s < \infty$*, then*

$$\sum_{n=1}^{\infty} \big(M(n)\big)^{r/2-1} \boldsymbol{P}\left(\|S_{k_n}\| > \varepsilon k_n^{r/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

The following corollary is a version of Corollary 4.1 of Hu et al. [3].

Corollary 1.6 (Tómács [6], Corollary 3.3). *Let* $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ *be an array of rowwise independent* B*-valued r.v.'s which is w.m.d. by the r.v.* X*. Let* $r \in \mathbb{R}$, $0 < t < s$ *and* $M_r \circ \psi \sim \Delta_2$*. Assume that* $\{\|S_{k_n}\| / k_n^{1/s}, n \in \mathbb{N}\}$ *is bounded in probability. If* $\boldsymbol{E} M_r(\psi(|X|^t)) < \infty$ *and* $\boldsymbol{E}|X|^s < \infty$*, then*

$$\sum_{n=1}^{\infty} k_n^{r-2} \boldsymbol{P}\left(\|S_{k_n}\| > \varepsilon k_n^{1/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

In Section 2 we give a sufficient condition for the boundedness in probability and in Section 3 we study two concrete sequences k_n in Corollary 1.5 and 1.6.

2. The boundedness in probability in case Banach space of type φ

If B has an appropriate geometric property, then a moment condition can imply the boundedness of $\{\|S_{k_n}\| / \gamma_{k_n}, n \in \mathbb{N}\}$.

Definition 2.1. A function $\varphi\colon [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if it is continuous, convex, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. For an Orlicz function φ the *Orlicz space* $l_\varphi(B)$ consists of those B-valued sequences $\{u_n, n \in \mathbb{N}\}$ for which

$$\sum_{n=1}^{\infty} \varphi\left(\|u_n\| / a\right) < \infty \quad \text{for some} \quad a > 0.$$

Let $\varepsilon_1, \varepsilon_2, \dots$ be independent r.v.'s with $\boldsymbol{P}(\varepsilon_n = 1) = \boldsymbol{P}(\varepsilon_n = -1) = 1/2$ for all $n \in \mathbb{N}$. B is said to be of *type* φ, if $\sum_{n=1}^{\infty} \varepsilon_n u_n$ converges in probability for all $\{u_n, n \in \mathbb{N}\} \in l_\varphi(B)$.

Definition 2.2. An Orlicz function φ is said to satisfy the Δ_2^0-condition ($\varphi \sim \Delta_2^0$) if there exist constants $c > 0$ and $t_0 > 0$ such that $\varphi(2t) \leqslant c\varphi(t)$ is satisfied for all $0 \leqslant t \leqslant t_0$.

Lemma 2.3. *Let φ be an Orlicz function and $\varphi \sim \Delta_2^0$. B is of type φ iff there exists a constant $c > 0$ such that*

$$\boldsymbol{E}\left\|\sum_{k=1}^{n} X_k\right\| \leqslant c\boldsymbol{E}\inf_{y>0}\left\{\frac{1}{y}\left(1+\sum_{k=1}^{n}\varphi(y\left\|X_k\right\|)\right)\right\}$$

for all $n \in \mathbb{N}$ and every independent B-valued r.v.' $X_1, \ldots, X_n$ with $\boldsymbol{E}X_k = \mathbf{0}$, $k = 1, \ldots, n$.

For the proof see Fazekas [4].

The following lemma is a generalization of Lemma 2.1 of Gut [2] and Lemma 2.7 (b) of Fazekas [5].

Lemma 2.4 (Tómács [6], Lemma 4.4). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of B-valued r.v.'s which is w.m.d. by the r.v. X and constant γ. If $\varphi \in \Phi_0$ then*

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\boldsymbol{E}\,\varphi(\|X_{nk}\|) \leqslant \max\{1,\gamma\}\boldsymbol{E}\,\varphi(|X|).$$

The following theorem show that in Theorem 1.3 we can write moment conditions instead of the boundedness of $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ if B is of type φ.

Theorem 2.5. *Let $\varphi \in \Phi_0$ be a submultiplicative Orlicz function, $\varphi \sim \Delta_2^0$ and let B be a space of type φ. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that the sequence $\{k_n\varphi(1/\gamma_{k_n}), n \in \mathbb{N}\}$ is bounded for some sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers. If $\boldsymbol{E}X_{nk} = \mathbf{0}$ for every $n \in \mathbb{N}$, $k = 1, \ldots, k_n$ and $\boldsymbol{E}\,\varphi(|X|) < \infty$, then $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ is bounded in probability.*

Proof. By Lemma 2.3 and 2.4 there exists a constant $c > 0$ such that

$$\begin{aligned}\boldsymbol{E}\frac{\|S_{k_n}\|}{\gamma_{k_n}} &\leqslant \frac{c}{\gamma_{k_n}}\boldsymbol{E}\inf_{y>0}\left\{\frac{1}{y}\left(1+\sum_{k=1}^{k_n}\varphi(y\left\|X_{nk}\right\|)\right)\right\}\\ &\leqslant c\boldsymbol{E}\left(1+\sum_{k=1}^{k_n}\varphi\left(\|X_{nk}\|/\gamma_{k_n}\right)\right)\\ &\leqslant c\big(1+\varphi\left(1/\gamma_{k_n}\right)\max\{1,\gamma\}k_n\boldsymbol{E}\,\varphi(|X|)\big).\end{aligned}$$

Thus Remark 1.1 implies the statement. □

3. Convergence rate theorems for two concrete sequences k_n

Lemma 3.1. *$f \sim \Delta_2$ iff there exist constants $k > 1$ and $c > 0$ such that*

$$f(kt) \leqslant cf(t) \quad \text{for all} \quad t > 0. \tag{3.1}$$

Proof. If $f \sim \Delta_2$ then in case $k = 2$ we get (3.1). Now suppose that there exist constants $k > 1$ and $c > 0$ such that the inequality (3.1) is true for all $t > 0$. Then we can obtain with induction that

$$f(k^n t) \leqslant c^n f(t) \quad \text{for all} \quad t > 0 \quad \text{and for all} \quad n \in \mathbb{N}.$$

It follows that there exists $n_0 \in \mathbb{N}$ such that

$$f(2t) \leqslant f(k^{n_0} t) \leqslant c^{n_0} f(t) \quad \text{for all} \quad t > 0.$$

Thus we get $f \sim \Delta_2$. □

The reader can readily verify the following lemma.

Lemma 3.2. *Let $g\colon [k_1, \infty) \to \mathbb{R}$ be a nondecreasing function which has the property that $g(k_n) \geqslant M_r(n)$ for all $n \in \mathbb{N}$. Then $M_r(\psi(x)) \leqslant g(x)$ for all $x \geqslant k_1$.*

Lemma 3.3. *Let $r \in \mathbb{R}$. Assume that there exists strictly increasing sequence $\{a_n, n \in \mathbb{N}\}$ of positive integers and there exist constants $k > 1$, $c > 0$ such that*

$$\frac{k_n}{k_{a_n}} \leqslant \frac{1}{k} \quad \textit{and} \quad \frac{M_r(a_n)}{M_r(n-1)} \leqslant c \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then $M_r \circ \psi \sim \Delta_2$.

Proof. Assume that $k_n \leqslant t < k_{n+1}$. Then Remark 1.4 implies

$$M_r(\psi(kt)) \leqslant M_r(\psi(kk_{n+1})) \leqslant M_r(\psi(k_{a_{n+1}})) = M_r(a_{n+1}) \leqslant cM_r(n) = cM_r(\psi(t)).$$

Similarly if $0 < t < k_1$ then

$$M_r(\psi(kt)) \leqslant M_r(\psi(kk_1)) \leqslant M_r(\psi(k_{a_1})) = M_r(a_1) \leqslant cM_r(0) = cM_r(\psi(t)).$$

It follows that $M_r(\psi(kt)) \leqslant cM_r(\psi(t))$ for all $t > 0$. Thus, by Lemma 3.1 we get the statement. □

Lemma 3.4. *Let $l \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + (ln)^k}{1^k + 2^k + \cdots + (n-1)^k} = \begin{cases} l^{k+1}, & \textit{if} \quad k > -1, \\ 1, & \textit{if} \quad k \leqslant -1. \end{cases}$$

Proof. It is easy to see that

$$x^{k+1}-(x-1)^{k+1}\leqslant (k+1)x^k\leqslant (x+1)^{k+1}-x^{k+1}\quad\text{for all}\quad x\geqslant 1, k\geqslant 0$$

and

$$(x+1)^{k+1}-x^{k+1}\leqslant (k+1)x^k\leqslant x^{k+1}-(x-1)^{k+1}\quad\text{for all}\quad x\geqslant 1, -1<k<0.$$

Apply these inequalities for $x=1,2,\dots,n$. Then we have

$$\lim_{n\to\infty}\frac{1^k+2^k+\dots+n^k}{n^{k+1}}=\frac{1}{k+1}\quad\text{for all}\quad k>-1,$$

which implies the statement for $k>-1$.

It is well known that $\frac{1}{1^c}+\frac{1}{2^c}+\cdots\frac{1}{n^c}$ is convergent if $c>1$. It follows that the statement is true in case $k<-1$ as well.

Finally in case $k=-1$ the inequalities

$$1+\frac{\frac{l-1}{l}}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}<\frac{1+\frac{1}{2}+\dots+\frac{1}{ln}}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}<1+\frac{l}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}$$

imply the statement. □

Lemma 3.5. *Let $k_1,d\in\mathbb{N}$, $q\in\mathbb{N}\setminus\{1\}$. If $k_n=k_1q^{n-1}$ or $k_n=k_1n^d$ then $M_r\circ\psi\sim\Delta_2$ for all $r\in\mathbb{R}$.*

Proof. In the first case, when $k_n=k_1q^{n-1}$, let $a_n=n+1$ and $k=q$. Then

$$\frac{k_n}{k_{a_n}}=\frac{k_1q^{n-1}}{k_1q^n}=\frac{1}{q}\leqslant\frac{1}{k}.$$

Let $Q=q^{r-1}$ and assume that $r>1$. In this case $|1/Q|<1$, thus we get

$$\frac{M_r(a_n)}{M_r(n-1)}=\frac{M_r(n+1)}{M_r(n-1)}=\frac{1+Q+\dots+Q^n}{1+Q+\dots+Q^{n-2}}=\frac{Q^2-\frac{1}{Q^{n-1}}}{1-\frac{1}{Q^{n-1}}}\to Q^2.$$

If $r<1$ then $1/Q>1$, thus

$$\frac{M_r(a_n)}{M_r(n-1)}=\frac{Q^2-\frac{1}{Q^{n-1}}}{1-\frac{1}{Q^{n-1}}}\to 1.$$

If $r=1$ then $Q=1$, so

$$\frac{M_r(a_n)}{M_r(n-1)}=\frac{n+1}{n-1}\to 1.$$

Thus we get that $\frac{M_r(a_n)}{M_r(n-1)}$ is bounded for all $r\in\mathbb{R}$. Hence conditions of Lemma 3.3 are satisfied, which implies the statement.

In the second case, when $k_n = k_1 n^d$, let $a_n = 2n$ and $k = 2^d$. Then

$$\frac{k_n}{k_{a_n}} = \frac{k_1 n^d}{k_1 (2n)^d} = \frac{1}{2^d} \leqslant \frac{1}{k}.$$

On the other hand it follows from Lemma 3.4 that $\frac{M_r(a_n)}{M_r(n-1)}$ is bounded for all $r \in \mathbb{R}$. So Lemma 3.3 implies the statement. □

Theorem 3.6. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 n^d\}$ ($k_1, d \in \mathbb{N}$ are fixed) be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $t > 0$, $r \geqslant 2d/(d+1)$, $s > t/r$ and $v = \max\{s, t(d+1)/(2d)\}$. If $\{\|S_{k_1 n^d}\| / n^{d/s}, n \in \mathbb{N}\}$ is bounded in probability and $\boldsymbol{E}|X|^v < \infty$, then*

$$\sum_{n=1}^{\infty} n^{(d+1)(r/2-1)} \boldsymbol{P}\left(\|S_{k_1 n^d}\| > \varepsilon n^{dr/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. We shall prove that conditions of Corollary 1.5 are satisfied. Let $k_n = k_1 n^d$. Then by Lemma 3.4 $\{M(n)/M(n-1), n \in \mathbb{N}\}$ is bounded. Let $Y = M^{r/2}(\psi(|X|^{t/r}))$. Now we turn to the proof of $\boldsymbol{E}Y < \infty$. It is well known that

$$1^d + \dots + n^d = a_1 n^{d+1} + a_2 n^d + \dots + a_{d+2}$$

for some $a_1, a_2, \dots, a_{d+2} \in \mathbb{R}$. Let

$$g\colon [k_1, \infty) \to \mathbb{R}, \quad g(x) = \sum_{i=1}^{d+2} |a_i| \left(k_1^{i-2} x^{d+2-i}\right)^{1/d}.$$

Then g is nondecreasing, $g(k_n) \geqslant M(n)$ and $g(x) \leqslant \text{const.} x^{(d+1)/d}$ for all $x \geqslant k_1$. Therefore by Lemma 3.2 we have

$$M^{r/2}\left(\psi(x^{t/r})\right) \leqslant \text{const.} x^{t(d+1)/(2d)} \quad \text{for all} \quad x^{t/r} \geqslant k_1.$$

It follows that

$$Y = Y\boldsymbol{I}(|X|^{t/r} < k_1) + Y\boldsymbol{I}(|X|^{t/r} \geqslant k_1) \leqslant k_1^{r/2} + \text{const.}|X|^{t(d+1)/(2d)},$$

where $\boldsymbol{I}(A)$ denotes the indicator function of the set A. So $\boldsymbol{E}Y < \infty$. By Lemma 3.5 $M \circ \psi \sim \Delta_2$. It is easy to see that the other conditions of Corollary 1.5 hold true as well, on the other hand $M(n) \geqslant \text{const.} n^{d+1}$. So this theorem is consequence of Corollary 1.5. □

Theorem 3.7. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 q^{n-1}\}$ ($k_1 \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\}$ are fixed) be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $w \geqslant 0$, $t > 0$, $s > t$ and $v = \max\{s, t(w+1)\}$. If $\{\|S_{k_1 q^{n-1}}\| / q^{n/s}, n \in \mathbb{N}\}$ is bounded in probability and $\boldsymbol{E}|X|^v < \infty$, then*

$$\sum_{n=1}^{\infty} q^{nw} \boldsymbol{P}\left(\|S_{k_1 q^{n-1}}\| > \varepsilon q^{n/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. We shall prove that conditions of Corollary 1.6 are satisfied. Let $k_n = k_1 q^{n-1}$, $r = w + 2$ and $Y = M_r(\psi(|X|^t))$. Then $M_r(n) = k_1^{r-1} \frac{Q^n - 1}{Q-1}$, where $Q = q^{r-1}$. Let

$$g\colon [k_1, \infty) \to \mathbb{R}, \quad g(x) = k_1^{r-1} \frac{Q^{1+\log(x/k_1)/\log q} - 1}{Q-1}.$$

Then g is nondecreasing, $g(k_n) = M_r(n)$ and $g(x) \leqslant \text{const.} x^{r-1}$ for all $x \geqslant k_1$. Therefore by Lemma 3.2 we have

$$M_r\left(\psi(x^t)\right) \leqslant \text{const.} x^{t(w+1)} \quad \text{for all} \quad x^t \geqslant k_1.$$

It follows that

$$Y = Y\boldsymbol{I}(|X|^t < k_1) + Y\boldsymbol{I}(|X|^t \geqslant k_1) \leqslant k_1^{r-1} + \text{const.}|X|^{t(w+1)}.$$

So $\boldsymbol{E}Y < \infty$. By Lemma 3.5 $M_r \circ \psi \sim \Delta_2$. The other conditions of Corollary 1.6 hold true as well. Thus Corollary 1.6 implies the statement. □

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