Pure powers in recurrence sequences

KÁLMÁN LIPTAI* and TIBOR TÓMÁCS

Abstract. Let G be a linear recursive sequence of order k satisfying the recursion $G_n=A_1G_{n-1}+\cdots+A_kG_{n-k}$. In the case $k=2$ it is known that there are only finitely many perfect powers in such a sequence.

Ribenboim and McDaniel proved for sequences with $k=2$, $G_0=0$ and $G_1=1$ that in general for a term G_n there are only finitely many terms G_m such that G_nG_m is a perfect square. P. Kiss proved that for any n there exists a number q_0, depending on G and n, such that the equation $G_nG_x=w^q$ in positive integers x,w,q has no solution with $x>n$ and $q>q_0$. We show that for any n there are only finitely many $x_1,x_2,\ldots,x_k,x,w,q$ positive integers such that $G_nG_{x_1}\cdots G_{x_k}G_x=w^q$ and some conditions hold.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1),$$

where A, B, R_0 and R_1 are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. α/β is not a root of unity, where α and β denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1, 1, 0, 1)$ and $R(2, 1, 0, 1)$ of the sequence R is called Fibonacci and Pell sequence, respectively.

Many results are known about relationship of the sequences R and perfect powers. For the Fibonacci sequence Cohn [2] and Wylie [23] showed that a Fibonacci number F_n is a square only when $n = 0, 1, 2$ or 12. Pethő [12], furthermore London and Finkelstein [9,10] proved that F_n is full cube only if $n = 0, 1, 2$ or 6. From a result of Ljunggren [8] it follows that a Pell number is a square only if $n = 0, 1$ or 7 and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results was showed by McDaniel and Ribenboim [11], Robbins [19,20] Cohn [3,4,5] and Pethő [15]. Shorey and Stewart [21] showed, that any non degenerate binary recurrence sequence contains only finitely many perfect powers which can be effictively determined. This results follows also from a result of Pethő [14].

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16975 and 020295.

Another type of problems was studied by Ribenboim and McDaniel. For a sequence R we say that the terms R_m, R_n are in the same square-class if there exist non zero integers x, y such that

$$R_m x^2 = R_n y^2,$$

or equivalently

$$R_m R_n = t^2,$$

where t is a positive rational integer.

A square-class is called trivial if it contains only one element. Ribenboim [16] proved that in the Fibonacci sequence the square-class of a Fibonacci number F_m is trivial, if $m \neq 1, 2, 3, 6$ or 12 and for the Lucas sequence $L(1, 1, 2, 1)$ the square-class of a Lucas number L_m is trivial if $m \neq 0, 1, 3$ or 6. For more general sequences $R(A, B, 0, 1)$, with $(A, B) = 1$, Ribenboim and McDaniel [17] obtained that each square class is finite and its elements can be effectively computed (see also Ribenboim [18]).

Further on we shall study more general recursive sequences.

Let $G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ be a k^{th} order linear recursive sequence of rational integers defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \qquad (n > k-1),$$

where $A_1, \ldots, A_k$ and $G_0, \ldots, G_{k-1}$ are not all zero integers. Denote by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ the distinct zeros of the polynomial $x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k$. Assume that $\alpha, \alpha_2, \ldots, \alpha_s$ has multiplicity $1, m_2, \ldots, m_s$ respectively and $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, s$. In this case, as it is known, the terms of the sequence can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \cdots + r_s(n)\alpha_s^n \qquad (n \geq 0), \tag{1}$$

where $r_i (i = 2, \ldots, s)$ are polynomials of degree $m_i - 1$ and the coefficients of the polynomials and a are elements of the algebraic number field $\mathbf{Q}(\alpha, \alpha_2, \ldots, \alpha_s)$. Shorey and Stewart [21] prowed that the sequence G does not contain q^{th} powers if q is large enough. This result follows also from [7] and [22], where more general theorems where showed.

Kiss [6] generalized the square-class notion of Ribenboim and McDaniel. For a sequence G we say that the terms G_m and G_n are in the same q^{th}-power class if $G_m G_n = w^q$, where w, q rational integers and $q \geq 2$.

In the above mentioned paper Kiss proved that for any term G_n of the sequence G there is no terms G_m such that $m > n$ and G_n, G_m are elements of the same q^{th}-power class if q sufficiently large.

The purpose of this paper to generalize this result. We show that the under certain conditions the number of the solutions of equation

$$G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q$$

where n is fixed, are finite.

We use a well known result of Baker [1].

Lemma. *Let $\gamma_1, \ldots, \gamma_v$ be non-zero algebraic numbers. Let $M_1, \ldots, M_v$ be upper bounds for the heights of $\gamma_1, \ldots, \gamma_v$, respectively. We assume that M_v is at least 4. Further let $b_1, \ldots, b_{v-1}$ be rational integers with absolute values at most B and let b_v be a non-zero rational integer with absolute value at most B'. We assume that B' is at least three. Let L defined by*

$$L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v,$$

where the logarithms are assumed to have their principal values. If $L \neq 0$, then

$$|L| > \exp(-C(\log B' \log M_v + B/B')),$$

where C is an effectively computable positive number depending on only the numbers $M_1, \ldots, M_{v-1}$, $\gamma_1, \ldots, \gamma_v$ and v (see Theorem 1 of [1] with $\delta = 1/B'$).

Theorem. *Let G be a k^{th} order linear recursive sequence satisfying the above conditions. Assume that $a \neq 0$ and $G_i \neq a\alpha^i$ for $i > n_0$. Then for any positive integer n, k and K there exists a number q_0, depending on n, G, K and k, such that the equation*

$$G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q \qquad (n \leq x_1 \leq \cdots \leq x_k < x) \tag{2}$$

in positive integer $x_1, x_2, \ldots, x_k, x, w, q$ has no solution with $x_k < Kn$ and $q > q_0$.

Proof of the theorem. We can assume, without loss of generality, that the terms of the sequence G are positive. We can also suppose that $n > n_0$ and n sufficiently large since otherwise our result follows from [20] and [7].

Let $x_1, x_2, \ldots, x_k, x, w, q$ positive integers satisfying (2) with the above conditions. Let ε_m be defined by

$$\varepsilon_m := \frac{1}{a} r_2(m) \Big(\frac{\alpha_2}{\alpha}\Big)^m + \frac{1}{a} r_3(m) \Big(\frac{\alpha_3}{\alpha}\Big)^m + \cdots + \frac{1}{a} r_s(m) \Big(\frac{\alpha_s}{\alpha}\Big)^m \qquad (m \geq 0).$$

By (1) we have

$$(1+\varepsilon_n)(1+\varepsilon_x)\prod_{i=1}^{k}(1+\varepsilon_{x_i})\,a^{k+2}\alpha^{n+x+x_1+\cdots+x_k}=w^q$$

from which

$$\begin{aligned}q\log w=(k+2)\log a+\left(n+x+\sum_{i=1}^{k}x_i\right)\log\alpha+\log(1+\varepsilon_n)\\+\log(1+\varepsilon_x)+\sum_{i=1}^{k}\log(1+\varepsilon_{x_i})\end{aligned}\tag{3}$$

follows. It is obvious that $x<n+x+\sum_{i=1}^{k}x_i<(k+2)x$. Using that $\log|1+\varepsilon_m|$ is bounded and $\lim_{m\to\infty}\frac{1}{a}r_i(m)\left(\frac{\alpha_i}{\alpha}\right)^m=0\quad(i=2,\ldots,s)$, we have

$$c_1\frac{x}{q}<\log w<c_2\frac{x}{q}\tag{4}$$

where c_1 and c_2 are constants.

Let L be defined by

$$L:=\left|\log\frac{w^q}{G_nG_{x_1}G_{x_2}\cdots G_{x_k}a\alpha^x}\right|=\left|\log(1+\varepsilon_x)\right|.$$

By the definition of ε_x and the properties of logarithm function there exists a constant c_3 that

$$L<e^{-c_3x}.\tag{5}$$

On the other hand, by the Lemma with $v=k+4, M_{k+4}=w, B'=q$ and $B=x$ we obtain the estimation

$$L=\left|q\log w-\log G_n-\sum_{i=1}^{k}\log G_{x_i}-\log a-x\log\alpha\right|>e^{-C(\log q\log w+x/q)}\tag{6}$$

where C depends on heights. By $x_k<Kn$ heights depend on $G_n,\ldots,G_{Kn}$, i.e. on n,K,k and on the parameters of the recurrence. By (4), (5) and (6) we have $c_3x<C(\log q\log w+x/q)<c_4\log q\log w$, i.e.

$$x<c_5\log q\log w\tag{7}$$

with some c_3, c_4, c_5. Using (4) and (7) we get $c_6 q \log w < x < c_5 \log q \log w$, i.e. $q < c_7 \log q$, where c_6 and c_7 are constants. But this inequality does not hold if $q > q_0 = q_0(G, n, K, k)$, which proves the theorem.

References

[1] A. BAKER, A sharpening of the bounds for linear forms in logarithms II, *Acta Arithm.* **24** (1973), 33–36.

[2] J. H. E. COHN, On square Fibonacci numbers, *J. London Math. Soc.* **39** (1964), 537–540.

[3] J. H. E. COHN, Squares in some recurrent sequences, *Pacific J. Math.* **41** (1972), 631–646.

[4] J. H. E. COHN, Eight Diophantine equations, *Proc. London Math. Soc.* **16** (1966), 153–166.

[5] J. H. E. COHN, Five Diophatine equations, *Math. Scand.* **21** (1967), 61–70.

[6] P. KISS, Pure powers and power classes in recurrence sequences, (to appear).

[7] P. KISS, Differences of the terms of linear recurrences, *Studia Sci. Math. Hungar.* **20** (1985), 285–293.

[8] W. LJUNGGREN, Zur Theorie der Gleichung $x^2 + 1 = Dy^4$, *Avh. Norske Vid Akad. Oslo* **5** (1942).

[9] J. LONDON and R. FINKELSTEIN, On Fibonacci and Lucas numbers which are perfect powers, *Fibonacci Quart.* **7** (1969) 476–481, 487, errata ibid **8** (1970) 248.

[10] J. LONDON and R. FINKELSTEIN, On Mordell's equation $y^2 - k = x^3$, *Bowling Green University Press* (1973).

[11] W. L. MCDANIEL and P. RIBENBOIM, Squares and double-squares in Lucas sequences, *C. R. Math. Acad. Sci. Soc. R. Canada* **14** (1992), 104–108.

[12] A. PETHŐ, Full cubes in the Fibonacci sequence, *Publ. Math. Debrecen* **30** (1983), 117–127.

[13] A. Pethő, The Pell sequence contains only trivial perfect powers, *Coll. Math. Soc. J. Bolyai, 60 sets, Graphs and Numbers,* Budapest, (1991), 561–568.

[14] A. Pethő, Perfect powers in second order linear recurrences, *J. Number Theory* **15** (1982), 5–13.

[15] A. Pethő, Perfect powers in second order recurrences, *Topics in Classical Number Theory,* Akadémiai Kiadó, Budapest, (1981), 1217–1227.

[16] P. Ribenboim, Square classes of Fibonacci and Lucas numbers, *Portugaliae Math.* **46** (1989), 159–175.

[17] P. Ribenboim and W. L. McDaniel, Square classes of Fibonacci and Lucas sequences, *Portugaliae Math.*, **48** (1991), 469–473.

[18] P. Ribenboim, Square classes of $(a^n-1)/(a-1)$ and a^n+1, *Sichuan Daxue Xunebar* **26** (1989), 196–199.

[19] N. Robbins, On Fibonacci numbers of the form px^2, where p is prime, *Fibonacci Quart.* **21** (1983), 266–271.

[20] N. Robbins, On Pell numbers of the form PX^2, where P is prime, *Fibonacci Quart.* **22** (1984), 340–348.

[21] T. N. Shorey and C. L. Stewart, On the Diophantine equation $ax^{2t}+bx^ty+cy^2=d$ and pure powers in recurrence sequences, *Math. Scand.* **52** (1983), 24–36.

[22] T. N. Shorey and C. L. Stewart, Pure powers in recurrence sequences and some related Diophatine equations, *J. Number Theory* **27** (1987), 324–352.

[23] O. Wylie, In the Fibonacci series $F_1=1, F_2=1, F_{n+1}=F_n+F_{n-1}$ the first, second and twelvth terms are squares, *Amer. Math. Monthly* **71** (1964), 220–222.

Kálmán Liptai and Tibor Tómács
Eszterházy Károly Teachers' Training College
Department of Mathematics
Leányka u. 4.
3301 Eger, Pf. 43.
Hungary
E-mail: liptaik@gemini.ektf.hu
tomacs@gemini.ektf.hu