Annales Mathematicae et Informaticae
32 *(2005) pp. 61–78.*

Solution of a sum form equation in the two dimensional closed domain case*

Imre Kocsis

Faculty of Engineering University of Debrecen
e-mail:kocsisi@mfk.unideb.hu

Abstract

In this note we give the solution of the sum form functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j)$$

arising in information theory (in characterization of so-called entropy of degree α), where $f : [0,1]^2 \to \mathbb{R}$ is an unknown function and the equation holds for all two dimensional complete probability distributions.

Key Words: Sum form equation, additive function, multiplicative function.

AMS Classification Number: 39B22

1. Introduction

In the following we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Throughout the paper we shall use the following notations: $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k$, $\underline{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^k$. For all $3 \leqslant n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ we define the sets $\Gamma_n^c[k]$ and $\Gamma_n^0[k]$ by

$$\Gamma_n^c[k] = \left\{ (p_1, \dots, p_n) : p_i \in [0,1]^k, i = 1, \dots, n, \sum_{i=1}^{n} p_i = \underline{1} \right\}$$

*This research has been supported by the Hungarian National Foundation for Scientific Research (OTKA), grant No. T-030082.

and

$$\Gamma_n^0[k] = \left\{(p_1, \ldots, p_n) : p_i \in]0,1[^k, i = 1, \ldots, n, \sum_{i=1}^{n} p_i = \underline{1}\right\},$$

respectively.

If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \in \mathbb{R}^k$ then $x \bullet y = \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_k y_k \end{pmatrix} \in \mathbb{R}^k$.

If we do not say else we denote the components of an element P of $\Gamma_n^c[2]$ or $\Gamma_n^0[2]$ by

$$P = (p_1, \ldots, p_n) = \begin{pmatrix} p_{11} & \ldots & p_{n1} \\ p_{12} & \ldots & p_{n2} \end{pmatrix}.$$

A function $A : \mathbb{R}^k \to \mathbb{R}$ is additive if $A(x + y) = A(x) + A(y)$, $x, y \in \mathbb{R}^k$, a function $M :]0,1[^k \to \mathbb{R}$ is multiplicative if $M(x \bullet y) = M(x)M(y)$, $x, y \in]0,1[^k$, a function $M : [0,1]^k \to \mathbb{R}$ is multiplicative if $M(\underline{0}) = 0$, $M(\underline{1}) = 1$, and $M(x \bullet y) = M(x)M(y)$, $x, y \in [0,1]^k$.

The functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \qquad \text{(E[k])}$$

will be denoted by $(E^c[k])$ if (E[k]) holds for all $(p_1, \ldots, p_n) \in \Gamma_n^c[k]$ and $(q_1, \ldots, q_m) \in \Gamma_m^c[k]$, and the function f is defined on $[0,1]^k$ (closed domain case), and by $(E^0[k])$ if (E[k]) holds for all $(p_1, \ldots, p_n) \in \Gamma_n^0[k]$ and $(q_1, \ldots, q_m) \in \Gamma_m^0[k]$, and f is defined on $]0,1[^k$ (open domain case). The solution of equation $(E^c[1])$ is given by Losonczi and Maksa in [3], while equation $(E^0[k])$ $(k \in \mathbb{N})$ is solved by Ebanks, Sahoo, and Sander in [2].

Theorem 1.1 (Losonczi, Maksa [3]). *Let $n \geqslant 3$ and $m \geqslant 3$ be fixed integers. A function $f : [0,1] \to \mathbb{R}$ satisfies $(E^c[1])$ if, and only if, there exist additive functions $A : \mathbb{R} \to \mathbb{R}$ and $D : \mathbb{R} \to \mathbb{R}$, a multiplicative function $M : [0,1] \to \mathbb{R}$, and $b \in \mathbb{R}$ such that $D(1) = 0$, $A(1) + nmb = (A(1) + nb)(A(1) + mb)$ and*

$$f(p) = A(p) + b, \quad p \in [0,1]$$

or

$$f(p) = D(p) + M(p), \quad p \in [0,1].$$

Theorem 1.2 (Ebanks, Sahoo, Sander [2]). *Let $k \geqslant 1$, $n \geqslant 3$, and $m \geqslant 3$ be fixed integers. A function $f :]0,1[^k \to \mathbb{R}$ satisfies $(E^0[k])$ if, and only if, there exist additive functions $A : \mathbb{R}^k \to \mathbb{R}$ and $D : \mathbb{R}^k \to \mathbb{R}$, a multiplicative function $M :]0,1[^k \to \mathbb{R}$ and $b \in \mathbb{R}$ such that $D(\underline{1}) = 0$, $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ and*

$$f(p) = A(p) + b, \quad p \in]0,1[^k$$

or

$$f(p) = D(p) + M(p), \quad p \in]0,1[^k.$$

The solution of equation ($E^c[k]$) is not known if $k \in \mathbb{N}$, $k \geq 2$. Our purpose is to solve equation ($E^c[2]$).

2. Preliminary results

Lemma 2.1. *Let $k \geqslant 1$, $n \geqslant 3$, and $m \geqslant 3$ be fixed integers. If the function $f : [0,1]^k \to \mathbb{R}$ satisfies ($E^c[k]$) and $A : \mathbb{R}^k \to \mathbb{R}$ is an additive function such that $A(\underline{1}) = 0$ then the function $g = f - A$ satisfies ($E^c[k]$), too.*

Proof.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} A(p_i \bullet q_j) =$$

$$\Big(\sum_{i=1}^{n} f(p_i) - \sum_{i=1}^{n} A(p_i)\Big)\Big(\sum_{i=1}^{n} f(q_j) - \sum_{i=1}^{n} A(q_j)\Big) = \sum_{i=1}^{n} g(p_i)\sum_{j=1}^{m} g(q_j).$$

□

Lemma 2.2. *If $A : \mathbb{R}^2 \to \mathbb{R}$ is additive, $M :]0,1[^2 \to \mathbb{R}$ is multiplicative, $H :]0,1[\to \mathbb{R}$, and $M\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + H(x)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$ then*

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2,$$

where $\mu :]0,1[\to \mathbb{R}$ is a multiplicative function or

$$M\begin{pmatrix} x \\ y \end{pmatrix} = y, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2.$$

Proof. Let $x, y, z \in]0,1[$. Then $A\begin{pmatrix} x \\ yz \end{pmatrix} + H(x) = M\begin{pmatrix} x \\ yz \end{pmatrix} =$ $M\begin{pmatrix} \sqrt{x} \\ y \end{pmatrix} M\begin{pmatrix} \sqrt{x} \\ z \end{pmatrix} = \left(A\begin{pmatrix} \sqrt{x} \\ y \end{pmatrix} + H(\sqrt{x})\right)\left(A\begin{pmatrix} \sqrt{x} \\ z \end{pmatrix} + H(\sqrt{x})\right)$. With fixed x and the notations $a_1(t) = A\begin{pmatrix} x \\ t \end{pmatrix}$, $t \in]0,1[$, $a_2(t) = A\begin{pmatrix} \sqrt{x} \\ t \end{pmatrix}$, $t \in]0,1[$ this implies that $a_1(yz) + H(x) = (a_2(y) + H(\sqrt{x}))(a_2(z) + H(\sqrt{x}))$, while with the substitutions $y = z = \sqrt{t}$, $a_1(t) + H(x) = (a_2(t) + H(\sqrt{x}))^2$, that is, $A\begin{pmatrix} 0 \\ t \end{pmatrix} = (a_2(t) + H(\sqrt{x}))^2 - A\begin{pmatrix} x \\ 0 \end{pmatrix} - H(x)$, $t \in]0,1[$. Since the function $t \to A\begin{pmatrix} 0 \\ t \end{pmatrix}$ is additive and $A\begin{pmatrix} 0 \\ t \end{pmatrix} \geqslant -A\begin{pmatrix} x \\ 0 \end{pmatrix} - H(x), t \in]0,1[$, there exists $c \in \mathbb{R}$ such that $A\begin{pmatrix} 0 \\ t \end{pmatrix} = ct$ (see Aczél [1]), thus $A\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ 0 \end{pmatrix} +$

cy, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$, furthermore $M\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ 0 \end{pmatrix} + H(x) + cy$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. Let $\mu(x) = A\begin{pmatrix} x \\ 0 \end{pmatrix} + H(x)$, $x \in]0,1[$ and let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in]0,1[^2$. Then $cy_1y_2 + \mu(x_1x_2) = M\begin{pmatrix} x_1x_2 \\ y_1y_2 \end{pmatrix} = M\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} M\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (cy_1 + \mu(x_1))(cy_2 + \mu(x_2))$. Thus $(c - c^2)y_1y_2 = \mu(x_1)\mu(x_2) - \mu(x_1x_2) + c(y_1\mu(x_2) + y_2\mu(x_1))$. Taking here the limit $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we have that μ is multiplicative and

$$c(1 - c)y_1y_2 = c(y_1\mu(x_2) + y_2\mu(x_1)).$$

This implies that either $c = 0$ and

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} x \in]0,1[^2$$

or $(1 - c)y_1y_2 = y_1\mu(x_2) + y_2\mu(x_1)$, $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in]0,1[^2$. Since μ is multiplicative, in this case we get that $c = 1$ and $A\begin{pmatrix} x \\ 0 \end{pmatrix} + H(x) = \mu(x) = 0$, $x \in]0,1[$. Thus

$$M\begin{pmatrix} x \\ y \end{pmatrix} = y, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2.$$

□

Lemma 2.3. *Suppose that $3 \leqslant n \in \mathbb{N}$, $3 \leqslant m \in \mathbb{N}$, $f : [0,1]^2 \to \mathbb{R}$ satisfies equation $(E^c[2])$ and*

$$K = (m - 1)f(\underline{0}) + f(\underline{1}) = 1. \tag{2.1}$$

Then $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$.

Proof. Substituting $P = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$,$Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$ in $(E^c[2])$, by (2.1), we have $(nm - 1)f(\underline{0}) + f(\underline{1}) = (n - 1)f(\underline{0}) + f(\underline{1})$ and, after some calculation, we get that $n(m-1)f(\underline{0}) = 0$. This and (2.1) imply that $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. □

3. The main result

Theorem 3.1. *Let $n \geqslant 3$ and $m \geqslant 3$ be fixed integers. A function $f : [0,1]^2 \to \mathbb{R}$ satisfies $(E^c[2])$ if, and only if, there exist additive functions $A, D : \mathbb{R}^2 \to \mathbb{R}$, a multiplicative function $M : [0,1]^2 \to \mathbb{R}$, and $b \in \mathbb{R}$ such that $D(\underline{1}) = 0$, $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ and*

$$f(p) = A(p) + b, \quad p \in [0,1]^2$$

or

$$f(p) = D(p) + M(p), \quad p \in [0,1]^2.$$

Proof. By Theorem 1.2, with $k = 2$ we have that there exist additive functions $A, D : \mathbb{R}^2 \to \mathbb{R}$, a multiplicative function $M :]0,1[^2 \to \mathbb{R}$ and $b \in \mathbb{R}$ such that $D(\underline{1}) = 0$, $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ and

$$f(p) = A(p) + b, \quad p \in]0,1[^2$$

or

$$f(p) = D(p) + M(p), \quad p \in]0,1[^2.$$

We prove that, beside the conditions of Theorem 3.1, f has similar form with the same $b \in \mathbb{R}$ and with the additive and multiplicative extensions of the functions A,D, and M onto the whole square $[0,1]^2$, respectively. To have this result we will apply special substitutions in equation ($E^c[2]$) to get information about the behavior of f on the boundary of $[0,1]^2$.

Case 1. $f(p) = A(p) + b, \quad p \in]0,1[^2$ and $A(\underline{1}) \neq 0$.

Subcase 1.A. $K \neq 1$ (see (2.1))

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, and $Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$ in ($E^c[2]$) we get that

$$n(m-1)f(\underline{0}) + f\begin{pmatrix} x \\ 0 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 1 \end{pmatrix} + (n-1)b = \left(f\begin{pmatrix} x \\ 0 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 1 \end{pmatrix} + (n-1)b\right)K.$$

Hence

$$f\begin{pmatrix} x \\ 0 \end{pmatrix} = A\begin{pmatrix} x \\ 0 \end{pmatrix} - A(\underline{1}) - (n-1)b + \frac{n(m-1)f(\underline{0})}{K-1} = A\begin{pmatrix} x \\ 0 \end{pmatrix} + b_{10}, \quad (3.1)$$

$x \in]0,1[$ for some $b_{10} \in \mathbb{R}$. A similar calculation shows that there exists $b_{20} \in \mathbb{R}$ such that

$$f\begin{pmatrix} 0 \\ y \end{pmatrix} = A\begin{pmatrix} 0 \\ y \end{pmatrix} + b_{20}, \quad y \in]0,1[. \quad (3.2)$$

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, and $Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$ in ($E^c[2]$) we get that

$$n(m-1)f(\underline{0}) + f\begin{pmatrix} x \\ 1 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 0 \end{pmatrix} + (n-1)b_{10} = \left(f\begin{pmatrix} x \\ 1 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 0 \end{pmatrix} + (n-1)b_{10}\right)K.$$

Thus

$$f\begin{pmatrix} x \\ 1 \end{pmatrix} = A\begin{pmatrix} x \\ 1 \end{pmatrix} - A(\underline{1}) - (n-1)b_{10} + \frac{n(m-1)f(\underline{0})}{K-1} = A\begin{pmatrix} x \\ 1 \end{pmatrix} + b_{11}, \quad (3.3)$$

$x \in]0,1[$ for some $b_{11} \in \mathbb{R}$. A similar calculation shows that there exists $b_{21} \in \mathbb{R}$ such that

$$f\begin{pmatrix} 1 \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ y \end{pmatrix} + b_{21}, \quad y \in]0,1[. \tag{3.4}$$

Now we show that $b = b_{10} = b_{11} = b_{20} = b_{21}$. Define the function $g : [0,1]^2 \to \mathbb{R}$ by $g\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} - \left(A\begin{pmatrix} x \\ y \end{pmatrix} - A(\underline{1})x\right)$. Then, by (3.1),(3.2),(3.3), and (3.4), $g\begin{pmatrix} x \\ y \end{pmatrix} = A(\underline{1})x + \delta$, $\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2 \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$, where $\delta \in \{b, b_{10}, b_{11}, b_{20}, b_{21}\}$, respectively. It follows from Lemma 2.1 that g satisfies equation $(E^c[2])$:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) \tag{3.5}$$

Thus, with the substitutions, $P = \begin{pmatrix} x_1 & \dots & x_n \\ r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$,
$Q = \begin{pmatrix} y_1 & \dots & y_m \\ s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$ in (3.5) we get that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g\begin{pmatrix} x_i y_j \\ rs \end{pmatrix} = \sum_{i=1}^{n} g\begin{pmatrix} x_i \\ r \end{pmatrix} \sum_{j=1}^{m} g\begin{pmatrix} y_j \\ s \end{pmatrix},$$

$(x_1, \dots, x_n) \in \Gamma_n^c[1], (y_1, \dots, y_n) \in \Gamma_m^c[1]$. Let $\zeta \in]0,1[$ be fixed and $G_\zeta(x) = g(x, \zeta)$, $x \in [0,1]$. Since g does not depend on its second variable if it is from $]0,1[$, G_ζ satisfies equation $(E^c[1])$. Concerning $G_\zeta(x) = A(\underline{1})x + b$, $x \in]0,1[$ and $A(\underline{1}) \neq 0$, by Theorem 1.1, we have that $G_\zeta(x) = A(\underline{1})x + b$, $x \in [0,1]$, that is, $b = b_{20} = b_{21}$. In a similar way we can get that $b = b_{10} = b_{11}$, that is,

$$g\begin{pmatrix} x \\ y \end{pmatrix} = A(\underline{1})x + b, \begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2 \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}. \tag{3.6}$$

Now we prove that (3.6) holds on $[0,1]^2$. Let $G_0(x) = g\begin{pmatrix} x \\ 0 \end{pmatrix}$, $x \in [0,1]$. $G_0(x) = A(\underline{1})x + b$, $x \in]0,1[$. Thus G_0 satisfies $(E^0[2])$. We show that G_0 satisfies $(E^c[2])$, too. Let $(p_1, \dots, p_n) = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \Gamma_n^c[2]$,
$(q_1, \dots, q_m) = \begin{pmatrix} y_1 & \dots & y_{m-1} & y_m \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \Gamma_m^c[2]$, $x_1, \dots x_n, y_1 \dots y_m \in [0,1[$. Since $g\begin{pmatrix} t \\ 0 \end{pmatrix} = g\begin{pmatrix} t \\ 1 \end{pmatrix}$, $t \in]0,1[$ we have that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G_0(x_i y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) =$$

$$\sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) = \sum_{i=1}^{n} G_0(x_i) \sum_{j=1}^{m} G_0(q_j). \tag{3.7}$$

Substituting $x_1 = \cdots = x_{n-2} = 0, x_{n-1} = x_n = \frac{1}{2}$, $y_1 = \cdots = y_m = \frac{1}{m}$ in (3.7) and using the equalities $G_0(x) = A(\underline{1})x + b$, $x \in]0,1[$ and $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ we get that

$$(G_0(0) - b)(nm - 2m - nA(\underline{1}) - nmb + 2A(\underline{1}) + 2mb) = 0.$$

An easy calculation shows that the condition $A(\underline{1}) \neq 0$ implies that $(nm - 2m - nA(\underline{1}) - nmb + 2A(\underline{1}) + 2mb) \neq 0$, that is $g(\underline{0}) = G_0(0) = b$.
The substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$ and $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} y_1 & \dots & y_m \\ v & \dots & v \end{pmatrix} \in \Gamma_m^c[2]$ in (3.5), using $G_0(0) = b$, imply that the function G_0 satisfies equation $(E^c[1])$ also in the remaining cases $x_1 = 1, x_2 = \cdots = x_n = 0$,$y_1 = 1, y_2 = \cdots = y_n = 0$ and $x_1 = 1, x_2 = \cdots = x_n = 0$,$(y_1, \dots, y_m) \in \Gamma_m^c[1]$. Thus, by Theorem 1.1, $G_0(x) = A(\underline{1})x + b$, $x \in [0,1]$, that is, $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = G_0(1) = A(\underline{1}) + b$. In a similar way we can get that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A(\underline{1}) + b$. Finally the following calculation proves that $g(\underline{1}) = A(\underline{1}) + b$. Substituting $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (\underline{1}, \underline{0}, \dots, \underline{0}) \in \Gamma_m^c[2]$ in (3.5) we have that $(A(\underline{1}) + nb)(g(\underline{1}) - A(\underline{1}) - b) = 0$. It is easy to see that the condition $A(\underline{1}) \neq 0$ implies that $A(\underline{1}) + nb \neq 0$ thus $g(\underline{1}) = A(\underline{1}) + b$.

Subcase 1.B. $K = 1$ (see (2.1))

In this case, by Lemma 2.3, $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. Substituting
$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in $(E^c[2])$ we get the following system of equations.

$$\begin{aligned} I. \quad & A(\underline{1}) + 4b = (A(\underline{1}) + 2b)^2 \\ II. \quad & A(\underline{1}) + 6b = (A(\underline{1}) + 2b)(A(\underline{1}) + 3b) \\ III. \quad & A(\underline{1}) + 9b = (A(\underline{1}) + 3b)^2. \end{aligned}$$

This and the condition $A(\underline{1}) \neq 0$ imply that $b = 0$, furthermore $A(\underline{1}) = 1$, that is, $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. Substituting $P = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in $(E^c[2])$ we get that $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$

$\left(f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}\right)^2$ thus $f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}\in\{0,1\}$, while with the substitutions $P=\begin{pmatrix}1&0&0\dots&0\\0&1&0\dots&0\end{pmatrix}\in\Gamma_n^0[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in $(E^c[2])$ we get that

$$\sum_{j=1}^m f\begin{pmatrix}q_{j1}\\0\end{pmatrix}+\sum_{j=1}^m f\begin{pmatrix}0\\q_{j2}\end{pmatrix}=\left(f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}\right)\sum_{j=1}^m f(q_j). \tag{3.8}$$

If $f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}=0$ then, with fixed $Q=(q_{12},\dots,q_{m2})$, (3.8) goes over into $\sum_{j=1}^m f\begin{pmatrix}q_{j1}\\0\end{pmatrix}=c$, $(q_{11},\dots,q_{m1})\in\Gamma_m^0[1]$ with some $c\in\mathbb{R}$, so, by Theorem 1.2, there exist additive function $a_{10}:\mathbb{R}\to\mathbb{R}$ and $b_{10}\in\mathbb{R}$ such that

$$f\begin{pmatrix}x\\0\end{pmatrix}=a_{10}(x)+b_{10},\quad x\in]0,1[. \tag{3.9}$$

In a similar way we can prove that there exist an additive function $a_{20}:\mathbb{R}\to\mathbb{R}$ and $b_{20}\in\mathbb{R}$ such that

$$f\begin{pmatrix}0\\y\end{pmatrix}=a_{20}(y)+b_{20},\quad y\in]0,1[. \tag{3.10}$$

If $f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}=1$ then (3.8) goes over into $\sum_{j=1}^m\left[f\begin{pmatrix}q_{j1}\\q_{j2}\end{pmatrix}-f\begin{pmatrix}q_{j1}\\0\end{pmatrix}-f\begin{pmatrix}0\\q_{j2}\end{pmatrix}\right]=0$, $(q_1,\dots,q_m)\in\Gamma_m^0[2]$. Thus there exist an additive function $A_0:\mathbb{R}^2\to\mathbb{R}$ and $b_0\in\mathbb{R}$ such that

$$f\begin{pmatrix}x\\y\end{pmatrix}-f\begin{pmatrix}x\\0\end{pmatrix}-f\begin{pmatrix}0\\y\end{pmatrix}=A_0\begin{pmatrix}x\\y\end{pmatrix}+b_0,\quad \begin{pmatrix}x\\y\end{pmatrix}\in]0,1[^2. \tag{3.11}$$

With the functions $a_{10}(x)=(A-A_0)\begin{pmatrix}x\\0\end{pmatrix}$, $x\in]0,1[$ and $a_{20}(y)=(A-A_0)\begin{pmatrix}0\\y\end{pmatrix}$, $y\in]0,1[$ we have that

$$f\begin{pmatrix}x\\0\end{pmatrix}=a_{10}(x)+\left(a_{20}(y)-f\begin{pmatrix}0\\y\end{pmatrix}+b_0\right),\ x\in]0,1[$$

and

$$f\begin{pmatrix}0\\y\end{pmatrix}=a_{20}(y)+\left(a_{10}(x)-f\begin{pmatrix}x\\0\end{pmatrix}+b_0\right),\ y\in]0,1[.$$

With fixed x and y, we obtain again that (3.9) and (3.10) hold with some $b_{10}\in\mathbb{R}$ and $b_{20}\in\mathbb{R}$, respectively.

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in $(E^c[2])$, after some calculation, we get that

$$f\begin{pmatrix} x \\ 1 \end{pmatrix} = A\begin{pmatrix} x \\ 1 \end{pmatrix}, \quad x \in]0,1[. \tag{3.12}$$

In a similar way we have that

$$f\begin{pmatrix} 1 \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ y \end{pmatrix}, \quad y \in]0,1[. \tag{3.13}$$

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_m^0[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} s & \dots & s \\ s & \dots & s \end{pmatrix}$ in $(E^c[2])$, after some calculation, we have that $b_{10} = 0$ and, in a similar way, we get that $b_{20} = 0$. Substituting $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $x \in]0,1[$ $Q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ y & 1-y & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$ in $(E^c[2])$, after some calculation, we have that

$$\left(a_{10}(x) - A\begin{pmatrix} x \\ 0 \end{pmatrix}\right) + \left(a_{20}(y) - A\begin{pmatrix} 0 \\ y \end{pmatrix} - 1\right) = a_{20}(y) - A\begin{pmatrix} 0 \\ y \end{pmatrix}.$$

This implies that either

$$a_{10}(x) = A\begin{pmatrix} x \\ 0 \end{pmatrix}, \quad x \in]0,1[\tag{3.14}$$

and

$$a_{20}(y) = A\begin{pmatrix} 0 \\ y \end{pmatrix}, \quad y \in]0,1[, \tag{3.15}$$

or none of these equations holds. It is easy to see that the later case is not possible. Thus (3.14) and (3.15) hold. Finally with the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_m^c[2]$, $Q = \begin{pmatrix} s & \dots & s \\ s & \dots & s \end{pmatrix}$ in $(E^c[2])$, after some calculation, we have that $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In a similar way we get that $f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

CASE 2.

$$f(x) = A(x) + b, \quad x \in]0,1[^2, \quad A(\underline{1}) = 0 \tag{3.16}$$

or

$$f(x) = D(x) + M(x), \quad x \in]0,1[^2, \quad D(\underline{1}) = 0. \tag{3.17}$$

Define the function g by $f - A$ if (3.16) holds and by $f - D$ if (3.17) holds. It is easy to see that we have to investigate the following three subcases.

SUBCASE 2.A. $g(x) = 0$, $x \in]0,1[^2$, when

$$f(x) = A(x) + b, \quad b = 0, \quad x \in]0,1[^2$$

or

$$f(x) = D(x) + M(x), \quad M(x) = 0, \quad x \in]0,1[^2,$$

SUBCASE 2.B. $g(x) = 1$, $x \in]0,1[^2$, when

$$f(x) = A(x) + b, \quad b = 1, \quad x \in]0,1[^2$$

or

$$f(x) = D(x) + M(x) \quad M(x) = 1, \quad x \in]0,1[^2,$$

SUBCASE 2.C. $g(x) = 0$, $x \in]0,1[^2$, $\quad M \neq 0, M \neq 1$, when

$$f(x) = D(x) + M(x), \quad x \in]0,1[^2, \quad M \neq 0, M \neq 1.$$

By Lemma 2.1, the function g satisfies ($E^c[2]$):

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) \tag{3.18}$$

SUBCASE 2.A. $g(x) = 0$, $x \in]0,1[^2$. With the substitutions
$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0\ldots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0\ldots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0\ldots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0\ldots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\ldots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\ldots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0\ldots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0\ldots & 0 \end{pmatrix} \in \Gamma_m^c[2]$
in (3.18), after some calculation, we have that $g(\underline{0}) = 0$. With the substitutions
$P = \begin{pmatrix} x & r & \ldots & r \\ 0 & u & \ldots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0, \quad x \in \left]0, \frac{1}{2}\right[, \tag{3.19}$$

while with the substitutions $P = \begin{pmatrix} x & r & \ldots & r \\ 0 & u & \ldots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \ldots, q_m) \in \Gamma_m^0[2]$ in (3.18) we have that

$$\sum_{j=1}^{n} g\begin{pmatrix} xq_{j1} \\ 0 \end{pmatrix} = 0, \quad (q_{11}, \ldots, q_{m1}) \in \Gamma_m^0[1].$$

Hence there exists additive function $a_x : \mathbb{R} \to \mathbb{R}$ such that

$$g\begin{pmatrix} q \\ 0 \end{pmatrix} = a_x\left(\frac{x}{q}\right) - \frac{a_x(1)}{n}, \quad q \in]0,x[, \tag{3.20}$$

where x is an arbitrary fixed element of $]0,1[$. It follows from (3.19) and (3.20) that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0, \quad x \in]0,1[. \tag{3.21}$$

In a similar way we get that

$$g\begin{pmatrix} 0 \\ y \end{pmatrix} = 0, \quad y \in]0,1[. \tag{3.22}$$

It is easy to see that

$$\left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix}, g\begin{pmatrix} 0 \\ 1 \end{pmatrix}, g\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \in \{(0,0,0),(0,0,1),(1,0,1),(0,1,1)\}. \tag{3.23}$$

Indeed, the substitutions
$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0\dots & 0 \\ 1 & 0\dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, and
$P = \begin{pmatrix} 1 & 0\dots & 0 \\ 1 & 0\dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0\dots & 0 \\ 1 & 0\dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$
in (3.18) imply that
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)^2$ thus $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \{0,1\}$,
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$,
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} g\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ thus if $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ then $g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$, and
$g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left(g\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^2$ thus $g\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \{0,1\}$,
respectively. In a similar way we get that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \{0,1\}$, and if $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$ then $g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$, respectively, that is, (3.23) holds.
Now we show that the statement of our theorem holds in each case given by (3.23). The substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 0 & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = g\begin{pmatrix} x \\ 1 \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ thus, if $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$, then $g\begin{pmatrix} x \\ 1 \end{pmatrix} = 1$, $x \in [0,1]$. In a similar way we have that, if $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$, then $g\begin{pmatrix} 1 \\ y \end{pmatrix} = 1$, $y \in [0,1]$. The substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ y & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix} x \\ 1 \end{pmatrix} g\begin{pmatrix} 1 \\ y \end{pmatrix} = 0$. Thus $g\begin{pmatrix} x \\ 1 \end{pmatrix} = 0$, $x \in [0,1]$ or $g\begin{pmatrix} 1 \\ y \end{pmatrix} = 0$, $y \in [0,1]$. In the remaining

case $g\begin{pmatrix}1\\0\end{pmatrix} = g\begin{pmatrix}0\\1\end{pmatrix} = 0$, substitute $P = \begin{pmatrix} x & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} y & s & \dots & s\\ 1 & 0 & \dots & 0\end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$ in (3.18). Then we have that $g\begin{pmatrix}xy\\1\end{pmatrix} = g\begin{pmatrix}x\\1\end{pmatrix} g\begin{pmatrix}y\\1\end{pmatrix}$, $x, y \in]0,1[$, that is, the function $\mu_1(x) = g\begin{pmatrix}x\\1\end{pmatrix}$, $x \in]0,1[$ is multiplicative. In a similar way we can see that the function $\mu_2(y) = g\begin{pmatrix}1\\y\end{pmatrix}$, $y \in]0,1[$ is multiplicative, too.

SUBCASE 2.B. $g(x) = 1, \quad x \in]0,1[^2$. The substitutions $P = \begin{pmatrix} x & r & \dots & r\\ 0 & u & \dots & u\end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18), imply that

$$\sum_{j=1}^{m} \left[g\begin{pmatrix} xq_{j1}\\ 0\end{pmatrix} - g\begin{pmatrix}x\\0\end{pmatrix}\right] = 0, \quad (q_{11}, \dots, q_{1m}) \in \Gamma_m^0[1].$$

Thus there exists an additive function $a_x : \mathbb{R} \to \mathbb{R}$ such that

$$g\begin{pmatrix}q\\0\end{pmatrix} = a_x\left(\frac{x}{q}\right) + g\begin{pmatrix}x\\0\end{pmatrix} - \frac{a_x(1)}{n}, \quad q \in]0,x[,$$

where x is an arbitrary fixed element of $]0,1[$. This implies that there exist additive function $a_1 : \mathbb{R} \to \mathbb{R}$ and $c_1 \in \mathbb{R}$ such that

$$g\begin{pmatrix}x\\0\end{pmatrix} = a_1(x) + c_1, \quad x \in]0,1[.$$

In a similar way we get that there exist additive function $a_2 : \mathbb{R} \to \mathbb{R}$ and $c_2 \in \mathbb{R}$ such that

$$g\begin{pmatrix}0\\y\end{pmatrix} = a_2(y) + c_2, \quad y \in]0,1[.$$

With the substitutions $P = \begin{pmatrix} x & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18) we get that

$$g\begin{pmatrix}x\\1\end{pmatrix} = \frac{m-1}{m} a_1(x-1) + 1, \quad x \in]0,1[.$$

Similarly we have that

$$g\begin{pmatrix}1\\y\end{pmatrix} = \frac{m-1}{m} a_2(y-1) + 1, \quad y \in]0,1[.$$

With the substitutions $P = \begin{pmatrix} 0 & r & \dots & r\\ 0 & u & \dots & u\end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 0 & s & \dots & s\\ 0 & v & \dots & v\end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), after some calculation, we get that $(g(\underline{0}))^2 = g(\underline{0})$, so $g(\underline{0}) \in \{0,1\}$.

If $g(\underline{0}) = 0$ then, with the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), we get that $(g(\underline{1}))^2 = g(\underline{1})$, so $g(\underline{1}) \in \{0, 1\}$. Furthermore, with the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0, 1[$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), we get that

$$a_1(x) = 0, \quad x \in]0, 1[.$$

In a similar way we obtain that

$$a_2(y) = 0, \quad y \in]0, 1[.$$

With the substitutions
$P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} y & s & \dots & s \\ 0 & v & \dots & v \end{pmatrix} \in \Gamma_m^c[2]$, $x, y \in]0, 1[$,
$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$ $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$, and
$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$
in (3.18), after some calculation, we get that
$c_1 = 0$ (a similar calculation shows that $c_2 = 0$),
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^2$, that is, $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \{0, 1\}$,
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$, and
$g(\underline{1}) = 1$, respectively.
If $g(\underline{0}) = 1$ then, with the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0, 1[$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), after some calculation, we get that $c_1 = 1$. In a similar way we have that $c_2 = 1$. The substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18) imply that $g(\underline{1}) = 1$. With the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0, 1[$, $Q = \begin{pmatrix} y & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0, 1[$, in (3.18) we get that

$$\frac{1}{m^2} a_1(x) a_1(y) = a_1(x)\left(1 + \frac{a_1(1)}{m}\right) +$$

$$a_1(y)\left(\frac{n}{m}+\frac{a_1(1)}{m}\right)-\frac{a_1(xy)}{m}+a_1(1)(1-n-m-a_1(1))$$

From this, with $y=\frac{1}{2}$, after some calculation, we get that

$$a_1(x)=\frac{ma_1(1)}{a_1(1)+m}\left(n+a_1(1)+\frac{2m^2-1}{2m-1}\right). \tag{3.24}$$

Since a_1 is additive and the right hand side of (3.24) does not depend on x we have that

$$a_1(x)=0, \quad x\in]0,1[.$$

In a similar way, we have that

$$a_2(y)=0, \quad y\in]0,1[.$$

With the substitutions $P=\begin{pmatrix}0 & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that $g\begin{pmatrix}0\\1\end{pmatrix}=1$. In a similar way, we get that $g\begin{pmatrix}1\\0\end{pmatrix}$. Thus

$$g(x)=1, \quad x\in[0,1]^2.$$

Subcase 2.C. $g(x)=M(x)$, $x\in]0,1[^2$, where $M:]0,1[^2\to\mathbb{R}$ is a multiplicative function which is different from the following four functions: $\begin{pmatrix}x\\y\end{pmatrix}\to 0$, $\begin{pmatrix}x\\y\end{pmatrix}\to 1$, $\begin{pmatrix}x\\y\end{pmatrix}\to x$, $\begin{pmatrix}x\\y\end{pmatrix}\to y$, $\begin{pmatrix}x\\y\end{pmatrix}\in]0,1[^2$. It is easy to check that this condition implies that there does not exist $c\in\mathbb{R}$ such that $\sum_{j=1}^n M(q_j)=c$ for all $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$.

With the substitutions $P=\begin{pmatrix}0 & r & \dots & r\\ 0 & u & \dots & u\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that

$$g(\underline{0})\left(\sum_{j=1}^n M(q_j)-m\right)=0.$$

Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^n M(q_j^0)\neq m$ thus $g(\underline{0})=0$. With the substitutions $P=\begin{pmatrix}1 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that $(g(\underline{1})-1)\sum_{j=1}^n M(q_j)=0$. Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^n M(q_j^0)\neq 0$ thus $g(\underline{1})=1$. The substitutions $P=\begin{pmatrix}1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=\begin{pmatrix}1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix}1\\0\end{pmatrix}+g\begin{pmatrix}0\\1\end{pmatrix}$ $=\left(g\begin{pmatrix}1\\0\end{pmatrix}+g\begin{pmatrix}0\\1\end{pmatrix}\right)^2$, that is, $g\begin{pmatrix}1\\0\end{pmatrix}+g\begin{pmatrix}0\\1\end{pmatrix}\in\{0,1\}$. The following calculation shows that, if there exists $x_0 \in]0,1[$ such that $g\begin{pmatrix} x_0 \\ 0 \end{pmatrix} \neq 0$, then there exists a multiplicative function $\mu :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. The substitutions $P = \begin{pmatrix} x_0 & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x_0 \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18), imply that

$$\sum_{j=1}^{m} \left[g\begin{pmatrix} x_0 q_{j1} \\ 0 \end{pmatrix} - g\begin{pmatrix} x_0 \\ 0 \end{pmatrix} M(q_j) \right] = 0, \quad Q = (q_1, \dots, q_m) \in \Gamma_m^0[2].$$

Thus there exists an additive function $A_1 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{g\begin{pmatrix} x_0 \\ 0 \end{pmatrix}} A_1\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{g\begin{pmatrix} x_0 \\ 0 \end{pmatrix}} \left[g\begin{pmatrix} x_0 x \\ 0 \end{pmatrix} - \frac{A(\underline{1})}{m} \right].$$

Hence there exist an additive function $A : \mathbb{R}^2 \to \mathbb{R}$ and a function $H :]0,1[\to \mathbb{R}$ such that

$$M\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + H(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2.$$

Since the case $M\begin{pmatrix} x \\ y \end{pmatrix} = y$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$ is excluded, by Lemma 2.2, there exists multiplicative function $\mu :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. In a similar way we can prove that, if there exists $y_0 \in]0,1[$ such that $g\begin{pmatrix} 0 \\ y_0 \end{pmatrix} \neq 0$, then there exists a multiplicative function $\mu :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(y)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$.

Now we show that $g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0$, $x \in]0,1[$ or $g\begin{pmatrix} 0 \\ y \end{pmatrix} = 0$, $y \in]0,1[$. Indeed, suppose that there exist $x_0 \in]0,1[$ and $y_0 \in]0,1[$ such that $g\begin{pmatrix} x_0 \\ 0 \end{pmatrix} \neq 0$ and $g\begin{pmatrix} 0 \\ y_0 \end{pmatrix} \neq 0$. Then there exist multiplicative functions $\mu_1 :]0,1[\to \mathbb{R}$ and $\mu_2 :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu_1(x) = \mu_2(y)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. This implies that $M\begin{pmatrix} x \\ y \end{pmatrix} = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$ or $M\begin{pmatrix} x \\ y \end{pmatrix} = 1$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$, which are excluded in this case.

If $g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0$, $x \in]0,1[$ and $g\begin{pmatrix} 0 \\ y \end{pmatrix} = 0$, $y \in]0,1[$ then substitute $P = \begin{pmatrix} 0 & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18). Thus we get that

$$g\begin{pmatrix}0\\1\end{pmatrix}\sum_{j=1}^{m}M(q_j)=0.$$

Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^m M(q_j^0)\neq 0$ therefore $g\begin{pmatrix}0\\1\end{pmatrix}=0$. In a similar way we have that $g\begin{pmatrix}1\\0\end{pmatrix}=0$. Substituting $P=\begin{pmatrix}x & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that

$$\left(g\begin{pmatrix}x\\1\end{pmatrix}-M\begin{pmatrix}x\\1\end{pmatrix}\right)\sum_{j=1}^{m}M(q_j)=0.$$

Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^m M(q_j^0)\neq 0$ therefore

$$g\begin{pmatrix}x\\1\end{pmatrix}=M\begin{pmatrix}x\\1\end{pmatrix},\quad x\in]0,1[.$$

In a similar way we have that

$$g\begin{pmatrix}1\\y\end{pmatrix}=M\begin{pmatrix}1\\y\end{pmatrix},\quad y\in]0,1[.$$

If there exists $x_0\in]0,1[$ such that $g\begin{pmatrix}x_0\\0\end{pmatrix}\neq 0$ and $g\begin{pmatrix}0\\y\end{pmatrix}=0$, $y\in]0,1[$ then, by Lemma 2.2, there exists a multiplicative function $\mu:]0,1[\to\mathbb{R}$ such that $M\begin{pmatrix}x\\y\end{pmatrix}=\mu(x)$, $\begin{pmatrix}x\\y\end{pmatrix}\in]0,1[^2$. Substituting $P=\begin{pmatrix}x & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $x\in]0,1[$ and $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18), we get that

$$\left(g\begin{pmatrix}x\\1\end{pmatrix}-\mu(x)\right)\sum_{j=1}^{m}\mu(q_{j1})=0.$$

Since there exists $(q_{11}^0,\dots,q_{m1}^0)\in\Gamma_m^0[1]$ such that $\sum_{j=1}^m \mu(q_{j1}^0)\neq 0$ thus $g\begin{pmatrix}x\\1\end{pmatrix}=\mu(x)$, $x\in]0,1[$.

The substitutions $P=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & r & \dots & r\end{pmatrix}\in\Gamma_n^c[2]$, $Q=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & s & \dots & s\end{pmatrix}\in\Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix}1\\0\end{pmatrix}=\left(g\begin{pmatrix}1\\0\end{pmatrix}\right)^2$, that is, $g\begin{pmatrix}1\\0\end{pmatrix}\in\{0,1\}$.

If $g\begin{pmatrix}1\\0\end{pmatrix}=1$ then, with the substitutions $P=\begin{pmatrix}1 & 0 & \dots & 0\\ x & r & \dots & r\end{pmatrix}\in\Gamma_n^c[2]$, $x\in]0,1[$, $Q=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & s & \dots & s\end{pmatrix}\in\Gamma_m^c[2]$ in (3.18), we get that $g\begin{pmatrix}1\\x\end{pmatrix}=1$, $x\in]0,1[$.

With the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 0 & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$.

With the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = g\begin{pmatrix} x \\ 1 \end{pmatrix} = \mu(x), \quad x \in]0,1[.$$

If $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ then, with the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^c[2]$ in (3.18), we get that $\sum_{j=1}^m g\begin{pmatrix} q_{j1} \\ 0 \end{pmatrix} = 0$, $(q_{11}, \dots, q_{1m}) \in \Gamma_m^c[1]$. Thus there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that $g\begin{pmatrix} x \\ 0 \end{pmatrix} = a(x) - \frac{a(1)}{m}$, $x \in [0,1]$. Since $0 = g(\underline{0}) = -\frac{a(1)}{m}$ we have that $a(1) = 0$ and

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = a(x), \quad x \in [0,1].$$

With the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix} (a(x) + \mu(1-x) - 1) = 0.$$

Since the function a is additive, the function μ is multiplicative and different from the functions $x \to 0$, $x \to 1$, and $x \to x$, there exists $x_0 \in]0,1[$ such that $a(x_0) + \mu(1-x_0) \neq 0$ thus

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

With the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ y & 1-y & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$ in (3.18) we get that $a(x) = 0$, $x \in]0,1[$. Substituting $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ x & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} y_1 & y_2 & \dots & y_m \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $y_1, \dots, y_m \in]0,1[$, in (3.18) we get that

$$\left(g\begin{pmatrix} 1 \\ x \end{pmatrix} - 1\right) \sum_{j=1}^m \mu(y_j) = 0.$$

Since there exists $(y_1^0, \ldots, y_m^0) \in \Gamma_m^0[1]$ such that $\sum_{j=1}^m \mu(y_j^0) \neq 0$ therefore
$g\begin{pmatrix} 1 \\ x \end{pmatrix} = 1$, $x \in]0,1[$. □

References

[1] Aczél, J. Lectures on Functional Equations and Their Applications, *Academic Press*, New York – London, 1966.

[2] Ebanks, B. R., Sahoo, P. K., Sander, W., Characterizations of information measures, *World Scientific*, Singapore – New Jersey – London – Hong Kong, 1998.

[3] Losonczi, L., Maksa, Gy., On some functional equation of information theory, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 73-82.

Imre Kocsis
H-4028, Debrecen
Ótemető u. 2-4.
Hungary