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GEODESICS AND DEFORMED PSEUDO–RIEMANNIAN MANIFOLDS WITH SYMMETRY

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Abstract. There is a general method, applicable in many situations, whereby a pseudo–Riemannian metric, invariant under the action of some Lie group, can be deformed to obtain a new metric whose geodesics can be expressed in terms of the geodesics of the old metric and the action of the Lie group. This method applied to Euclidean space and the unit sphere produces new examples of complete Riemannian metrics whose geodesics are expressible in terms of elementary functions.

1. Introduction

Riemannian manifolds whose geodesics are given by explicit formulas are rare and worthy of notice. For instance, by deforming the unit 2–sphere in a special way, Faridi and Schucking [2] found a one–parameter family of rotationally symmetric surfaces whose geodesics are expressible by means of elementary functions. These deformed spheres can be obtained by applying the general deformation method that is described herein. Under reasonable hypotheses, starting with a pseudo–Riemannian metric on a manifold which is invariant under the action of a Lie group G, one can construct a family of G–invariant metrics whose geodesics are expressible in terms of the geodesics of the original metric, the exponential map of G, and the action of G. In this way new examples of complete n–dimensional Riemannian manifolds whose geodesics are expressible in terms of elementary functions are discovered by deforming the standard metrics on S^n or $\mathbf{R}^n$.

This construction utilizes ideas from geometrical mechanics. Especially, the geodesic equations are viewed as a Hamiltonian system on the cotangent bundle. Enough terminology and results from mechanics are reviewed to enable a description of the construction and a proof of the formula for geodesics. Still, the reader who is unacquainted with this material should consult [1] for complete details and a systematic exposition.

2. Notation, Background, and Preliminaries

Let Q be a smooth finite–dimensional manifold with tangent bundle TQ and cotangent bundle T^*Q. The cotangent bundle has a natural symplectic structure that induces the Poisson bracket $\{f, g\}$ of two smooth real–valued functions f and g on T^*Q. The Poisson bracket operation turns the smooth real–valued functions on T^*Q, the so–called Hamiltonian functions, into a Lie algebra. The Hamiltonian vector field X_H on T^*Q, associated to the Hamiltonian function $H: T^*Q \to \mathbf{R}$, is defined so that

$$X_H(f) = \{f, H\}$$

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for every smooth function $f\colon T^*Q \to \mathbf{R}$.

Let G be a Lie group with Lie algebra $\mathcal{G}$. The dual space of $\mathcal{G}$ will be denoted $\mathcal{G}^*$.

Suppose G acts smoothly on Q. The action lifts to a symplectic action of G on T^*Q which makes the canonical projection map $\pi\colon T^*Q \to Q$ G–equivariant. That is, $\pi \circ \mathbf{L}_a = L_a \circ \pi$ for all $a \in G$, where L_a and $\mathbf{L}_a$ denote left translation by $a \in G$ on Q and T^*Q respectively. Thus if $\xi \in \mathcal{G}$, then ξ_Q is π–related to ξ_{T^*Q}, where ξ_Q and ξ_{T^*Q} denote the vector fields that ξ generates through the actions of G on Q and T^*Q respectively.

The action of G by cotangent lift has a momentum map

$$J\colon T^*Q \to \mathcal{G}^*$$

defined by

$$J(\alpha)(\xi) = \alpha(\xi_Q)$$

for $\alpha \in T^*Q$ and $\xi \in \mathcal{G}$. For each $\xi \in \mathcal{G}$, the smooth function

$$\hat{J}(\xi)\colon T^*Q \to \mathbf{R}$$

is defined by $\hat{J}(\xi)(\alpha) = J(\alpha)(\xi)$ for $\alpha \in T^*Q$.

Fact 1. *The momentum map enjoys the following properties.*

1. *J restricts to a linear map on each fiber of T^*Q.*
2. *J is Ad^*–equivariant.*
3. *$X_{\hat{J}(\xi)} = \xi_{T^*Q}$ for all $\xi \in \mathcal{G}$.*
4. *If H is a G–invariant Hamiltonian function, then J is an integral of the Hamiltonian vector field X_H.*

For details see Corollaries 4.2.9, 4.2.11, and 4.2.14 of [1]. Item (4) is called Noether's Theorem.

Fact 2. *Pseudo–Riemannian metrics on Q are in one–to–one correspondence with hyperregular Hamiltonian functions on T^*Q which restrict to a quadratic function on each fiber. Moreover, G–invariant metrics correspond to G–invariant Hamiltonian functions.*

Indeed, if $\mathcal{L}\colon TQ \to T^*Q$ denotes the Legendre transformation, or "flat map", associated to the pseudo–Riemannian metric g, and defined by

$$\mathcal{L}(v)(-) = g(v, -)$$

for $v \in TQ$, then $\mathcal{L}$ is a vector bundle isomorphism, and the function $H\colon T^*Q \to \mathbf{R}$ defined by

$$H(\alpha) = \frac{1}{2} g(\mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha)),$$

is the Hamiltonian function that corresponds to g. (See Theorem 3.6.9 and Example 3.6.10 in [1].) Note that a Hamiltonian function which restricts to a quadratic function on every fiber of T^*Q is hyperregular if and only if the fiberwise quadratic functions are all nondegenerate quadratic functions, meaning that the corresponding symmetric bilinear forms are nondegenerate.

By Theorem 3.6.2 and Theorem 3.7.1 in [1], the geodesic spray of g on TQ is $\mathcal{L}$–related to the Hamiltonian vector field X_H on T^*Q associated to H. Thus $\mathcal{L}$ carries the geodesic flow on TQ onto the Hamiltonian flow ϕ_t generated by X_H. Thus the geodesics of g are the base projections of the integral curves of X_H. This immediately implies the formula for the geodesics of g in terms of ϕ_t given next.

Proposition 1. *Suppose H is the Hamiltonian function that corresponds to the pseudo–Riemannian metric g, and let $\mathcal{L}\colon TQ \to T^*Q$ be the Legendre transformation associated to g. Let $\exp\colon TQ \to Q$ be the exponential map of g and ϕ_t the Hamiltonian flow generated by X_H. Then*

$$\exp(tv) = \pi(\phi_t(\mathcal{L}(v)))$$

for all $v \in TQ$ and $t \in \mathbf{R}$ where defined.

Given a quadratic function $C\colon \mathcal{G}^* \to \mathbf{R}$, the corresponding symmetric bilinear form B on $\mathcal{G}^*$ satisfies $C(\mu) = \frac{1}{2}B(\mu,\mu)$ for all $\mu \in \mathcal{G}^*$. Using the canonical isomorphism $\mathcal{G} \approx \mathcal{G}^{**}$, for each $\mu \in \mathcal{G}^*$, there is a unique $\hat{C}(\mu) \in \mathcal{G}$ defined by the condition

$$\nu(\hat{C}(\mu)) = B(\mu,\nu)$$

for all $\nu \in \mathcal{G}^*$.

3. The Main Result

Theorem 1. *Let Q be a smooth manifold, and let G be a Lie group acting smoothly on Q. Suppose $H_0\colon T^*Q \to \mathbf{R}$ is a hyperregular, fiberwise quadratic, G–invariant Hamiltonian function corresponding to a G–invariant pseudo–Riemannian metric g_0 on Q. Let $C\colon \mathcal{G}^* \to \mathbf{R}$ be an $Ad(G)^*$–invariant quadratic function. Define*

$$H_C = H_0 + C \circ J.$$

Then H_C is fiberwise quadratic and G–invariant.

Assume H_C is hyperregular, so that it corresponds to a G–invariant pseudo–Riemannian metric g_C on Q. Then

$$\exp_C(tv) = L_{\exp(t\hat{C}(J(\mathcal{L}_C v)))} \exp_0(t(\mathcal{L}_0^{-1}\mathcal{L}_C v))$$

for all $v \in TQ$, and $t \in \mathbf{R}$ where defined. Here $\exp\colon \mathcal{G} \to G$, $\exp_0\colon TQ \to Q$, and $\exp_C\colon TQ \to Q$ are the exponential maps of G, g_0, and g_C respectively, and $\mathcal{L}_0$ and $\mathcal{L}_C$ are the Legendre maps of g_0 and g_C respectively.

Proof: $C \circ J$ is fiberwise quadratic because J is fiberwise linear and C is quadratic. It is G–invariant because J is Ad^*–equivariant and C is $Ad(G)^*$–invariant. Thus H_C is fiberwise quadratic and G–invariant being the sum of two such functions. Therefore, assuming H_C is hyperregular, H_C corresponds to a G–invariant pseudo–Riemannian metric on Q by Fact 2.

Because H_0 is a G–invariant Hamiltonian, Noether's theorem (Fact 1(4)) implies J is an integral of X_{H_0}, which in turn implies $C \circ J$ is an integral of X_{H_0}. Thus $\{C \circ J, H_0\} = 0$, or equivalently X_{H_0} and $X_{C\circ J}$ are commuting Hamiltonian vector fields. But

$$X_{H_C} = X_{H_0} + X_{C\circ J}.$$

Therefore

$$\phi_t^C = \phi_t^0 \circ \psi_t^C = \psi_t^C \circ \phi_t^0$$

where ϕ_t^0, ϕ_t^C, and ψ_t^C denote the Hamiltonian flows generated by X_{H_0}, X_{H_C}, and $X_{C\circ J}$ respectively.

Lemma 1. *Let $\alpha \in T^*Q$, and set $\mu = J(\alpha)$.*

$$\psi_t^C(\alpha) = \mathbf{L}_{\exp(t\hat{C}(\mu))}(\alpha)$$

for all $t \in \mathbf{R}$. In particular, $X_{C\circ J}$ is a complete vector field.

Proof of Lemma: Pick a basis $\xi_1, \dots, \xi_k$ of $\mathcal{G}$. Let B be the symmetric bilinear form on $\mathcal{G}^*$ corresponding to C. Then B can be written in tensor form

$$B = \sum b^{ij} \xi_i \otimes \xi_j$$

where $b^{ij} = b^{ji} \in \mathbf{R}$. Thus, by definition of $\hat{J}$,

$$C \circ J = \frac{1}{2} \sum b^{ij} \hat{J}(\xi_i) \hat{J}(\xi_j)$$

and, by definition of $\hat{C}(\mu)$,

$$\hat{C}(\mu) = \sum b^{ij} \mu(\xi_i) \xi_j$$

for $\mu \in \mathcal{G}^*$. Let $f \colon T^*Q \to \mathbf{R}$ be smooth. Now calculate using in turn the definition of the Hamiltonian vector field, the derivation property of Poisson brackets, the symmetry of b^{ij}, and Fact 1(3).

$$\begin{aligned}
X_{C \circ J}(f) &= \{f, C \circ J\} \\
&= \frac{1}{2} \sum b^{ij} \{f, \hat{J}(\xi_i) \hat{J}(\xi_j)\} \\
&= \frac{1}{2} \sum b^{ij} (\hat{J}(\xi_i) \{f, \hat{J}(\xi_j)\} + \hat{J}(\xi_j) \{f, \hat{J}(\xi_i)\}) \\
&= \sum b^{ij} \hat{J}(\xi_i) \{f, \hat{J}(\xi_j)\} \\
&= \sum b^{ij} \hat{J}(\xi_i) (\xi_j)_{T^*Q}(f)
\end{aligned}$$

Let $\alpha \in T^*Q$, and set $\mu = J(\alpha)$. Then $\hat{J}(\xi_i)(\alpha) = J(\alpha)(\xi_i) = \mu(\xi_i)$. Thus evaluating $X_{C \circ J}$ at α gives

$$X_{C \circ J} = \sum b^{ij} \mu(\xi_i) (\xi_j)_{T^*Q} = \hat{C}(\mu)_{T^*Q}.$$

On the other hand, since $C \circ J$ is G–invariant, Noether's theorem implies J is an integral for $X_{C \circ J}$. This means

$$J(\psi_t^C(\alpha)) = J(\alpha) = \mu$$

for all $t \in \mathbf{R}$ where defined. In conclusion, the integral curve of $X_{C \circ J}$ through α equals the integral curve of $\hat{C}(\mu)_{T^*Q}$ through α. This completes the proof of the lemma.

To finish the proof of the Theorem, apply Proposition 1, Lemma 1, and the G–equivariance of π, and calculate.

$$\begin{aligned}
\exp_C(tv) &= \pi(\phi_t^C(\mathcal{L}^C(v))) \\
&= \pi(\psi_t^C \phi_t^0(\mathcal{L}_C(v))) \\
&= \pi(\mathbf{L}_{\exp(t\hat{C}(\mu))} \phi_t^0(\mathcal{L}_C(v))) \\
&= L_{\exp(t\hat{C}(\mu))} \pi(\phi_t^0(\mathcal{L}_C(v))) \\
&= L_{\exp(t\hat{C}(\mu))} \exp_0(t\mathcal{L}^{-1}(\mathcal{L}_C(v)))
\end{aligned}$$

for all $v \in TQ$, and $t \in \mathbf{R}$ where defined, and where $\mu = J(\mathcal{L}_C(v)) = J(\phi_t^0(\mathcal{L}_C(v)))$ for all t because J is an integral of X_{H_0}. In particular, if g_0 is geodesically complete, then so is g_C.

There is a deep structural relation between the dynamics of H_0 and H_C that must account, at least in part, for the simple formula relating the geodesics of g_0 and g_C. The reduced Hamiltonian functions of H_0 and H_C on every reduced phase space of T^*Q differ only by a constant and hence have identical dynamics. (See section 4.3 of [1].)

4. Concluding Remarks and Examples

Two requirements must be met if the deformation described above will produce new examples. First, $\mathcal{G}^*$ must admit a non–zero $Ad(G)^*$–invariant quadratic function C. Second, $H_0 + C \circ J$ must be hyperregular. The two requirements will be addressed in turn.

In general $\mathcal{G}^*$ need not admit a non–zero $Ad(G)^*$–invariant quadratic function. Here is an example I learned from Brad Currey. Consider the 2–dimensional solvable matrix group

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x > 0, y \in \mathbf{R} \right\}.$$

There are no non–constant analytic $Ad(G)^*$–invariant functions on $\mathcal{G}^*$ since one of the co–adjoint orbits is open in $\mathcal{G}^*$. In particular, there is no non–trivial invariant quadratic function.

Yet, there are natural situations in which invariant quadratic functions exist:

1. If $\mathcal{G}$ admits a nondegenerate $Ad(G)$–invariant symmetric bilinear form, then dually there is an $Ad(G)^*$–invariant symmetric bilinear form and corresponding quadratic function on $\mathcal{G}^*$. This occurs if G is semi–simple, by taking the Killing form, or if G is compact, by averaging any inner product on $\mathcal{G}$ over the adjoint action.
2. If G has a non–trivial center (of positive dimension), then any $\xi \neq 0$ in the center of $\mathcal{G}$ is fixed under the adjoint action. Thus the function $C \colon \mathcal{G}^* \to \mathbf{R}$ defined by $C(\alpha) = (\alpha(\xi))^2$ is an $Ad(G)^*$–invariant quadratic function on $\mathcal{G}^*$. It is well known that every nilpotent Lie group has a non–trivial center.

Even if there exists a non–zero $Ad(G)^*$–invariant quadratic function C, H_C need not be hyperregular. Let $Q = S^2$ and, let $G = S^1$ act by rotations around the north–south axis. Fix $\lambda \in \mathbf{R}$, and let $C(\mu) = -\frac{1}{2}\lambda\mu^2$ for $\mu \in \mathcal{G}^* = \mathbf{R}$. Inspection reveals that H_C is hyperregular only if $\lambda < 1$. This gives rise to the "λ–sphere" defined in [2].

However H_C is hyperregular in the following two cases:

1. If g is positive definite and C is a positive quadratic function, then H_C is positive definite on each fiber and therefore hyperregular.
2. If Q is compact and if C is sufficiently close to zero, then restricted to each fiber of T^*Q, H_C will be sufficiently close to the nondegenerate quadratic function H_0 to be nondegenerate itself. Therefore, H_C will be hyperregular.

Example: Let Q be the unit n–sphere S^n, and let G be a compact connected subgroup of $SO(n+1)$ acting in the usual way. Let C_0 be an $Ad(G)^*$–invariant positive quadratic function. Let H_0 be the Hamiltonian function for S^n. Because S^n is compact, there is a constant $k > 0$ such that for all $\lambda < k$, the function

$$H_\lambda = H_0 - \lambda C_0 \circ J$$

is hyperregular and hence corresponds to some Riemannian metric on S^n. The Theorem implies that the geodesics of this metric are expressible in terms of elementary functions because (1) the geodesics of the unit n–sphere are so expressible, and (2) the exponential map of matrix groups are so expressible.

Non–compact complete examples can be obtained by deforming the flat metric metric on $\mathbf{R}^n$. Let G be a compact connected subgroup of $SO(n)$. Let H_0 be the Hamiltonian function for $\mathbf{R}^n$, and let C_0 be a positive quadratic $Ad(G)^*$–invariant function on $\mathcal{G}^*$. Then

$$H_\lambda = H_0 + \lambda C_0 \circ J$$

is hyperregular if $\lambda \geq 0$ because H_λ will be positive definite on each fiber of T^*Q. Just as in the previous example, the Riemannian metric that corresponds to H_λ,

$\lambda \geq 0$, will be complete, and its geodesics will be expressible in terms of elementary functions.

References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics, 2nd. ed.* Benjamin/Cummings, Reading, 1978.
[2] A. Faridi and E. Schucking, Geodesics and deformed spheres. *Proc. A.M.S.* **100** (1987) 522–524.

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