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Approximation In Weighted Generalized Grand Lebesgue Spaces*

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Abstract

In the present paper the best approximation of the functions in generalized grand Lebesgue spaces have been investigated. We study the inverse problem of approximation theory in generalized grand Lebesgue spaces.

1 Introduction

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$, a function ω is called a *weight* on $\mathbb{T}$ if $\omega : \mathbb{T} \to [0, \infty]$ is measurable and $\omega^{-1}(\{0, \infty\})$ has measure zero (with respect to Lebesgue measure). Let ω be a 2π periodic weight function. We denote by $L^p_\omega(\mathbb{T})$, $1 < p < \infty$, the weighted Lebesgue space of all measurable functions on $\mathbb{T}$ for which

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, \omega dx \right)^{1/p} < \infty.$$

We define a class $L^{p),\theta}_\omega(\mathbb{T})$, $\theta > 0$, of 2π periodic measurable functions on $\mathbb{T}$ satisfying the condition

$$\sup_{0<\varepsilon<p-1} \left\{ \frac{\varepsilon^\theta}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)} < \infty.$$

The class $L^{p),\theta}_\omega(\mathbb{T})$, $\theta > 0$, is a Banach space with respect to the norm

$$\|f\|_{L^{p),\theta}_\omega(\mathbb{T})} := \sup_{0<\varepsilon<p-1} \left\{ \varepsilon^\theta \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)} . \tag{1}$$

The class $L^{p),\theta}_\omega(\mathbb{T})$ with the norm (1) is called the weighted generalized grand Lebesgue space. Note that non- weighted grand Lebesgue space $L^{p)}(\mathbb{T})$ was introduced by Iwaniec

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and Sbordone [11]. Information about properties of these spaces can be found in [9], [11], [12] and [20]. The embeddings

$$L^{p}(\mathbb{T}) \subset L^{p)}(\mathbb{T}) \subset L^{p-\varepsilon}$$

hold. According to [9] $L^{p}(\mathbb{T})$ is not dense in $L^{p)}(\mathbb{T})$. Also, if $\theta_1 < \theta_2$ and $1 < p < \infty$, for weighted generalized grand Lebesgue space, the following relations hold:

$$L^{p}_{\omega}(\mathbb{T}) \subset L^{p),\theta_1}_{\omega}(\mathbb{T}) \subset L^{p),\theta_2}_{\omega}(\mathbb{T}) \subset L^{p-\varepsilon}_{\omega}(\mathbb{T})$$

The closure of the space $L^{p}(\mathbb{T})$ is denoted by $L^{p),\theta}_{\omega}(\mathbb{T})$ with the norm $\mathcal{L}^{p;\theta}_{\omega}(\mathbb{T})$, $\theta > 0$.

Let $1 < p < \infty$ and let $A_p(\mathbb{T})$ be the collection of all weights on $\mathbb{T}$, satisfying the condition

$$\sup_{I} \left(\frac{1}{|I|} \int_{I} \omega(x)^{p} dx \right)^{1/p} \left(\frac{1}{|I|} \int_{I} [\omega(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty, \tag{2}$$

where the supremum is taken over all intervals I with length $|I| \le 2\pi$. The condition (2) is called the *Muckenhoupt* -A_p condition [22] and the weight functions which belong to $A_p(\mathbb{T})$, $(1 < p < \infty)$, are called the *Muckenhoupt weights.*

For $f \in L^{p),\theta}_{\omega}$, $\theta > 0$, $1 < p < \infty$, we set

$$\Delta^{r}_{t} f(x) := \sum_{k=0}^{r} (-1)^{r+k+1} \binom{r}{k} f(x+kt),\ t > 0,$$

where $r \in \mathbb{N}$. Let $1 < p < \infty$ and let $\omega \in A_p(\mathbb{T})$. For $f \in \mathcal{L}^{p),\theta}_{\omega}(\mathbb{T})$, $\theta > 0$ we define the *rth mean modulus* $\Omega_r(f,\cdot)_{p),\theta,\omega} : [0,\infty) \longrightarrow [0,\infty)$ of f by

$$\Omega_r(f,\cdot)_{p),\theta,\omega} = \sup_{|h| \le \delta} \|\sigma^{r}_{h} f(x)\|_{L^{p),\theta}_{\omega}(\mathbb{T})}\ \delta > 0,$$

where

$$\sigma^{r}_{h}(f)(x) := \frac{1}{h} \int_{0}^{h} |\Delta^{r}_{t} f(x)|\, dt.$$

Note that if $\omega \in A_p(\mathbb{T})$ and $0 < \delta < \infty$, then according to [20] we get

$$\sup_{|h| \le \delta} \|\sigma^{r}_{h}(f)(x)\|_{L^{p),\theta}_{\omega}(\mathbb{T})} \le c\, \|f\|_{L^{p),\theta}_{\omega}(\mathbb{T})} < \infty.$$

The modulus of continuity $\Omega(f,\cdot)_{p),\theta,\omega}$ is a non-decreasing, nonnegative function of $\delta > 0$ and

$$\lim_{\delta \to 0} \Omega_r(f,\cdot)_{p),\theta,\omega} = 0, \quad \Omega_r(f+g,\cdot)_{p),\theta,\omega} \le \Omega(f,\cdot)_{p),\theta,\omega} + \Omega(g,\cdot)_{p),\theta,\omega},$$

for $f, g \in \mathcal{L}^{p),\theta}_{\omega}(\mathbb{T}), \theta > 0$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f) \tag{3}$$

be the Fourier series of the function $f \in L_1(T)$, where $A_k(x,f) := a_k(f)\cos kx + b_k(f)\sin kx$, $\alpha_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(T)$. The *n-th partial sums* of the series (3) is defined by

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,f),\ n = 1,2,...$$

The best approximation of $f \in \mathcal{L}_\omega^{p),\theta}$, $\theta > 0$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{p),\theta,\omega} := \inf\left\{\|f - T_n\|_{L_\omega^{p),\theta}(\mathbb{T})} : T_n \in \Pi_n\right\}.$$

We use the constants $c, c_1, c_2, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.We need the following theorem:

THEOREM 1.1 ([15]). Let $1 < p < \infty$, $\theta > 0$, $\omega \in A_p(\mathbb{T})$ and $r \in \mathbb{N}$. Then, for $f \in \mathcal{L}_\omega^{p),\theta}$, $\theta > 0$ we have

$$\Omega_r\left(f, \frac{1}{n}\right)_{p),\theta,\omega} \leq \frac{c_1}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_{p),\theta,\omega} \tag{4}$$

with a constant $c_1 > 0$ independent of n.

2 Main Results

The problems of approximation theory in grand Lebesgue spaces have been investigated by several authors (see, for example, [3–7, 15, 18]). In the present paper, we investigate the problem of the best approximation for functions of a subspace of the weighted generalized grand Lebesgue spaces. We prove an inverse theorem of approximation theory in weighted generalized Lebesgue spaces. Similar problems in different spaces have been investigated in [1, 2, 8, 10, 13, 14, 16, 17, 21–28]. Our main results are the following.

THEOREM 2.1. Let $1 < p < \infty$, $\theta > 0$, $\omega \in A_p(\mathbb{T})$ and $r \in \mathbb{N}$. Let $f \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \tag{5}$$

be its Fourier series and let

$$\sum_{n=1}^{\infty} E_n(f)_{p),\theta,\omega} n^{\alpha-1} < \infty, \tag{6}$$

where $\alpha \in \mathbb{R}$. Then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} n^\alpha (a_n\cos nx + b_n\sin nx) \tag{7}$$

is the Fourier series of the function $\widetilde{f} \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ and for every $\widetilde{f} \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ the estimates

$$\begin{aligned} & E_n(\widetilde{f})_{p),\theta,\omega} \\ \leq \; & c_2 \left[E_n(f)_{p),\theta,\omega} n^{\alpha} + \sum_{k=n+1}^{\infty} E_k(f)_{p),\theta,\omega} k^{\alpha-1} \right], n = 1, 2, ..., \end{aligned} \tag{8}$$

$$E_0(\widetilde{f})_{p),\theta,\omega} \leq c_3 \left[E_0(f)_{p),\theta,\omega} + \sum_{k=1}^{\infty} E_k(f)_{p),\theta,\omega} k^{\alpha-1} \right] \tag{9}$$

hold with the constants c_2, $c_3 > 0$, nondependent on f and n.

COROLLARY 2.2. Under the conditions of Theorem 1.1 the estimate

$$\Omega_r \left(\widetilde{f}, \delta \right)_{p),\theta,\omega} \leq c_4 \left\{ \frac{1}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_{\nu}(f)_{p),\theta,\omega} + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \right\}$$

holds with a constant $c_4 > 0$, depending on α and r.

Note that similar estimates in the different spaces for modulus of continuity were proved in [2, 16, 19].

3 Proofs of the Main Results

Proof of Theorem 2.1. Let s_n and $\widetilde{s}_n$ be the nth partial sums of (5) and (7), respectively and let $\mu_n = n^{\alpha}$ $(n = 1, 2, ...)$. By Abel transform

$$\widetilde{s}_m - f = \sum_{i=1}^{m-1} (s_i - f) \Delta\mu_i + (s_m - f) \quad m = 1, 2, ...,$$

where $\Delta\mu_i = \mu_i - \mu_{i+1}$. It is clear that $|\Delta\mu_i| \leq c i^{\alpha-1}$. Then for a fixed $n = 1, 2, ...$ and for every $k = 0, 1, ...$ we obtain

$$\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n} = \sum_{i=2^k n}^{2^{k+1}n-1} (s_i - f) \Delta\mu_i + (s_{2^{k+1}n} - f)\mu_{2^{k+1}n} - (s_{2^k n} - f)\mu_{2^k n} \; . \tag{10}$$

According to [27] the inequality

$$\| f - s_n \|_{L_{\omega}^{p),\theta}(\mathbb{T})} \leq c_5 E_n(f)_{p),\theta,\omega} \tag{11}$$

holds. Then from (10) and (11) we get

$$\begin{aligned} \left\| \widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n} \right\|_{L_{\omega}^{p),\theta}(\mathbb{T})} & \leq c_6 \sum_{l=2^k n}^{2^{k+1}n-1} E_l(f)_{p),\theta,\omega} l^{\alpha-1} + c_7 E_{2^k n}(f)_{p),\theta,\omega} (2^k n)^{\alpha} \\ & \leq c_8 2^k n E_{2^k n}(f)_{p),\theta,\omega} (2^k n)^{\alpha-1} + c_9 (2^k n)^{\alpha} E_{2^k n}(f)_{p),\theta,\omega} \\ & = c_{10} (2^k n)^{\alpha} E_{2^k n}(f)_{p),\theta,\omega} \; . \end{aligned}$$

The last inequality yields

$$\sum_{k=0}^{\infty} \left\| \widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n} \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c_{11} \sum_{k=0}^{\infty} (2^k n)^\alpha \ E_{2^k n}(f)_{p),\theta,\omega}. \tag{12}$$

On the other hand the following inequality holds [8, p.209] :

$$\sum_{k=1}^{\infty} (2^k n)^\alpha E_{2^k n}(f)_{p),\theta,\omega} \leq c_{12} \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p),\theta,\omega}. \tag{13}$$

Using (12) and (13) we have

$$\sum_{k=0}^{\infty} \left\| \widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n} \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c_{13} \left[E_n(f)_{p),\theta,\omega} n^\alpha + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p),\theta,\omega} \right]. \tag{14}$$

By (6), it follows that the series

$$\widetilde{s}_n + \sum_{k=o}^{\infty} (\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n})$$

converges in the sense of the metric $L_\omega^{p),\theta}$ to some function $\widetilde{f} \in L_\omega^{p),\theta}$. It is clear that the series (7) is the Fourier series of the function $\widetilde{f}$. We can write the following inequality

$$E_n(\widetilde{f})_{p),\theta,\omega} \leq \left\| \widetilde{f} - \widetilde{s}_n \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq \sum_{k=0}^{\infty} \left\| \widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n} \right\|_{L_\omega^{p),\theta}(\mathbb{T})}.$$

Now combining (14) and last relation we obtain the inequality (8) of Theorem 2.1.

Now, we estimate $E_0(\widetilde{f})_{p),\theta,\omega}$. The inequality

$$E_n(\widetilde{f})_{p),\theta,\omega}. \leq \left\| \widetilde{f} - \frac{a_0}{2} \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq \left\| \widetilde{f} - \widetilde{s}_1 \right\|_{L_\omega^{p),\theta}(\mathbb{T})} + \left\| \widetilde{s}_1 - \frac{a_0}{2} \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \tag{15}$$

holds. From (11) and (8) we have

$$\begin{aligned} \left\| \widetilde{f} - \widetilde{s}_1 \right\|_{L_\omega^{p),\theta}(\mathbb{T})} &\leq c_{14} E_1(\widetilde{f})_{p),\theta,\omega} \\ &\leq c_{15} \left[E_1(f)_{p),\theta,\omega} + \sum_{k=2}^{\infty} E_k(f)_{p),\theta,\omega} k^{\alpha-1} \right]. \end{aligned} \tag{16}$$

It is known that

$$\left\| \widetilde{s}_1 - \frac{a_0}{2} \right\|_{L_\omega^{p),\theta}(\mathbb{T})} = \left\| a_1 \cos x + b_1 \sin x \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq 2\pi \left(|a_1| + |b_1| \right). \tag{17}$$

We choose the number t_0, such that $\left\| \widetilde{f} - t_0 \right\|_{L_\omega^{p),\theta}(\mathbb{T})} = E_0(\widetilde{f})_{p),\theta,\omega}$. Then we obtain

$$\begin{aligned} \pi |a_1| &= \left| \int_0^{2\pi} \widetilde{f}(x) \cos x dx \right| = \left| \int_0^{2\pi} \left[\widetilde{f}(x) - t_0 \right] \cos x dx \right| \\ &\leq c_{16} \left\| \widetilde{f} - t_0 \right\|_{L_\omega^{p),\theta}(\mathbb{T})} = c_{16} E_0(\widetilde{f})_{p),\theta,\omega}. \end{aligned}$$

The last inequality yields

$$|a_1| \leq \frac{c_{16}}{\pi} E_0(\widetilde{f})_{p),\theta,\omega} \ . \tag{18}$$

Similar to the above, we obtain

$$|b_1| \leq \frac{c_{17}}{\pi} E_0(\widetilde{f})_{p),\theta,\omega} \ . \tag{19}$$

Using (15), (16)-(19), we obtain the inequality (9) of Theorem 2.1.

Proof of Corollary 2.2. Taking the relations (4), (8) and (9) into account we get

$$\begin{aligned}
& \Omega_r \left(\widetilde{f}, \delta\right)_{p),\theta,\omega} \\
\leq \quad & \frac{c_{18}}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r-1} E_\nu(\widetilde{f})_{p),\theta,\omega} \\
= \quad & \frac{c_{19}}{n^r} \left\{ E_0(\widetilde{f})_{p),\theta,\omega} + \sum_{\nu=1}^{n} (\nu+1)^{r-1} E_\nu(\widetilde{f})_{p),\theta,\omega} \right\} \\
\leq \quad & \frac{c_{20}}{n^r} \left[E_0(\widetilde{f})_{p),\theta,\omega} + \sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p),\theta,\omega} \right] \\
& + \frac{c_{21}}{n^r} \sum_{\nu=1}^{n} (\nu+1)^{r-1} \left[E_\nu(f)_{p),\theta,\omega} + \sum_{s=\nu}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \right] \\
\leq \quad & \frac{c_{22}}{n^r} \left[\sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p),\theta,\omega} + \sum_{\nu=1}^{n} (\nu+1)^{2r-1} \sum_{s=\nu}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \right] \\
\leq \quad & \frac{c_{23}}{n^r} \left[\sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p),\theta,\omega} \right] \\
& + \frac{c_{24}}{n^r} \sum_{\nu=1}^{n} (\nu+1)^{r-1} \left[\sum_{s=\nu}^{n} s^{\alpha-1} E_s(f)_{p),\theta,\omega} + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \right] \\
\leq \quad & c_{25} \left\{ \frac{1}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p),\theta,\omega} + \frac{1}{n^{2r}} \sum_{s=1}^{n} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \sum_{\nu=1}^{s} \nu^{\alpha-1} \right\} \\
& + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \\
\leq \quad & c_{26} \left\{ \frac{1}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p),\theta,\omega} + \frac{1}{n^{2r}} \sum_{s=1}^{n} s^{r+\alpha-1} E_s(f)_{p),\theta,\omega} \right\} \\
& + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \\
\leq \quad & c_{27} \left\{ \frac{1}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p),\theta,\omega} + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{p),\theta,\omega} \right\},
\end{aligned}$$

which finishes the proof.

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