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CERTAIN ESTIMATES FOR DOUBLE SINE SERIES WITH MULTIPLE-MONOTONE COEFFICIENTS

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Abstract. In this paper we obtain estimates of the sum of double sine series near the origin, with multiple-monotone coefficients tending to zero. These estimates extend some results of Telyakovski [11] and Popov [7] from single to multidimensional case.

1. Introduction and Preliminaries

Many authors considered the sine series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx$$

with monotone coefficients tending to zero. Young [13] was the first to consider the problem of estimates of $g(x)$ for $x \to 0$ expressed in terms of the coefficients a_n. Then Salem [8], [9], Shogunbekov [10], Aljančić, Bojanić and Tomić [1] considered this problem, as well. Later Telyakovski in [11] has proved the following fact:

Theorem T. *Assume that* $a_n \downarrow 0$. *Then for* $x \in \left(\frac{\pi}{m+1}, \frac{\pi}{m}\right] \equiv I_m, m = 1, 2, \dots$ *the following estimate is valid:*

$$g(x) = \sum_{n=1}^{m} n a_n x + O\left(\frac{1}{m^3} \sum_{n=1}^{m} n^3 a_n\right).$$

Likewise, among others Popov [7] has proved

Theorem P. *For any nonincreasing sequence of positive numbers* a_k *tending to zero, the following inequality holds:*

$$-\frac{1}{2} a_1 \sin\frac{x}{2} \le g(x), \quad \forall x \in (0, \pi].$$

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The problem shown above is considered by the present author [4]–[6] as well. The object of this work is to extend these two theorems, formulated above, from single to multidimensional case. In fact, we shall investigate the behavior near the origin of the sum of double sine series with multiple-monotone coefficients.

Let us now consider the following double sine series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}\sin mx \sin ny \tag{1.1}$$

whose coefficients satisfy conditions $a_{m,n}\to 0$ for $m\to\infty$ and all n fixed, and for $n\to\infty$ and all m fixed.

For $k_1\geq 0, k_2\geq 0$ denote

$$\begin{aligned}
\triangle_{k_1,0}a_{m,n} &= \sum_{i=0}^{k_1}(-1)^i\binom{k_1}{i}a_{m+i,n},\\
\triangle_{0,k_2}a_{m,n} &= \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}a_{m,n+j},\\
\triangle_{k_1,k_2}a_{m,n} &= \sum_{i=0}^{k_1}\sum_{j=0}^{k_2}(-1)^{i+j}\binom{k_1}{i}\binom{k_2}{j}a_{m+i,n+j}.
\end{aligned}$$

Parallel with series (1.1) we consider the series of the form

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)\overline{B}_n^{k_2}(y), \tag{1.2}$$

where

$$\begin{aligned}
&\overline{B}_r^{1}(x)\equiv\widetilde{D}_r(x)=\sin x+\sin 2x+\cdots+\sin rx, r\geq 1;\\
&\overline{B}_r^{\nu}(x)=\sum_{\mu=1}^{r}\overline{B}_\mu^{\nu-1}(x),\ (\nu=2,3,\dots).
\end{aligned}$$

Let

$$\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}c_{\mu,\nu} \tag{1.3}$$

be double numerical series and

$$S_{m,n}=\sum_{\nu=1}^{n}\sum_{\mu=1}^{m}c_{\mu,\nu}$$

its rectangular partial sums.

If there exists a number S such that for all $\varepsilon>0$ there exist natural numbers k and l such that

$$|S_{m,n}-S|<\varepsilon$$

for all $n > k$ and $m > l$, then series (1.3) converges in Pringsheim's sense to the number S (see [12], page 27).

It's well-known (see [3]) that if sequence of numbers $\{a_{k,l}\}$ satisfies conditions (A) and $\triangle_{k_1,k_2} a_{k,l} \geq 0$ for all k and l, then:

Lemma 1.1. (1) *Series (1.1) converges almost everywhere in Pringsheim's sense, in other words there exists a function $g(x,y)$ such that the sum of series (1.1) is $g(x,y)$.*

(2) *Series (1.2) converges almost everywhere in Pringsheim's sense to $g(x,y)$.*

Throughout this paper the O expressions contain positive constants and they may depend only on k_1 and k_2.

2. Main Results

To prove our main results we need first the following lemma.

Lemma 2.1. *For $\nu = 1, 2, 3, \dots$ and $x \in (0, \pi]$ the following inequality is true*

$$\overline{B}_r^{\nu}(x) \geq -\frac{r^{\nu-1}}{2} \sin \frac{x}{2}.$$

Proof. For the proof we shall use mathematical induction. Namely, for $\nu = 1$ we have

$$\begin{aligned}\overline{B}_r^{1}(x) \equiv \widetilde{D}_r(x) &= \frac{\cos \frac{x}{2} - \cos \left(r + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}} \geq \frac{\cos \frac{x}{2} - 1}{2 \sin \frac{x}{2}} \\ &= -\frac{1}{2} \tan \frac{x}{4} \geq -\frac{1}{2} \sin \frac{x}{2}.\end{aligned}$$

Assuming that

$$\overline{B}_r^{\nu-1}(x) \geq -\frac{r^{\nu-2}}{2} \sin \frac{x}{2}$$

we obtain

$$\overline{B}_r^{\nu}(x) = \sum_{\mu=1}^{r} \overline{B}_{\mu}^{\nu-1}(x) \geq -\frac{1}{2} \sin \frac{x}{2} \sum_{\mu=1}^{r} \mu^{\nu-2} \geq -\frac{r^{\nu-1}}{2} \sin \frac{x}{2},$$

which completes the proof of the Lemma 2.1. □

We shall prove two theorems. The first theorem gives an estimate of the sum $g(x,y)$ near the origin from above, while the second one gives the estimate from below.

Theorem 2.2. *Assume that $\{a_{m,n}\}$ satisfies conditions (A) and $\triangle_{k_1,k_2} a_{m,n} \geq 0$, for $k_1 \geq 1, k_2 \geq 1$. Then for $x \in I_r$ and $y \in I_\ell$, $(r, \ell = 1, 2, \dots)$ the following* *estimate is valid*

$$g(x,y)=\sum_{m=1}^{r}\sum_{n=1}^{\ell}mnxya_{m,n} \tag{2.1}$$
$$+O\Bigg(\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(rn)^2+(mn)^2+(\ell m)^2}{(r\ell)^3}mna_{m,n}$$
$$+\sum_{i=1}^{k_1}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(\ell^2+n^2)\,m^3n}{r^{4-i}\ell^3}\triangle_{i-1,0}a_{m,n}$$
$$+\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(r^2+m^2)\,mn^3}{r^3\ell^{4-j}}\triangle_{0,j-1}a_{m,n}$$
$$+\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}\Bigg).$$

Proof. By the Lemma 1.1 we have

$$g(x,y)=\sum_{n=1}^{\ell}\sum_{m=1}^{r}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)\overline{B}_n^{k_2}(y) \tag{2.2}$$
$$+\sum_{n=\ell+1}^{\infty}\sum_{m=1}^{r}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)\overline{B}_n^{k_2}(y)$$
$$+\sum_{n=1}^{\ell}\sum_{m=r+1}^{\infty}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)\overline{B}_n^{k_2}(y)$$
$$+\sum_{n=\ell+1}^{\infty}\sum_{m=r+1}^{\infty}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)\overline{B}_n^{k_2}(y)$$
$$=A_{\ell,r}^{k_1,k_2}(x,y)+A_{\infty,r}^{k_1,k_2}(x,y)+A_{\ell,\infty}^{k_1,k_2}(x,y)+A_{\infty,\infty}^{k_1,k_2}(x,y).$$

The expression $A_{\ell,r}^{k_1,k_2}(x,y)$ can be written as follows

$$A_{\ell,r}^{k_1,k_2}(x,y) \tag{2.3}$$
$$=\sum_{n=1}^{\ell}\left[\sum_{m=1}^{r}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)\right]\overline{B}_n^{k_2}(y)=\sum_{n=1}^{\ell}H_{r,n}^{k_1,k_2}(x)\overline{B}_n^{k_2}(y),$$

where $H_{r,n}^{k_1,k_2}(x)=\sum_{m=1}^{r}\triangle_{k_1,k_2}a_{m,n}\overline{B}_m^{k_1}(x)$.

Since

$$\overline{B}_m^{s}(x)-\overline{B}_{m-1}^{s}(x)=\overline{B}_m^{s-1}(x),\ (s=2,3,\dots k_1),$$

then

$$\begin{aligned}
H_{r,n}^{k_1,k_2}(x) &= \sum_{m=1}^{r} \left[\triangle_{k_1-1,k_2} a_{m,n} - \triangle_{k_1-1,k_2} a_{m+1,n}\right] \overline{B}_m^{k_1}(x) \\
&= \sum_{m=1}^{r} \triangle_{k_1-1,k_2} a_{m,n} \left[\overline{B}_m^{k_1}(x) - \overline{B}_{m-1}^{k_1}(x)\right] - \triangle_{k_1-1,k_2} a_{r+1,n} \overline{B}_r^{k_1}(x) \\
&= \sum_{m=1}^{r} \triangle_{k_1-1,k_2} a_{m,n} \overline{B}_m^{k_1-1}(x) - \triangle_{k_1-1,k_2} a_{r+1,n} \overline{B}_r^{k_1}(x) \\
&= \sum_{m=1}^{r} \triangle_{k_1-2,k_2} a_{m,n} \overline{B}_m^{k_1-2}(x) - \sum_{i=k_1-1}^{k_1} \triangle_{i-1,k_2} a_{r+1,n} \overline{B}_r^{i}(x) \\
&\vdots \\
&= \sum_{m=1}^{r} \triangle_{0,k_2} a_{m,n} \sin mx - \sum_{i=1}^{k_1} \triangle_{i-1,k_2} a_{r+1,n} \overline{B}_r^{i}(x).
\end{aligned}$$

Applying the relation $\sin u = u + O(u^3)$, as $u \to 0$, $\triangle_{i-1,k_2} a_{r,n} \geq \triangle_{i-1,k_2} a_{r+1,n}$ by our assumption, the well-known estimate $\overline{B}_r^{i}(x) = O\left((r+1)^i\right)$ for $i \geq 1$, $x \in [0,\pi]$, and $x \in I_r$ we obtain

$$\begin{aligned}
(2.4)\quad & H_{r,n}^{k_1,k_2}(x) \\
&= \sum_{m=1}^{r} m \triangle_{0,k_2} a_{m,n} x + O\left(\frac{1}{r^3} \sum_{m=1}^{r} m^3 \triangle_{0,k_2} a_{m,n} + \sum_{i=1}^{k_1} r^i \triangle_{i-1,k_2} a_{r,n}\right).
\end{aligned}$$

In a similar way using the same arguments as for $H_{r,n}^{k_1,k_2}(x)$ we arrive at the following estimates ($y \in I_\ell$):

$$\begin{aligned}
\sum_{n=1}^{\ell} \triangle_{0,k_2} a_{m,n} \overline{B}_n^{k_2}(y) &= \sum_{n=1}^{\ell} \triangle_{0,0} a_{m,n} \sin ny - \sum_{j=1}^{k_2} \triangle_{0,j-1} a_{m,\ell+1} \overline{B}_\ell^{j}(y) \\
&= \sum_{n=1}^{\ell} n a_{m,n} y + O\left(\frac{1}{\ell^3} \sum_{n=1}^{\ell} n^3 a_{m,n} + \sum_{j=1}^{k_2} \ell^j \triangle_{0,j-1} a_{m,\ell}\right),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\ell} \triangle_{i-1,k_2} a_{r,n} \overline{B}_n^{k_2}(y) &= \sum_{n=1}^{\ell} n \triangle_{i-1,0} a_{r,n} y \\
&\quad + O\left(\frac{1}{\ell^3} \sum_{n=1}^{\ell} n^3 \triangle_{i-1,0} a_{r,n} + \sum_{j=1}^{k_2} \ell^j \triangle_{i-1,j-1} a_{r,\ell}\right).
\end{aligned}$$

Using (2.3), (2.4), and last estimates we get

$$
\begin{aligned}
(2.5)\; A_{\ell,r}^{k_1,k_2}(x,y) &= \sum_{m=1}^{r} mx \sum_{n=1}^{\ell} \triangle_{0,k_2} a_{m,n} \overline{B}_n^{k_2}(y) \\
&\quad + O\Bigg(\frac{1}{r^3} \sum_{m=1}^{r} m^3 \sum_{n=1}^{\ell} \triangle_{0,k_2} a_{m,n} \overline{B}_n^{k_2}(y) \\
&\quad + \sum_{i=1}^{k_1} r^i \sum_{n=1}^{\ell} \triangle_{i-1,k_2} a_{r,n} \overline{B}_n^{k_2}(y) \Bigg) \\
&= \sum_{m=1}^{r} \sum_{n=1}^{\ell} mnxy a_{m,n} + O\Bigg(\frac{1}{r\ell^3} \sum_{m=1}^{r} \sum_{n=1}^{\ell} mn^3 a_{m,n} \\
&\quad + \frac{1}{r} \sum_{m=1}^{r} \sum_{j=1}^{k_2} m\ell^j \triangle_{0,j-1} a_{m,\ell} + \frac{1}{r^3\ell} \sum_{m=1}^{r} \sum_{n=1}^{\ell} m^3 n a_{m,n} \\
&\quad + \frac{1}{(r\ell)^3} \sum_{m=1}^{r} \sum_{n=1}^{\ell} (mn)^3 a_{m,n} + \frac{1}{r^3} \sum_{m=1}^{r} \sum_{j=1}^{k_2} m^3 \ell^j \triangle_{0,j-1} a_{m,\ell} \\
&\quad + \frac{1}{\ell} \sum_{n=1}^{\ell} \sum_{i=1}^{k_1} n r^i \triangle_{i-1,0} a_{r,n} + \frac{1}{\ell^3} \sum_{n=1}^{\ell} \sum_{i=1}^{k_1} n^3 r^i \triangle_{i-1,0} a_{r,n} \\
&\quad + \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} r^i \ell^j \triangle_{i-1,j-1} a_{r,\ell} \Bigg).
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{1}{r} \sum_{m=1}^{r} \sum_{j=1}^{k_2} m\ell^j \triangle_{0,j-1} a_{m,\ell} &= \frac{1}{r} \sum_{m=1}^{r} \sum_{j=1}^{k_2} m\ell^{j-1} \ell \triangle_{0,j-1} a_{m,\ell} \\
&\le \frac{1}{r} \sum_{m=1}^{r} \sum_{j=1}^{k_2} m\ell^{j-1} \frac{4}{\ell^3} \sum_{n=1}^{\ell} n^3 \triangle_{0,j-1} a_{m,\ell} \\
&\le \sum_{j=1}^{k_2} \frac{4}{r\ell^{4-j}} \sum_{m=1}^{r} \sum_{n=1}^{\ell} mn^3 \triangle_{0,j-1} a_{m,n},
\end{aligned}
$$

and in a similar manner we can find

$$\frac{1}{r^3}\sum_{m=1}^{r}\sum_{j=1}^{k_2} m^3\ell^j \triangle_{0,j-1}a_{m,\ell} \le \sum_{j=1}^{k_2}\frac{4}{r^3\ell^{4-j}}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3\triangle_{0,j-1}a_{m,n},$$

$$\frac{1}{\ell}\sum_{n=1}^{\ell}\sum_{i=1}^{k_1} nr^i \triangle_{i-1,0}a_{r,n} \le \sum_{i=1}^{k_1}\frac{4}{r^{4-i}\ell}\sum_{m=1}^{r}\sum_{n=1}^{\ell}m^3n\triangle_{i-1,0}a_{m,n},$$

$$\frac{1}{\ell^3}\sum_{n=1}^{\ell}\sum_{i=1}^{k_1} n^3r^i \triangle_{i-1,0}a_{r,n} \le \sum_{i=1}^{k_1}\frac{4}{r^{4-i}\ell^3}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3\triangle_{i-1,0}a_{m,n},$$

$$\sum_{i=1}^{k_1}\sum_{j=1}^{k_2} r^i\ell^j \triangle_{i-1,j-1}a_{r,\ell} \le \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\frac{16}{r^{4-i}\ell^{4-j}}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3\triangle_{i-1,j-1}a_{m,n}.$$

Therefore,

$$\begin{aligned}(2.6)\qquad A^{k_1,k_2}_{\ell,r}(x,y) &= \sum_{m=1}^{r}\sum_{n=1}^{\ell} mnxya_{m,n}\\
&+O\Bigg(\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(rn)^2+(mn)^2+(\ell m)^2}{(r\ell)^3}mna_{m,n}\\
&+\sum_{i=1}^{k_1}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(\ell^2+n^2)\,m^3n}{r^{4-i}\ell^3}\triangle_{i-1,0}a_{m,n}\\
&+\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(r^2+m^2)\,mn^3}{r^3\ell^{4-j}}\triangle_{0,j-1}a_{m,n}\\
&+\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}\Bigg).\end{aligned}$$

Now, we estimate $A^{k_1,k_2}_{\infty,r}(x,y)$. Indeed, since $\overline{B}^{k_2}_n(y)=O\left(y^{-k_2}\right)$ for $y\in(0,\pi]$, (2.4), and $x\in I_r$, $y\in I_\ell$, then

$$\begin{aligned}(2.7)\quad A^{k_1,k_2}_{\infty,r}(x,y) &= O\left(y^{-k_2}\right)\sum_{m=1}^{r}\triangle_{k_1,k_2-1}a_{m,\ell+1}\overline{B}^{k_1}_m(x)\\
&= O\left(\ell^{k_2}\right)\sum_{m=1}^{r}\triangle_{k_1,k_2-1}a_{m,\ell}\overline{B}^{k_1}_m(x)\\
&= O\Bigg(\frac{1}{r}\sum_{m=1}^{r}m\ell^{k_2}\triangle_{0,k_2-1}a_{m,\ell}\\
&\quad+\frac{1}{r^3}\sum_{m=1}^{r}m^3\ell^{k_2}\triangle_{0,k_2-1}a_{m,\ell}+\sum_{i=1}^{k_1}r^i\ell^{k_2}\triangle_{i-1,k_2-1}a_{r,\ell}\Bigg).\end{aligned}$$

It is not difficult to prove that

$$\frac{1}{r}\sum_{m=1}^{r} m\ell^{k_2}\triangle_{0,k_2-1}a_{m,\ell}\le\sum_{j=1}^{k_2}\frac{4}{r\ell^{4-j}}\sum_{m=1}^{r}\sum_{n=1}^{\ell} mn^3\triangle_{0,j-1}a_{m,n}.$$

and

$$\frac{1}{r^3}\sum_{m=1}^{r} m^3\ell^{k_2}\triangle_{0,k_2-1}a_{m,\ell}\le\sum_{j=1}^{k_2}\frac{4}{r^3\ell^{4-j}}\sum_{m=1}^{r}\sum_{n=1}^{\ell} (mn)^3\triangle_{0,j-1}a_{m,n}.$$

From last two estimates we have

$$\begin{aligned}\frac{1}{r}\sum_{m=1}^{r} m\ell^{k_2}\triangle_{0,k_2-1}a_{m,\ell}&+\frac{1}{r^3}\sum_{m=1}^{r} m^3\ell^{k_2}\triangle_{0,k_2-1}a_{m,\ell}\\ &\le\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{4\left(r^2+m^2\right)mn^3}{r^3\ell^{4-j}}\triangle_{0,j-1}a_{m,n},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{k_1} r^i\ell^{k_2}\triangle_{i-1,k_2-1}a_{r,\ell}&=\sum_{i=1}^{k_1} r^{i-1}r\ell^{k_2}\triangle_{i-1,k_2-1}a_{r,\ell}\\ &\le\sum_{i=1}^{k_1} r^{i-1}\ell^{k_2}\frac{4}{r^3}\sum_{m=1}^{r} m^3\triangle_{i-1,k_2-1}a_{m,\ell}\\ &\le\sum_{i=1}^{k_1} r^{i-1}\ell^{k_2-1}\frac{16}{(r\ell)^3}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3\triangle_{i-1,k_2-1}a_{m,\ell}\\ &\le\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{16(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}.\end{aligned}$$

Therefore, from these estimates and (2.7) we obtain

$$\begin{aligned}A^{k_1,k_2}_{\infty,r}(x,y)=O\Bigg(&\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{\left(r^2+m^2\right)mn^3}{r^3\ell^{4-j}}\triangle_{0,j-1}a_{m,n}\\ &+\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}\Bigg).\end{aligned}\tag{2.8}$$

In a similar manner we can find the following estimate

$$(2.9)\quad A_{\ell,\infty}^{k_1,k_2}(x,y)=O\Bigg(\sum_{i=1}^{k_1}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(n^2+\ell^2)\,m^3n}{r^{4-i}\ell^3}\triangle_{i-1,0}a_{m,n} \\ +\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}\Bigg).$$

In the end from $\overline{B}_m^{k_1}(x)=O\left(x^{-k_1}\right),\overline{B}_n^{k_2}(y)=O\left(y^{-k_2}\right)$ for $x,y\in(0,\pi]$, and $x\in I_r$, $y\in I_\ell$ we obtain

$$(2.10)\quad A_{\infty,\infty}^{k_1,k_2}(x,y)=O\left(x^{-k_1}y^{-k_2}\sum_{n=\ell+1}^{\infty}\sum_{m=r+1}^{\infty}\triangle_{k_1,k_2}a_{m,n}\right) \\ =O\Big(r^{k_1}\ell^{k_2}\triangle_{k_1-1,k_2-1}a_{r,\ell}\Big).$$

By virtue of the monotonicity of $\triangle_{k_1,k_2}a_{r,\ell}$, for $k_1\geq 1,k_2\geq 1$, we get

$$\begin{aligned}(2.11)\quad r^{k_1}\ell^{k_2}\triangle_{k_1-1,k_2-1}a_{r,\ell}&=r^{k_1-1}\ell^{k_2-1}r\ell\triangle_{k_1-1,k_2-1}a_{r,\ell}\\ &\leq\frac{16}{r^{4-k_1}\ell^{4-k_2}}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3\,\triangle_{k_1-1,k_2-1}a_{r,\ell}\\ &\leq\frac{16}{r^{4-k_1}\ell^{4-k_2}}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3\,\triangle_{k_1-1,k_2-1}a_{m,n}\\ &\leq\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{16\,(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}.\end{aligned}$$

Thus, from (2.10) and (2.11) we get

$$(2.12)\qquad A_{\infty,\infty}^{k_1,k_2}(x,y)=O\left(\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{(mn)^3}{r^{4-i}\ell^{4-j}}\triangle_{i-1,j-1}a_{m,n}\right).$$

Inserting (2.6), (2.8), (2.9) and (2.12) into (2.2) we obtain (2.1). □

As a direct consequence of the theorem 2.2 is the following corollary:

Corollary 2.3. [4] *Assume that* $\{a_{m,n}\}$ *satisfies conditions (A) and* $\triangle_{1,1}a_{m,n} \geq 0$. *Then for* $x \in I_r$ *and* $y \in I_\ell$, $(r, \ell = 1, 2, \dots)$ *the following estimate is valid*

$$g(x,y) = \sum_{m=1}^{r}\sum_{n=1}^{\ell} mnxya_{m,n} + O\Bigg(\frac{1}{r\ell^3}\sum_{m=1}^{r}\sum_{n=1}^{\ell} mn^3a_{m,n} + \frac{1}{r^3\ell}\sum_{m=1}^{r}\sum_{n=1}^{\ell} m^3na_{m,n} + \frac{1}{(r\ell)^3}\sum_{m=1}^{r}\sum_{n=1}^{\ell}(mn)^3a_{m,n}\Bigg). \tag{2.13}$$

Now we prove the statement that gives the estimate of $g(x,y)$ from below.

Theorem 2.4. *Assume that* $a_{k,l}$ *satisfy conditions (A) and* $\triangle_{k_1,k_2}a_{k,l} \geq 0$. *Then the following estimate is valid*

$$\frac{1}{4}\sin\frac{x}{2}\sin\frac{y}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{k_1-1}n^{k_2-1}\triangle_{k_1,k_2}a_{m,n} \leq g(x,y). \tag{2.14}$$

Proof. Based on Lemma 1.1 and Lemma 2.1 we immediately obtain (2.14). □

Theorem 2.4, for $k_1 = 1$, $k_2 = 1$, implies the following statement proved in [4].

Corollary 2.5. *Assume that* $a_{k,l}$ *satisfy conditions (A) and* $\triangle_{1,1}a_{k,l} \geq 0$. *Then the following estimate is valid*

$$\frac{a_{1,1}}{4}\sin\frac{x}{2}\sin\frac{y}{2} \leq g(x,y).$$

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