

# A characterization of the identity function

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**Abstract.** We prove that if a multiplicative function  $f$  satisfies the equation  $f(n^2 + m^2 + 3) = f(n^2 + 1) + f(m^2 + 2)$  for all positive integers  $n$  and  $m$ , then either  $f(n)$  is the identity function or  $f(n^2 + m^2 + 3) = f(n^2 + 1) = f(m^2 + 2) = 0$  for all positive integers.

Throughout this paper  $\mathbf{N}$  denotes the set of positive integers and let  $\mathcal{M}$  be the set of complex valued multiplicative functions  $f$  such that  $f(1) = 1$ .

In 1992, C. Spiro [3] showed that if  $f \in \mathcal{M}$  is a function such that  $f(p + q) = f(p) + f(q)$  for all primes  $p$  and  $q$ , then  $f(n) = n$  for all  $n \in \mathbf{N}$ . Recently, in the paper [2] written jointly with J. M. de Koninck and I. Kátai we proved that if  $f \in \mathcal{M}$ ,  $f(p + n^2) = f(p) + f(n^2)$  holds for all primes  $p$  and  $n \in \mathbf{N}$ , then  $f(n)$  is the identity function. It follows from results of [1] that a completely multiplicative function  $f$  satisfies the equation  $f(n^2 + m^2) = f(n^2) + f(m^2)$  for all  $n, m \in \mathbf{N}$  if and only if  $f(2) = 2$ ,  $f(p) = p$  for all primes  $p \equiv 1 \pmod{4}$  and  $f(q) = q$  or  $f(q) = -q$  for all primes  $p \equiv 3 \pmod{4}$ .

The purpose of this note is to prove the following

**Theorem.** Assume that  $f \in \mathcal{M}$  satisfies the condition

$$(1) \quad f(n^2 + m^2 + 3) = f(n^2 + 1) + f(m^2 + 2)$$

for all  $n, m \in \mathbf{N}$ . Then either

$$(2) \quad f(n^2 + 1) = f(m^2 + 2) = f(n^2 + m^2 + 3) = 0 \quad \text{for all } n, m \in \mathbf{N},$$

or  $f(n) = n$  for all  $n \in \mathbf{N}$ .

**Corollary.** If  $f \in \mathcal{M}$  satisfies the condition (1) and  $f(n_0^2 + 1) \neq 0$  for some  $n_0 \in \mathbf{N}$ , then  $f(n)$  is the identity function.

First we prove the following lemma.

**Lemma.** Assume that the conditions of Theorem 1 are satisfied. Then either (2) is satisfied for all  $n \in \mathbf{N}$  or the conditions

$$(3) \quad \begin{aligned} f(n^2 + 1) &= n^2 + 1, & f(m^2 + 2) &= m^2 + 2 & \text{and} \\ f(n^2 + m^2 + 3) &= n^2 + m^2 + 3 \end{aligned}$$

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simultaneously hold for all  $n, m \in \mathbf{N}$ .

**Proof.** From (1), we have

$$f(n^2 + 1) + f(m^2 + 2) = f(m^2 + 1) + f(n^2 + 2)$$

for all  $n, m \in \mathbf{N}$ , and so

$$(4) \quad f(n^2 + 2) - f(n^2 + 1) = f(3) - f(2) := D \quad \text{for all } n \in \mathbf{N}.$$

Thus, the last relation together with (1) implies that

$$(5) \quad f(n^2 + m^2 + 3) = f(n^2 + 1) + f(m^2 + 1) + D$$

holds for all  $n, m \in \mathbf{N}$ . Let  $S_j := f(j^2 + 1)$ . It follows from (5) that if  $k, l, u$  and  $v \in \mathbf{N}$  satisfy the condition

$$k^2 + l^2 = u^2 + v^2,$$

then

$$f(k^2 + 1) + f(l^2 + 1) + D = f(u^2 + 1) + f(v^2 + 1) + D,$$

which shows that

$$(6) \quad k^2 + l^2 = u^2 + v^2 \quad \text{implies} \quad S_k + S_l = S_u + S_v.$$

We shall prove that

$$(7) \quad S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n$$

holds for all  $n \in \mathbf{N}$ .

Since

$$(2j + 1)^2 + (j - 2)^2 = (2j - 1)^2 + (j + 2)^2$$

and

$$(2j + 1)^2 + (j - 7)^2 = (2j - 5)^2 + (j + 5)^2,$$

we get from (6) that

$$(8) \quad S_{2j+1} + S_{j-2} = S_{2j-1} + S_{j+2}$$

and

$$S_{2j+1} + S_{j-7} = S_{2j-5} + S_{j+5}.$$

These with (8) imply that

$$\begin{aligned} S_{j+5} - S_{j+2} + S_{j-2} - S_{j-7} &= S_{2j-1} - S_{2j-5} \\ &= S_{j+1} - S_{j-3} + S_{2j-3} - S_{2j-5} = S_{j+1} - S_{j-3} + S_j - S_{j-4}, \end{aligned}$$

which proves (7) with  $n = j - 7$ .

By (8), we have

$$S_7 = S_{2 \cdot 3+1} = 2S_5 - S_1,$$

$$S_9 = S_{2 \cdot 4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2 \cdot 5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Finally, by using (6) and the facts

$$8^2 + 1^2 = 7^2 + 4^2, \quad 10^2 + 5^2 = 11^2 + 2^2 \quad \text{and} \quad 12^2 + 1^2 = 9^2 + 8^2,$$

we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1.$$

Thus, to complete the proof of the lemma, by using (1), (4), (5) and (7), it is enough to prove that either  $S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = 0$  or

$$(9) \quad S_j = j^2 + 1 \quad \text{for} \quad j = 1, 2, 3, 4, 5, 6.$$

Repeated use of (1), using the multiplicativity of  $f$ , gives  $S_1 = f(1^2 + 1) = f(2)$ ,

$$(10) \quad S_2 = f(2^2 + 1) = f(5) = f(1^2 + 1^2 + 3) = f(2) + f(3),$$

$$(11) \quad S_3 = f(3^2 + 1) = f(10) = f(2)f(5) = f(2)^2 + f(2)f(3).$$

and thus

$$f(11) = f(2^2 + 2^2 + 3) = f(5) + f(6) = f(2) + f(3) + f(2)f(3).$$

On the other hand, it follows from (4) that

$$\begin{aligned} f(11) &= f(3^2 + 2) = f(10) + D = f(2)f(5) + D \\ &= f(2)^2 + f(2)f(3) + f(3) - f(2), \end{aligned}$$

which, together with the last relation, implies

$$(12) \quad f(2)^2 = 2f(2),$$

and

$$f(13) = f(1^2 + 3^2 + 3) = f(2) + f(11) = 2f(2) + f(2)f(3) + f(3).$$

Finally, the relation (10) together with the fact

$$f(8) = f(1^2 + 2^2 + 3) = f(2) + f(6) = f(5) + f(3)$$

show that

$$(13) \quad f(2)f(3) = 2f(3).$$

Moreover

$$(14) \quad S_5 = f(5^2 + 1) = f(26) = f(2)f(13) = 4f(2) + 6f(3),$$

$$(15) \quad \begin{aligned} S_6 &= f(6^2 + 1) = f(37) = f(3^2 + 5^2 + 3) \\ &= f(11) + f(26) = 5f(2) + 9f(3), \end{aligned}$$

$$(16) \quad 2f(17) = f(4^2 + 4^2 + 3) - D = f(35) - D = f(5)f(7) - D,$$

and

$$(17) \quad f(3)f(7) = f(21) = f(3^2 + 3^2 + 3) = 2f(10) + D = 3f(2) + 5f(3).$$

The equation (12) shows that either  $f(2) = 0$  or  $f(2) = 2$ . Assume that  $f(2) = 0$ . Then (13) implies that  $f(3) = 0$  and so, by using (10)–(17) we have

$$S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = 0,$$

from which follows that (2) is true.

Assume now that  $f(2) = 2$ . In this case we have  $f(5) = 2 + f(3)$ ,  $f(8) = 2 + 2f(3)$ . We shall prove that  $f(3) = 3$ . It follows from (15) and using the fact

$$f(37) = f(1^2 + 6^2 + 3) - f(3) = f(5)f(8) - f(3) = 2f(3)^2 + 5f(3) + 4$$

that

$$(17) \quad 2f(3)^2 - 4f(3) - 6 = 0.$$

On the other hand, from (4) we infer that

$$f(6)f(11) - f(3)f(13) = f(66) - f(65) = f(3) - f(2),$$

consequently

$$3f(3)^2 - 7f(3) - 6 = 0.$$

This together with (17) proves that  $f(3) = 3$ , and so (10)–(17) imply that

$$S_j = j^2 + 1 \quad (j = 1, 2, 3, 4, 5, 6).$$

This completes the proof of (9) and so the lemma is proved.

### Proof of the theorem

In the proof of the theorem, using the lemma, we can assume that (3) is satisfied, that is

$$(18) \quad \begin{aligned} f(n^2 + 1) &= n^2 + 1, \quad f(m^2 + 2) = m^2 + 2 \quad \text{and} \\ f(n^2 + m^2 + 3) &= n^2 + m^2 + 3. \end{aligned}$$

It is clear from (18) that  $f(n) = n$  for all  $n \leq 7$ .

Assume that  $f(n) = n$  for all  $n < T$ , where  $T > 7$ . We shall prove that  $f(T) = T$ . It is clear that  $T$  must be a prime power, that is  $T = q^\alpha$  with  $\alpha \in \mathbf{N}$  and some prime  $q$ .

It is easily seen that if  $\alpha = 1$ , then  $q > 7$  and there are positive integers  $n, m \leq \frac{q-1}{2}$  such that  $n^2 + m^2 + 3 = qN$ ,  $(q, N) = 1$  and  $N < q$ . Thus, we have  $f(q) = q$ .

Assume now that  $\alpha \geq 2$  and  $q > 3$ . We consider the congruence

$$n^2 + m^2 + 3 \equiv 0 \pmod{q^\alpha}.$$

Let

$$\mathcal{A}_q(3) := \left\{ 1 \leq m \leq q-1 : \left( \frac{-m^2-3}{q} \right) = 1 \right\}.$$

Then we have

$$\begin{aligned}
\sharp \mathcal{A}_q(3) &= \sum_{\substack{m=1 \\ (m^2+3, q)=1}}^{q-1} \frac{1}{2} \left( 1 + \left( \frac{-m^2-3}{q} \right) \right) = \sum_{m=0}^{q-1} \frac{1}{2} \left( 1 + \left( \frac{-m^2-3}{q} \right) \right) \\
&\quad - \sum_{\substack{m=0 \\ q|m^2+3}}^{q-1} \frac{1}{2} \left( 1 + \left( \frac{-m^2-3}{q} \right) \right) - \frac{1}{2} \left( 1 + \left( \frac{-3}{q} \right) \right) \\
&= \frac{1}{2} \left( q - \left( \frac{-1}{q} \right) - 2 - 2 \left( \frac{-3}{q} \right) \right).
\end{aligned}$$

By our assumption, the last relation implies that  $\sharp \mathcal{A}_q(3) \geq 1$ . Thus, there are integers  $m \in \{1, \dots, q-1\}$ ,  $1 \leq n_1 \leq q^\alpha - 1$ ,  $(n_1, q) = 1$  and  $1 \leq n_2 := q^\alpha - n_1 \leq q^\alpha - 1$  such that

$$n_i^2 + m^2 + 3 = q^\alpha N_i \quad (i = 1, 2).$$

It follows from the above relations that

$$q^\alpha(N_2 - N_1) = (q^\alpha - n_1)^2 - n_1^2 = q^{2\alpha} - 2q^\alpha n_1,$$

that is

$$N_2 - N_1 = q^\alpha - 2n_1.$$

Since  $(n_1, q) = 1$ , we obtain that at least one of  $N_1$  or  $N_2$  is coprime to  $q$ . Let  $n \in \{n_1, n_2\}$  and  $N \in \{N_1, N_2\}$  such that  $n^2 + m^2 + 3 = q^\alpha N$ ,  $(N, q) = 1$ . Then  $\alpha \geq 2$  implies that

$$N \leq \frac{1}{q^\alpha} \left[ (q^\alpha - 1)^2 + (q - 1)^2 + 3 \right] < q^\alpha.$$

Thus,

$$Nf(q^\alpha) = f(N)f(q^\alpha) = f(Nq^\alpha) = f(n^2 + m^2 + 3) = n^2 + m^2 + 3 = Nq^\alpha,$$

which shows that  $f(q^\alpha) = q^\alpha$  as we wanted to establish.

To complete the proof of the theorem, it remains to consider the cases  $q = 2$  and  $q = 3$ . Let  $q = 2$  and  $T = 2^\alpha$ , where  $\alpha \geq 3$ .

Since  $-7 \equiv 1 \pmod{8}$ , we have  $-7$  is a quadratic residue modulo  $2^\alpha$  and therefore there exists  $n_\alpha \in [0, 2^{\alpha-1} - 1]$  such that  $n_\alpha^2 + 7 = n_\alpha^2 + 2^2 + 3 \equiv 0 \pmod{2^\alpha}$ , and consequently,  $[n_\alpha + 2^{\alpha-1}]^2 + 7 \equiv 0 \pmod{2^\alpha}$ . Define  $N_1$ ,

and  $N_2$  by  $n_\alpha^2 + 7 = 2^\alpha N_1$  and  $[n_\alpha + 2^{\alpha-1}]^2 + 7 = 2^\alpha N_2$ . We easily deduce from these two equations and the fact  $7 < T = 2^\alpha$  that

$$N_1 < 2^\alpha, \quad N_2 < 2^\alpha \quad \text{and} \quad N_2 - N_1 = n_\alpha + 2^{\alpha-2}.$$

It follows from the last relation and the fact 2 does not divide  $n_\alpha$  that one of  $N_1$  or  $N_2$  is odd, and so  $f(2^\alpha) = 2^\alpha$ .

Finally, let  $q = 3$  and  $T = 3^\alpha$ , where  $\alpha > 1$ . We consider the congruence

$$n^2 + 2 \equiv 0 \pmod{3^\alpha}.$$

Similarly as above, one can deduce that there are positive integers  $n, N \in \mathbf{N}$  such that  $n^2 + 2 = 3^\alpha N$ ,  $(N, 3) = 1$  and  $N < 3^\alpha$ . Thus these together with (18) implies that  $f(3^\alpha) = 3^\alpha$ .

The theorem is proved.

### References

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