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Third-Order BVP With Advanced Arguments And Stieltjes Integral Boundary Conditions*

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Abstract

A class of third-order boundary value problem with advanced arguments and Stieltjes integral boundary conditions is discussed. Some existence criteria of at least three positive solutions are established. The main tool used is a fixed point theorem due to Avery and Peterson.

1 Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [6].

Recently, third-order boundary value problems (BVPs for short) with integral boundary conditions, which cover third-order multi-point BVPs as special cases, have attracted much attention from many authors, see [1, 3, 4, 5, 9, 10, 11] and the references therein. In particular, in 2012, Jankowski [9] studied the existence of multiple positive solutions to the following BVP

$$\begin{cases} u'''(t) + h(t)f(t, u(\alpha(t))) = 0, \ t \in (0, 1), \\ u(0) = u''(0) = 0, \ u(1) = \beta u(\eta) + \lambda[u], \end{cases} \tag{1}$$

where λ denoted a linear functional on $C[0, 1]$ given by

$$\lambda[u] = \int_0^1 u(t) d\Lambda(t) \tag{2}$$

involving a Stieltjes integral with a suitable function Λ of bounded variation. The measure $d\Lambda$ could be a signed one. The situation with a signed measure $d\Lambda$ was first

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discussed in [12, 13] for second-order differential equations; it was also discussed in [7, 8] for second-order impulsive differential equations.

Among the boundary conditions in (1), only $u(1)$ is related to a Stieltjes integral. A natural question is that whether we can obtain similar results when $u(0)$ is also related to a Stieltjes integral. To answer this question, in this paper, we are concerned with the following third-order BVP with advanced arguments and Stieltjes integral boundary conditions

$$\begin{cases} u'''(t) + f(t, u(\alpha(t))) = 0, \ t \in (0,1), \\ u(0) = \gamma u(\eta) + \lambda[u], \ u''(0) = 0, \ u(1) = \beta u(\eta) + \lambda[u]. \end{cases} \tag{3}$$

Throughout this paper, we always assume that $\alpha : [0,1] \to [0,1]$ is continuous and $\alpha(t) \geq t$ for $t \in [0,1]$, $0 < \eta < 1$, $0 \leq \gamma < \beta < 1$, Λ is a suitable function of bounded variation and $\lambda[u]$ is defined as in (2). It is important to indicate that it is not assumed that $\lambda[u]$ is positive to all positive u.

In order to obtain our main results, we need the following concepts and Avery and Peterson fixed point theorem [2].

Let E be a real Banach space and K be a cone in E.

A map Θ is said to be a nonnegative continuous convex functional on K if $\Theta : K \to [0, \infty)$ is continuous and

$$\Theta(tu + (1-t)v) \leq t\Theta(u) + (1-t)\Theta(v)$$

for all $u, v \in K$ and $t \in [0,1]$.

Similarly, A map Φ is said to be a nonnegative continuous concave functional on K if $\Phi : K \to [0, \infty)$ is continuous and

$$\Phi(tu + (1-t)v) \geq t\Phi(u) + (1-t)\Phi(v)$$

for all $u, v \in K$ and $t \in [0,1]$.

Let φ and Θ be nonnegative continuous convex functionals on K, Φ be a nonnegative continuous concave functional on K and Ψ be a nonnegative continuous functional on K. For positive numbers a, b, c, d, we define the following sets:

$$K(\varphi, d) = \{u \in K : \varphi(u) < d\},$$

$$K(\varphi, \Phi, b, d) = \{u \in K : b \leq \Phi(u), \varphi(u) \leq d\},$$

$$K(\varphi, \Theta, \Phi, b, c, d) = \{u \in K : b \leq \Phi(u), \ \Theta(u) \leq c, \varphi(u) \leq d\}$$

and

$$R(\varphi, \Psi, a, d) = \{u \in K : a \leq \Psi(u), \varphi(u) \leq d\}.$$

THEOREM 1 (Avery and Peterson fixed point theorem). Let E be a real Banach space and K be a cone in E. Let φ and Θ be nonnegative continuous convex functionals on K, Φ be a nonnegative continuous concave functional on K, and Ψ be a nonnegative continuous functional on K satisfying $\Psi(ku) \leq k\Psi(u)$ for $0 \leq k \leq 1$, such that for some positive numbers M and d,

$$\Phi(u) \leq \Psi(u) \text{ and } \|u\| \leq M\varphi(u)$$

for all $u \in \overline{K(\varphi, d)}$. Suppose $S : \overline{K(\varphi, d)} \to \overline{K(\varphi, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

(C1) $\{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\} \neq \phi$ and $\Phi(Su) > b$ for $u \in K(\varphi, \Theta, \Phi, b, c, d)$;

(C2) $\Phi(Su) > b$ for $u \in K(\varphi, \Phi, b, d)$ with $\Theta(Su) > c$; and

(C3) $\theta \notin R(\varphi, \Psi, a, d)$ and $\Psi(Su) < a$ for $u \in R(\varphi, \Psi, a, d)$ with $\Psi(u) = a$.

Then S has at least three fixed points $u_1,\ u_2,\ u_3 \in \overline{K(\varphi, d)}$, such that

$$b < \Phi(u_1),$$

$$a < \Psi(u_2) \text{ with } \Phi(u_2) < b$$

and

$$\Psi(u_3) < a.$$

2 Main Results

Let $\Delta = 1 - \gamma - (\beta - \gamma)\eta$. Then $\Delta > 0$.

LEMMA 1. For any $y \in C[0, 1]$, the BVP

$$\begin{cases} u'''(t) = -y(t),\ t \in (0, 1), \\ u(0) = \gamma u(\eta) + \lambda[u],\ u''(0) = 0,\ u(1) = \beta u(\eta) + \lambda[u] \end{cases} \tag{4}$$

has the unique solution

$$\begin{aligned} u(t) \quad = \quad & \frac{1 - (\beta - \gamma)\eta + (\beta - \gamma)t}{\Delta}\lambda[u] + \frac{\gamma + (\beta - \gamma)t}{\Delta}\int_0^1 k(\eta, s)y(s)ds \\ & + \int_0^1 k(t, s)y(s)ds \end{aligned}$$

for $t \in [0, 1]$ where

$$k(t, s) = \frac{1}{2}\begin{cases} (1 - t)(t - s^2),\ 0 \le s \le t \le 1, \\ t(1 - s)^2,\ 0 \le t \le s \le 1. \end{cases}$$

PROOF. By integrating the differential equation in (4) three times from 0 to t and using the boundary condition $u''(0) = 0$, we get

$$u(t) = u(0) + u'(0)t - \frac{1}{2}\int_0^t (t - s)^2 y(s)ds,\ t \in [0, 1]. \tag{5}$$

So,

$$u'(0) = u(1) - u(0) + \frac{1}{2}\int_0^1 (1 - s)^2 y(s)ds. \tag{6}$$

In view of (5), (6) and the boundary conditions $u(0) = \gamma u(\eta) + \lambda[u]$ and $u(1) = \beta u(\eta) + \lambda[u]$, we have

$$u(t) = [\gamma + (\beta - \gamma)t]u(\eta) + \lambda[u] + \int_0^1 k(t,s)y(s)ds, \ t \in [0,1]. \tag{7}$$

So,

$$u(\eta) = \frac{1}{\Delta}\lambda[u] + \frac{1}{\Delta}\int_0^1 k(\eta,s)y(s)ds. \tag{8}$$

Substituting (8) into (7), we get

$$\begin{aligned} u(t) &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] + \frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)y(s)ds \\ &\quad + \int_0^1 k(t,s)y(s)ds \end{aligned}$$

for $t \in [0,1]$.

LEMMA 2 [9]. $0 \le k(t,s) \le \frac{1}{2}(1+s)(1-s)^2$ for $(t,s) \in [0,1]\times[0,1]$.

Throughout, we assume that the following conditions are fulfilled:

(H1) $f \in C([0,1]\times[0,+\infty), \ [0,+\infty))$;

(H2)

$$\int_0^1 d\Lambda(t) \ge 0, \ \int_0^1 t d\Lambda(t) \ge 0, \ \kappa(s) = \int_0^1 k(t,s)d\Lambda(t) \ge 0, \ s \in [0,1].$$

For convenience, we denote

$$\rho = [1-(\beta-\gamma)\eta]\int_0^1 d\Lambda(t) + (\beta-\gamma)\int_0^1 t d\Lambda(t)$$

and

$$\rho' = \gamma\int_0^1 d\Lambda(t) + (\beta-\gamma)\int_0^1 t d\Lambda(t).$$

Obviously, $\rho, \ \rho' \ge 0$. In the remainder of this paper, we always assume that $\rho < \Delta$.

Let $C[0,1]$ be equipped with the maximum norm. Then $C[0,1]$ is a Banach space. Define

$$K = \left\{u \in C[0,1] : u(t) \ge 0, \ t \in [0,1], \ \min_{t\in[\eta,1]} u(t) \ge \Gamma\|u\|, \ \lambda[u] \ge 0\right\},$$

where

$$\Gamma = \min\left\{\frac{\beta(1-\eta)}{1-\beta\eta}, \ \frac{\beta\eta}{1-\gamma(1-\eta)}\right\}.$$

Then K is a cone in $C[0,1]$.

Now, we define operators T and S on K by

$$(Tu)(t) = \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] + (Fu)(t),\ t\in[0,1]$$

and

$$(Su)(t) = \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu] + (Fu)(t),\ t\in[0,1],$$

where

$$(Fu)(t) = \frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds + \int_0^1 k(t,s)f(s,u(\alpha(s)))ds$$

for $t \in [0,1]$.

LEMMA 3. $T,\ S : K \to K$.

PROOF. Let $u \in K$. Then it is easy to verify that

$$(Tu)''(t) = -\int_0^t f(s,u(\alpha(s)))ds \le 0,\ t\in[0,1],$$

which shows that Tu is concave down on $[0,1]$. In view of

$$(Fu)(0) = \frac{\gamma}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds \ge 0$$

and

$$(Fu)(1) = \frac{\beta}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds \ge 0,$$

we have

$$(Tu)(0) = \frac{1-(\beta-\gamma)\eta}{\Delta}\lambda[u] + (Fu)(0) \ge 0$$

and

$$(Tu)(1) = \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}\lambda[u] + (Fu)(1) \ge 0.$$

So, $(Tu)(t) \ge 0$, $t \in [0,1]$.

Now, we prove that $\min_{t\in[\eta,1]}(Tu)(t) \ge \Gamma\|Tu\|$. To do it we consider two cases:

Case 1. Let $(Tu)(\eta) \le (Tu)(1)$. Then $\min_{t\in[\eta,1]}(Tu)(t) = (Tu)(\eta)$ and there exists $\bar{t} \in [\eta,1]$ such that $\|Tu\| = (Tu)(\bar{t})$. Moreover,

$$\frac{(Tu)(\bar{t})-(Tu)(0)}{\bar{t}-0} \le \frac{(Tu)(\eta)-(Tu)(0)}{\eta-0}.$$

So,

$$\|Tu\| \le \frac{1}{\eta}(Tu)(\eta) - \frac{1-\eta}{\eta}(Tu)(0),$$

which together with

$$(Tu)(0) = \gamma(Tu)(\eta) + \lambda[u] \tag{9}$$

implies that

$$\|Tu\| \leq \frac{1-\gamma(1-\eta)}{\eta}(Tu)(\eta),$$

i.e.,

$$\min_{t\in[\eta,1]} (Tu)(t) \geq \frac{\eta}{1-\gamma(1-\eta)}\|Tu\|. \tag{10}$$

Case 2. Let $(Tu)(\eta) > (Tu)(1)$ and $\|Tu\| = (Tu)(\bar{t})$. Note that in this case $\min\limits_{t\in[\eta,1]} (Tu)(t) = (Tu)(1)$.

If $\bar{t} \in [0, \eta]$, then

$$\frac{(Tu)(1) - (Tu)(\bar{t})}{1-\bar{t}} \geq \frac{(Tu)(1) - (Tu)(\eta)}{1-\eta}.$$

So,

$$\|Tu\| \leq \frac{1}{1-\eta}(Tu)(\eta) - \frac{\eta}{1-\eta}(Tu)(1),$$

which together with

$$(Tu)(\eta) = \frac{1}{\beta}\Big((Tu)(1) - \lambda[u]\Big) \tag{11}$$

implies that

$$\|Tu\| \leq \frac{1-\beta\eta}{\beta(1-\eta)}(Tu)(1),$$

i.e.,

$$\min_{t\in[\eta,1]} (Tu)(t) \geq \frac{\beta(1-\eta)}{1-\beta\eta}\|Tu\|. \tag{12}$$

If $\bar{t} \in (\eta, 1)$, then

$$\frac{(Tu)(\bar{t}) - (Tu)(\eta)}{\bar{t}-\eta} \leq \frac{(Tu)(\eta) - (Tu)(0)}{\eta - 0}.$$

So,

$$\|Tu\| \leq \frac{1}{\eta}(Tu)(\eta) - \frac{1-\eta}{\eta}(Tu)(0),$$

which together with (9) and (11) implies that

$$\|Tu\| \leq \frac{1-\gamma(1-\eta)}{\beta\eta}(Tu)(1),$$

i.e.,

$$\min_{t\in[\eta,1]} (Tu)(t) \geq \frac{\beta\eta}{1-\gamma(1-\eta)}\|Tu\|. \tag{13}$$

It follows from (10), (12) and (13) that

$$\min_{t\in[\eta,1]}(Tu)(t)\geq \Gamma\|Tu\|.$$

Finally, we need to show that $\lambda[Tu]\geq 0$. In view of

$$\begin{aligned}\lambda[Fu] &= \int_0^1 \frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))dsd\Lambda(t)\\ &\quad+\int_0^1\int_0^1 k(t,s)f(s,u(\alpha(s)))dsd\Lambda(t)\\ &= \frac{\rho'}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds+\int_0^1 \kappa(s)f(s,u(\alpha(s)))ds\\ &\geq 0,\end{aligned}$$

we have

$$\lambda[Tu]=\frac{\rho}{\Delta}\lambda[u]+\lambda[Fu]\geq 0.$$

This shows that $T:K\to K$. Similarly, we can prove that $S:K\to K$.

LEMMA 4. The operators T and S have the same fixed points in K.

PROOF. Suppose that $u\in K$ is a fixed point of S. Then

$$\begin{aligned}\lambda[u] &= \int_0^1\left(\frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu]+(Fu)(t)\right)d\Lambda(t)\\ &= \frac{\Delta}{\Delta-\rho}\lambda[Fu],\end{aligned}$$

which shows that

$$\lambda[Fu]=\frac{\Delta-\rho}{\Delta}\lambda[u].$$

So,

$$\begin{aligned}u(t) &= (Su)(t)\\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu]+(Fu)(t)\\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u]+(Fu)(t)\\ &= (Tu)(t),\ t\in[0,1],\end{aligned}$$

which indicates that u is a fixed point of T. Suppose that $u\in K$ is a fixed point of T. Then

$$\begin{aligned}\lambda[u] &= \int_0^1\left(\frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u]+(Fu)(t)\right)d\Lambda(t)\\ &= \frac{\rho}{\Delta}\lambda[u]+\lambda[Fu],\end{aligned}$$

which shows that

$$\lambda[u] = \frac{\Delta}{\Delta - \rho}\lambda[Fu].$$

So,

$$\begin{aligned} u(t) &= (Tu)(t) \\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u]+(Fu)(t) \\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu]+(Fu)(t) \\ &= (Su)(t),\ t\in[0,1], \end{aligned}$$

which indicates that u is a fixed point of S.

LEMMA 5. $T, S : K \to K$ is completely continuous.

PROOF. First, by LEMMA 3, we know that $T(K) \subset K$. Next, we show that T is compact. Let $D \subset K$ be a bounded set. Then there exists $M_1 > 0$ such that $\|u\| \le M_1$ for any $u \in D$. Since Λ is a function of bounded variation, there exists $M_2 > 0$ such that $v_{\Delta'} = \sum_{i=1}^{n} |\Lambda(t_i) - \Lambda(t_{i-1})| \le M_2$ for any partition $\Delta^{'} : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. Let

$$M_3 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0, M_1]\}.$$

Then for any $u \in D$,

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]}(Tu)(t) \\ &\le \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}\lambda[u]+\frac{\beta}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds \\ &\quad +\frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s,u(\alpha(s)))ds \\ &\le \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}M_1M_2+\frac{\beta M_3}{\Delta}\int_0^1 k(\eta,s)ds+\frac{5}{24}M_3, \end{aligned}$$

which shows that $T(D)$ is uniformly bounded.

On the other hand, for any $\varepsilon > 0$, since $k(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, there exists $\delta_1(\varepsilon) > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1(\varepsilon)$,

$$|k(t_1,s) - k(t_2,s)| < \frac{\varepsilon}{3M_3},\ s \in [0,1].$$

Let $\delta = \min\left\{\delta_1(\varepsilon), \frac{\varepsilon\Delta}{3(\beta-\gamma)M_1M_2}, \frac{\varepsilon\Delta}{3(\beta-\gamma)M_3\int_0^1 k(\eta,s)ds}\right\}$. Then for any $u \in D$, $t_1, t_2 \in$

$[0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
& |(Tu)(t_1) - (Tu)(t_2)| \\
= & \Big|\frac{(\beta-\gamma)(t_1-t_2)}{\Delta}\lambda[u] + \frac{(\beta-\gamma)(t_1-t_2)}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds \\
& + \int_0^1 (k(t_1,s)-k(t_2,s))f(s,u(\alpha(s)))ds\Big| \\
\leq & \frac{(\beta-\gamma)|t_1-t_2|}{\Delta}\lambda[u] + \frac{(\beta-\gamma)|t_1-t_2|}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds \\
& + \int_0^1 |k(t_1,s)-k(t_2,s)|f(s,u(\alpha(s)))ds \\
\leq & \frac{(\beta-\gamma)|t_1-t_2|M_1M_2}{\Delta} + \frac{(\beta-\gamma)|t_1-t_2|M_3}{\Delta}\int_0^1 k(\eta,s)ds \\
& + M_3\int_0^1 \big|k(t_1,s)-k(t_2,s)\big|ds \\
< & \ \varepsilon,
\end{aligned}$$

which shows that $T(D)$ is equicontinuous. It follows from Arzela-Ascoli theorem that $T(D)$ is relatively compact. Thus, we have shown that T is a compact operator.

Finally, we prove that T is continuous. Suppose that $u_n, u \in K$ and $\lim_{n\to\infty} u_n = u$. Then there exists $M_4 > 0$ such that $\|u\| \leq M_4$ and $\|u_n\| \leq M_4$ $(n = 1, 2, \cdots)$. For any $\varepsilon > 0$, since $f(s,x)$ is uniformly continuous on $[0,1]\times[0,M_4]$, there exists $\delta > 0$ such that for any x_1, $x_2 \in [0,M_4]$ with $|x_1 - x_2| < \delta$,

$$|f(s,x_1) - f(s,x_2)| < \frac{\varepsilon}{\frac{2\beta}{\Delta}\int_0^1 k(\eta,s)ds + \frac{5}{12}}, \ s \in [0,1]. \tag{14}$$

At the same time, since $\lim_{n\to\infty} u_n = u$, there exists positive integer N such that for any $n > N$,

$$\|u_n - u\| < \min\left\{\delta, \ \frac{\varepsilon\Delta}{2[1+(\beta-\gamma)(1-\eta)]|\Lambda(1)-\Lambda(0)|}\right\}. \tag{15}$$

It follows from (14) and (15) that for any $n > N$,

$$\begin{aligned} & \|Tu_n - Tu\| \\ = & \max_{t\in[0,1]} |(Tu_n)(t) - (Tu)(t)| \\ \leq & \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}|\lambda[u_n] - \lambda[u]| + \frac{\beta}{\Delta}\int_0^1 k(\eta,s)|f(s,u_n(\alpha(s))) - f(s,u(\alpha(s)))|ds \\ & + \frac{1}{2}\int_0^1 (1+s)(1-s)^2|f(s,u_n(\alpha(s))) - f(s,u(\alpha(s)))|ds \\ \leq & \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}\|u_n - u\||\Lambda(1) - \Lambda(0)| \\ & + \int_0^1 \big(\frac{\beta}{\Delta}k(\eta,s) + \frac{1}{2}(1+s)(1-s)^2\big)|f(s,u_n(\alpha(s))) - f(s,u(\alpha(s)))|ds \\ < & \varepsilon, \end{aligned}$$

which indicates that T is continuous. Therefore, $T : K \to K$ is completely continuous. Similarly, we can prove that $S : K \to K$ is also completely continuous.

For convenience, we denote

$$D_1 = \frac{\rho'}{\Delta}\int_0^1 k(\eta,s)ds + \int_0^1 \kappa(s)ds,\ D_2 = \frac{\beta}{\Delta}\int_0^1 k(\eta,s)ds + \frac{5}{24},$$

$$D_3 = \frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)ds + \int_\eta^1 \kappa(s)ds \text{ and } D_4 = \frac{1}{\Delta}\int_\eta^1 k(\eta,s)ds.$$

Let

$$\mu > \frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}D_1 + D_2 \text{ and } 0 < L < \beta\big(\frac{D_3}{\Delta-\rho} + D_4\big).$$

THEOREM 2. Assume that there exist positive constants a, b and d with $a < b < \frac{b}{\Gamma} \leq d$ such that

(A1) $f(t,u) \leq \frac{d}{\mu}$ for $(t,u) \in [0,1] \times [0,d]$,

(A2) $f(t,u) \geq \frac{b}{L}$ for $(t,u) \in [\eta,1] \times [b,\frac{b}{\Gamma}]$, and

(A3) $f(t,u) \leq \frac{a}{\mu}$ for $(t,u) \in [0,1] \times [0,a]$.

Then the BVP (3) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_i\| \leq d$ $(i = 1,2,3)$ and

$$\min_{t\in[\eta,1]} u_1(t) > b,\ \|u_2\| > a \text{ with } \min_{t\in[\eta,1]} u_2(t) < b,\ \|u_3\| < a.$$

PROOF. For $u \in K$, we define

$$\Phi(u) = \min_{t\in[\eta,1]} u(t) \text{ and } \varphi(u) = \Theta(u) = \Psi(u) = \|u\|.$$

Then it is easy to know that Φ is a nonnegative continuous concave functional on K and φ, Θ and Ψ are nonnegative continuous convex functionals on K. In order to apply Theorem 1 to prove our main results, we use the operator S and take $c = b/\Gamma$.

We first assert that $S : \overline{K(\varphi, d)} \to \overline{K(\varphi, d)}$. Indeed, if $u \in \overline{K(\varphi, d)}$, then $0 \le u(t) \le d$, $t \in [0, 1]$, which together with $(A1)$ implies that

$$\begin{aligned} \lambda[Fu] &= \frac{\rho'}{\Delta}\int_0^1 k(\eta, s) f(s, u(\alpha(s)))ds + \int_0^1 \kappa(s) f(s, u(\alpha(s)))ds \\ &\le \frac{D_1 d}{\mu} \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\|Fu\| \\ = &\max_{t\in[0,1]}\left(\frac{\gamma + (\beta-\gamma)t}{\Delta}\int_0^1 k(\eta, s) f(s, u(\alpha(s)))ds + \int_0^1 k(t, s) f(s, u(\alpha(s)))ds\right) \\ \le &\frac{\beta}{\Delta}\int_0^1 k(\eta, s) f(s, u(\alpha(s)))ds + \frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s, u(\alpha(s)))ds \\ \le &\frac{D_2 d}{\mu}. \end{aligned} \tag{17}$$

In view of (16) and (17), we have

$$\varphi(Su) = \|Su\| \le \frac{1 + (\beta-\gamma)(1-\eta)}{\Delta - \rho}\lambda[Fu] + \|Fu\| \le \left(\frac{1 + (\beta-\gamma)(1-\eta)}{\Delta - \rho} D_1 + D_2\right)\frac{d}{\mu} \le d.$$

This indicates that $S : \overline{K(\varphi, d)} \to \overline{K(\varphi, d)}$.

Next, we assert that $\{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\} \ne \phi$ and $\Phi(Su) > b$ for $u \in K(\varphi, \Theta, \Phi, b, c, d)$. In fact, the constant function $\frac{b+c}{2} \in \{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\}$. Moreover, for $u \in K(\varphi, \Theta, \Phi, b, c, d)$, we know that $b \le u(\alpha(t)) \le c$ for $t \in [\eta, 1]$, which together with $(A2)$ implies that

$$\begin{aligned} \lambda[Fu] &= \frac{\rho'}{\Delta}\int_0^1 k(\eta, s) f(s, u(\alpha(s)))ds + \int_0^1 \kappa(s) f(s, u(\alpha(s)))ds \\ &\ge \frac{\rho'}{\Delta}\int_\eta^1 k(\eta, s) f(s, u(\alpha(s)))ds + \int_\eta^1 \kappa(s) f(s, u(\alpha(s)))ds \\ &\ge \frac{D_3 b}{L} \end{aligned} \tag{18}$$

and

$$\begin{aligned} (Fu)(\eta) &= \frac{1}{\Delta}\int_0^1 k(\eta, s) f(s, u(\alpha(s)))ds \\ &\ge \frac{1}{\Delta}\int_\eta^1 k(\eta, s) f(s, u(\alpha(s)))ds \\ &\ge \frac{D_4 b}{L}. \end{aligned} \tag{19}$$

In view of (18) and (19), we see that

$$\begin{aligned}
\Phi(Su) &= \min_{t\in[\eta,1]}(Su)(t)\\
&= \min\Big((Su)(\eta),(Su)(1)\Big)\\
&= \min\Big((Su)(\eta),\beta(Su)(\eta)+\frac{\Delta}{\Delta-\rho}\lambda[Fu]\Big)\\
&\geq \beta(Su)(\eta)\\
&= \beta\Big(\frac{1}{\Delta-\rho}\lambda[Fu]+(Fu)(\eta)\Big)\\
&\geq \beta\Big(\frac{D_3}{\Delta-\rho}+D_4\Big)\frac{b}{L}\\
&> b,
\end{aligned}$$

as required.

Thirdly, we assert that $\Phi(Su) > b$ for $u \in K(\varphi,\Phi,b,d)$ with $\Theta(Su) > c$. To see this, we suppose $u \in K(\varphi,\Phi,b,d)$ and $\Theta(Su) = \|Su\| > c$. Then

$$\Phi(Su) = \min_{t\in[\eta,1]}(Su)(t) \geq \Gamma\|Su\| > \Gamma c = b.$$

Finally, we assert that $\theta \notin R(\varphi,\Psi,a,d)$ and $\Psi(Su) < a$ for $u \in R(\varphi,\Psi,a,d)$ with $\Psi(u) = a$. Indeed, it follows from $\Psi(\theta) = 0 < a$ that $\theta \notin R(\varphi,\Psi,a,d)$. Moreover, for $u \in R(\varphi,\Psi,a,d)$ and $\Psi(u) = a$, we know that $0 \leq u(t) \leq a$ for $t \in [0,1]$, which together with $(A3)$ implies that

$$\begin{aligned}
\lambda[Fu] &= \frac{\rho'}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds+\int_0^1\kappa(s)f(s,u(\alpha(s)))ds\\
&\leq \frac{D_1 a}{\mu}
\end{aligned}\qquad (20)$$

and

$$\begin{aligned}
&\|Fu\|\\
= &\max_{t\in[0,1]}\Big(\frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds+\int_0^1 k(t,s)f(s,u(\alpha(s)))ds\Big)\\
\leq &\frac{\beta}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))ds+\frac{1}{2}\int_0^1(1+s)(1-s)^2 f(s,u(\alpha(s)))ds\\
\leq &\frac{D_2 a}{\mu}.
\end{aligned}\qquad (21)$$

In view of (20) and (21), we have

$$\begin{aligned}
\Psi(Su) &= \|Su\| \\
&\leq \frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}\lambda[Fu] + \|Fu\| \\
&\leq \left(\frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}D_1 + D_2\right)\frac{a}{\mu} \\
&< a,
\end{aligned}$$

as required.

To sum up, all the hypotheses of Theorem 1 are satisfied. Hence, the BVP (3) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_i\| \leq d$ $(i = 1, 2, 3)$ and

$$\min_{t\in[\eta,1]} u_1(t) > b,\ \|u_2\| > a \text{ with } \min_{t\in[\eta,1]} u_2(t) < b,\ \|u_3\| < a.$$

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