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GENERALIZED FIBONACCI-TYPE NUMBERS AS MATRIX DETERMINANTS

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Abstract. In this note we construct such matrix determinants of complex entries which are equal to the numbers defined by Fibonacci-type linear recursions of order $k\geq 2$.

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1. Introduction

Let $k \geq 2$ be an integer. The recursive sequence $\{G_n\}_{n=2-k}^{\infty}$ of order k is defined for every $n \geq 2$ by the recursion

$$G_n = p_1 G_{n-1} + p_2 G_{n-2} + \cdots + p_k G_{n-k}, \tag{1}$$

where p_i $(1 \leq i \leq k)$ and G_j $(2-k \leq j \leq 1)$ are given complex numbers and $p_1 p_k G_1$ is not equal to zero. For brevity, we will use the formula

$$G_n = G_n\big(p_1, p_2, \ldots, p_k, G_{2-k}, G_{3-k}, \ldots, G_1\big),$$

as well. In the case $k = 2$ we get the wellknown family of second order linear recurrences of complex numbers. The two most important sequences from this family are the Fibonacci $\{F_n\}$ and the Lucas $\{L_n\}$ sequences, where

$$F_n = G_n(1,1,0,1) \text{ and } L_n = G_n(1,1,2,1),$$

respectively.

The close connections between the Fibinacci (and Lucas) numbers and suitable matrix determinants have been known for ages. For example, it is known that for $k \geq 1$ F_k is equal to the following tridiagonal matrix determinant of $k \times k$:

$$F_k = \det \begin{pmatrix} 1 & i & & & & \\ i & 1 & i & & & \\ & i & 1 & i & & \\ & & i & 1 & \ddots & \\ & & & \ddots & \ddots & i \\ & & & & i & 1 \end{pmatrix}.$$

Recently, some papers have been publicated in this field. (For more information about the list of these papers see [1].) One of the latest such papers was written by Nathan D. Cahill and Darren A. Narayan [1]. They have constructed such family of tridiagonal matrix determinants of $k \times k$ which generate any arbitrary linear subsequence

$$F_{\alpha k+\beta} \text{ or } L_{\alpha k+\beta} \ (k = 1, 2, \ldots)$$

of the Fibonacci or Lucas numbers. For example,

$$F_{4k-2} = \det \begin{pmatrix} 1 & 0 & & & & \\ 0 & 8 & 1 & & & \\ & 1 & 7 & 1 & & \\ & & 1 & 7 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 7 \end{pmatrix}.$$

The aim of this note is to investigate suitable matrix determinants of $n \times n$ which form the terms G_n of the Fibonacci-type sequences defined by (1). In this paper we suppose that in (1) $p_1 \neq 0, p_j = 0$ $(2 \leq j \leq k-1$ for $3 \leq k), p_k = \pm 1,$ and $G_1 \neq 0$, that is we deal with the family of sequences

$$G_n = G_n\big(p_1, 0, \ldots, 0, \pm 1, G_{2-k}, G_{3-k}, \ldots, G_1\big). \tag{2}$$

(Naturally, the sign $\pm$ in (2) is fixed in a given sequence.)

For our aim we construct the matrix $\mathbf{A}_{n\times n} = (a_{t,j})$ of complex numbers by the following forms: $a_{1,1} = G_1$, $a_{1,j} = -e^{j+1}G_{j-k}$ $(2 \leq j \leq k)$, $a_{j+1,j} = -e^3$ $(1 \leq j \leq n-1)$, $a_{j,k+j-1} = -e^{k+1}$ $(2 \leq j \leq n+1-k), a_{j,j} = p_1$ $(2 \leq j \leq n)$ and the other entries are equal to 0. That is,

$$\mathbf{A}_{n\times n} \tag{3}$$

$$= \begin{pmatrix} G_1 & -e^3G_{2-k} & -e^4G_{3-k} & \cdots & -e^{k+1}G_0 & 0 & 0 & \cdots & 0 & 0 \\ -e^3 & p_1 & 0 & \cdots & 0 & -e^{k+1} & 0 & \cdots & 0 & 0 \\ 0 & -e^3 & p_1 & \cdots & 0 & 0 & -e^{k+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -e^3 & p_1 \end{pmatrix}$$

where $e = -1$ if $p_k = -1$ and $e = -i$ if $p_k = 1$.

2. Result

We shall prove the following theorem.

Theorem. *Let the squence* $\{G_n\}_{n=2-k}^{\infty}$ *be defined by (2), where* $p_1G_1 \neq 0, p_k = \pm 1$ *and* $k \geq 2$. *Let the matrix* $\mathbf{A}_{n\times n}$ *be defined by (3). Then for every* $n \geq 1$

$$G_n = \det(\mathbf{A}_{n\times n}).$$

Remark. In the case $k = 2$ our matrices $\mathbf{A}_{n\times n}$ are of tridiagonal ones.

Proof. First we consider the case $1 \leq n \leq k$. Then, for $n = 1$

$$\det(\mathbf{A}_{1\times 1}) = G_1.$$

If $n = 2$ or 3, then

$$\det\begin{pmatrix} G_1 & -e^3G_{2-k} \\ -e^3 & p_1 \end{pmatrix} = p_1G_1 - e^6G_{2-k}$$

$$= p_1G_1 + p_kG_{2-k} = G_2$$

and

$$\det\begin{pmatrix} G_1 & -e^3G_{2-k} & -e^4G_{3-k} \\ -e^3 & p_1 & 0 \\ 0 & -e^3 & p_1 \end{pmatrix}$$

$$= p_1G_2 - e^4G_{3-k}e^6 = p_1G_2 - e^2G_{3-k} = p_1G_2 + p_kG_{3-k} = G_3.$$

Suppose that $G_{n-j} = \det(\mathbf{A}_{n-j\times n-j})$ $(j = 1, 2, 3)$ holds for an integer n, where $4 \leq n < k$. Then, developing the determinant

$$\det(\mathbf{A}_{n\times n}) = \det\begin{pmatrix} G_1 & -e^3G_{2-k} & -e^4G_{3-k} & \cdots & -e^nG_{n-1-k} & -e^{n+1}G_{n-k} \\ -e^3 & p_1 & 0 & \cdots & 0 & 0 \\ 0 & -e^3 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -e^3 & p_1 \end{pmatrix}$$

with respect to the last column, we have

$$\det(\mathbf{A}_{n\times n}) = p_1G_{n-1} - (-1)^{n+1}e^{n+1}G_{n-k}\left(-e^3\right)^{n-1}$$

$$= p_1G_{n-1} + (-1)^{2n+1}e^{4n-2}G_{n-k} = p_1G_{n-1} + p_kG_{n-k} = G_n.$$

That is, our theorem holds for every n, if $1 \leq n \leq k$.

Now, we shall deal with the case $n > k$. If $n = k + 1$ then

$$\det\left(\mathbf{A}_{k+1\times k+1}\right) = \det\begin{pmatrix} G_1 & -e^3G_{2-k} & -e^4G_{3-k} & \cdots & -e^{k+1}G_0 & 0 \\ -e^3 & p_1 & 0 & \cdots & 0 & -e^{k+1} \\ 0 & -e^3 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -e^3 & p_1 \end{pmatrix}$$

$$= p_1G_k + e^3\det\begin{pmatrix} G_1 & -e^3G_{2-k} & -e^4G_{3-k} & \cdots & -e^kG_{-1} & 0 \\ -e^3 & p_1 & 0 & \cdots & 0 & -e^{k+1} \\ 0 & -e^3 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -e^3 & 0 \end{pmatrix}.$$

Developing successively the resulting determinants with respect to their last rows, we have

$$\det\left(\mathbf{A}_{n\times n}\right) = p_1G_k + \left(e^3\right)^{k-1}\det\begin{pmatrix} G_1 & 0 \\ -e^3 & -e^{k+1} \end{pmatrix}$$

$$= p_1G_k - e^{3k-3}e^{k+1}G_1 = p_1G_k + p_kG_1 = G_{k+1}.$$

Let us suppose that $\det\left(\mathbf{A}_{n-j\times n-j}\right) = G_{n-j} \quad (1 \le j \le k)$ holds for an integer $n \ge k+2$. In this case

$$\det\left(\mathbf{A}_{n\times n}\right)$$

$$= \det\begin{pmatrix} G_1 & -e^3G_{2-k} & \cdots & -e^{k+1}G_0 & 0 & 0 & \cdots & 0 & 0 \\ -e^3 & p_1 & \cdots & 0 & -e^{k+1} & 0 & \cdots & 0 & 0 \\ 0 & -e^3 & \cdots & 0 & 0 & -e^{k+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -e^3 & p_1 \end{pmatrix}$$

$$= p_1G_{n-1} + e^3\det\begin{pmatrix} G_1 & -e^3G_{2-k} & \cdots & -e^{k+1}G_0 & 0 & \cdots & 0 & 0 \\ -e^3 & p_1 & \cdots & 0 & -e^{k+1} & \cdots & 0 & 0 \\ 0 & -e^3 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -e^3 & 0 \end{pmatrix}.$$

Now, develop successively the resulting determinants with respect to their last rows. Then one can get the following equalities:

$$\det\left(\mathbf{A}_{n\times n}\right) = p_1G_{n-1} + \left(e^3\right)^{k-1}\left(-e^{k+1}\right)G_{n-k}$$

$$= p_1G_{n-1} - e^2G_{n-k} = p_1G_{n-1} + p_kG_{n-k} = G_n.$$

This completes the proof of the Theorem.

Reference

[1] Cahill, N. D., Narayan, D. A., Fibonacci and Lucas Numbers as Tridiagonal Matrix Determinant, *The Fibonacci Quarterly* **42** (2004), 216–221.

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