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ELEMENTARY PROBLEMS WHICH ARE EQUIVALENT TO THE GOLDBACH'S CONJECTURE

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Abstract. We denote by $\{p_1=2,\ p_2=3,\ p_3=5,\ldots,\ p_k,\ldots\}$ the sequence of increasing primes, and for each positive integer $k\geq 1$ let

$$S(k):=\min\{2n>p_k\colon\ 2n-p_1,\ 2n-p_2,\ldots,\ 2n-p_k \text{ all are composite numbers}\}.$$

We prove that the following conjectures are equivalent to the Goldbach's conjecture.

Conjecture B. For every positive integer k, we have

$$S(k)\geq p_{k+1}+3.$$

Conjecture C. For every positive integer k, the number $S(k)$ is the sum of two odd primes.

1. Introduction

Goldbach wrote a letter to Euler in 1742 suggesting that every integer $n>5$ is the sum of three primes. Euler replied that this is equivalent to the following statement:

Conjecture A. *Every even integer $2n>4$ is the sum of two odd primes.*

This is now known as Goldbach's conjecture. A. Schinzel showed that Goldbach's conjecture is equivalent to every integer $n>17$ is the sum of three distinct primes. It has been proven that every even integer is the sum of at most six primes [2] (Goldbach suggests two) and in 1966 Chen proved every sufficiently large even integers is the sum of a prime plus a number with no more than two prime factors. In 1993 Sinisalo [5] verified Goldbach's conjecture for all integers less than $4\cdot 10^{11}$. More recently Jean-Marc Deshouillers, Yannick Saouter and Herman te Riele [1] have verified this up to 10^{14} with the help of a Cray C90 and various workstations. In July 1998, Joerg Richstein [4] completed a verification to $4\cdot 10^{14}$ and placed a list of champions online. See the monograf of P. Ribenboim [3] for more information.

In the following, we shall denote by $\mathcal{P}$ the set of all increasing primes, that is

$$\mathcal{P}:=\{p_1=2,\ p_2=3,\ p_3=5,\ldots,\ p_k,\ \ldots\}.$$

For each positive integer $k \geq 1$, let

$$\mathcal{A}_k := \{2n > p_k\colon\ 2n - p_1, 2n - p_2, \ldots, 2n - p_k \text{ all are composite numbers}\}.$$

Since $p_1 \cdots p_k \in\ \mathcal{A}_k \subseteq \mathbf{N}$, therefore $\mathcal{A}_k$ has a minimum element. Let

$$S(k) := \min \mathcal{A}_k.$$

We shall prove that the following conjectures are equivalent to Conjecture A.

Conjecture B. *For every positive integer k, we have*

$$S(k)\ \geq\ p_{k+1}\ +\ 3.$$

Conjecture C. *For every positive integer k, the number $S(k)$ is the sum of two odd primes.*

The purpose of this note is to prove the following

Theorem. *We have*

(a) *Every even integer $2n\ >\ 4$ is the sum of two odd primes if and only if*

$$S(k)\ \geq\ p_{k+1}\ +\ 3. \tag{1}$$

holds for every positive integer k.

(b) *Every even integer $2n\ >\ 4$ is the sum of two odd primes if and only if the number $S(k)$ is the sum of two odd primes for all positive integers k.*

In the other words, Conjectures A, B and C are equivalent.

2. Lemmas

In the following we denote by G the set of all even positive integers which are the sums of two odd primes. Goldbach's conjecture states that G contains all even integers $2n \geq 6$.

Lemma 1. *We have*

$$\{\ 2n\colon\ 6\ \leq 2n\ \leq p_k + 3\ \}\ \subset\ G \quad \textit{if and only if} \quad \{2n\colon\ 6\ \leq 2n\ < S(k)\}\ \subset\ G.$$

Proof. It follows from the definition of $S(k)$ that $S(k)\ \geq p_k + 9$, consequently

$$\{2n\colon\ 6\ \leq 2n\ \leq p_k + 3\}\ \subset\ G \quad \text{if} \quad \{2n\colon\ 6\ \leq 2n\ < S(k)\}\ \subset\ G.$$

Now assume that $\{2n\colon 6 \le 2n \le p_k+3\} \subset G$. Let $2N$ be an even integer with $6 \le 2N < S(k)$. If $2N \le p_k+3$, then we have $2N \in G$ by our assumption. Let $p_k+3 < 2N < S(k)$. Hence

$$2N-p_1 > 2N-p_2 > \cdots > 2N-p_k > 3.$$

On the other hand, the conditions $2N < S(k)$ and $S(k) = \min \mathcal{A}_k$ yield

$$2N \notin \mathcal{A}_k.$$

Since

$$\mathcal{A}_k = \{2n > p_k\colon 2n-p_1,\ 2n-p_2,\ \ldots,\ 2n-p_k \text{ all are composite numbers}\},$$

the last relations imply that

$$2N-p_i \quad \text{is a prime for some} \quad p_i \in \{p_1,\ p_2,\ p_3, \ldots,\ p_k\}.$$

Consequently, $2N \in G$, and so Lemma 1 is proved.

Lemma 2. *Let k be a positive integer. Then*

$$\{2n\colon S(k) \le 2n < S(k+1)\} \subset G \quad \textit{if and only if} \quad S(k) \ge p_{k+1}+3.$$

Proof. Assume that $S(k) \ne S(k+1)$ and $\{2n\colon S(k) \le 2n < S(k+1)\} \subset G$. Then we have $S(k) = p+q$ for for some primes p and q. Since the numbers $S(k)-p$ and $S(k)-q$ are primes, we infer from the definition of $S(k)$ that $p > p_k$ and $q > p_k$. Consequently, $S(k) = p+q \ge 2p_k+4 \ge p_{k+1}+3$.

Now assume that $S(k) \ne S(k+1)$ and $S(k) > p_{k+1}+3$. Let $2N$ be an even integer for which $S(k) \le 2N < S(k+1)$ is satisfied. As we have seen in the proof of Lemma 1, in this case we also have $2N \notin \mathcal{A}_{k+1}$ and

$$2N-p_1 > 2N-p_2 > \ldots > 2N-p_k > 2N-p_{k+1} \ge S(k)-p_{k+1} > 3.$$

Consequently,

$$2N-p_i \quad \text{is a prime for some} \quad p_i \in \{p_1,\ p_2,\ p_3, \cdots,\ p_k,\ p_{k+1}\},$$

which shows that $2N \in G$.

Finally, in the case $S(k) = S(k+1)$ we also have that $S(k) = S(k+1) \ge p_{k+1}+9 > p_{k+1}+1$ by the definition of $S(k+1)$.

The proof of Lemma 2 is finished.

3. Proof of the theorem

Proof of (a). Assume that every even integer $2n > 4$ is the sum of two odd primes. In this case we infer from Lemma 2 that $S(k) \geq p_{k+1}+3$. Thus, Conjecture A implies Conjecture B.

Now we assume that Conjecture B is true, that is (1) holds for every positive integer k. Hence, Lemma 2 shows that

$$\{2n\colon 6 \leq 2n < S(k+1)\} \subset G \tag{2}$$

holds for all positive integers k.

Finally, let $2n > 4$ be any even integer. It is clear to see from the definition of $S(k)$ that $S(k) > p_k$. Hence

$$S(k) \to \infty \quad \text{as} \quad k \to \infty.$$

Consequently, $S(\ell) > 2n$ is true for some positive integer ℓ, and so we get from (2) that $2n \in G$. The proof of the the part (a) of the theorem is completed.

Proof of (b). It is obvious that Conjecture C is a consequence of Conjecture A.

Assume now that the conjecture C is true, that is, for each positive integer k, we have $S(k) = p + q$ for for some primes p and q. Since the numbers $S_k - p$ and $S(k) - q$ are primes, we also have $p > p_k$ and $q > p_k$. Consequently,

$$S(k) = p + q > 2p_k \geq p_{k+1} + 1,$$

and so Conjecture B is true. This with (a) completes the proof of (b). The assertion (b) is proved.

The proof of the theorem is finished.

References

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