

General solution of the differential equation $y''(x) - (y'(x))^2 + x^2 e^{y(x)} = 0$

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Abstract. In this note we prove that the general solution of the differential equation $y''(x) - (y'(x))^2 + x^2 e^{y(x)} = 0$, $x > 0$ is the function $y(x) = -\ln W(x)$, where $W(x) = \frac{1}{12}x^4 + Ax + B$ and A, B are arbitrary constants.

1. Introduction

In this note we prove that the general solution of the differential equation

$$(1) \quad y''(x) - (y'(x))^2 + x^2 e^{y(x)} = 0, \quad x > 0$$

is the function

$$(2) \quad y(x) = -\ln W(x), \quad \text{where } W(x) = \frac{1}{12}x^4 + Ax + B$$

and A, B are arbitrary constants. First, we note that such type of differential equations as (1) are difficult to solve. For example, E. Y. RODIN (see [1], p. 474, Unsolved problems, SIAM 81-17) posed the following problem. Find the general solution of the differential equation:

$$(3) \quad y''(x) + x^2 e^{y(x)} = 0, \quad x > 0.$$

We prove that (1) has general solution given by (2), however we can't find the general solution of (3).

2. The Result

We prove the following theorem:

Theorem. *The general solution of the differential equation (1) is the function*

$$y(x) = \ln \left(\frac{1}{12}x^4 + Ax + B \right)$$

where A, B are arbitrary constants.

Proof. Putting $y(x) = \ln z(x)$ we obtain

$$(4) \quad y'(x) = \frac{z'(x)}{z(x)}$$

and consequently we have

$$(5) \quad y''(x) = \frac{z''(x)}{z(x)} - \left(\frac{z'(x)}{z(x)} \right)^2.$$

Since $y(x) = \ln z(x)$, then $e^{y(x)} = z(x)$ and by (4) and (5) it follows that (1) can be reduced to the following form:

$$(6) \quad \frac{z''(x)}{z^2(x)} - 2 \frac{(z'(x))^2}{z^3(x)} = -x^2.$$

Integrating (6) with respect to x we obtain:

$$\int \left(\frac{z''(x)}{z^2(x)} - 2 \frac{(z'(x))^2}{z^3(x)} \right) dx = -\frac{1}{3}x^3 + C_1.$$

Denote by

$$(7) \quad f(z(x)) = \frac{z''(x)}{z^2(x)} - 2 \frac{(z'(x))^2}{z^3(x)}.$$

Then we see that the function

$$(8) \quad \frac{z'(x)}{z^2(x)} = F(z(x))$$

satisfies the following condition

$$F'(z(x)) = \left(\frac{z'(x)}{z^2(x)} \right)' = \frac{z''(x)}{z^2(x)} - 2 \frac{(z'(x))^2}{z^3(x)} = f(z(x))$$

and therefore by (7) and (8) it follows that

$$(9) \quad \frac{z'(x)}{z^2(x)} = -\frac{1}{3}x^3 + C_1.$$

Integrating the last equality with respect to x we obtain

$$(10) \quad \int \frac{z'(x)}{z^2(x)} dx = -\frac{1}{12}x^4 + C_1x + C_2.$$

On the other hand it easy to see that

$$(11) \quad \left(-\frac{1}{z(x)} \right)' = \frac{z'(x)}{z^2(x)}$$

and consequently by (10) and (11) it follows that

$$-\frac{1}{z(x)} = -\frac{1}{12}x^4 + C_1x + C_2$$

and we have

$$y(x) = \ln z(x) = -\ln \left(\frac{1}{12}x^4 + Ax + B \right) = -\ln W(x)$$

where $A = -C_1$, $B = -C_2$. The proof of the theorem is complete.

References

- [1] S. RABINOWITZ, Index to Mathematical Problems, Westford, Massachusetts, USA, 1992.

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