ON THE CALCULUS OF PROBABILITIES

By

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Dr. Lloyd having taken the Chair, the President gave an account of some researches in the Calculus of Probabilities.

Many questions in the mathematical theory of probabilities conduct to approximate expressions of the form

$$p = \frac{2}{\sqrt{\pi}} \int_0^t dt\, e^{-t};$$

that is

$$p = \theta(t),$$

θ being the characteristic of a certain function which has been tabulated by Encke in a memoir on the Method of Least Squares, translated from the Berlin Ephemeris, in vol. ii part 7 of Taylor's Scientific Memoirs; p being the probability sought, and t an auxiliary variable.

Sir William Hamilton proposes to treat the equation

$$p = \theta(t)$$

as being in all cases rigorous, by suitably determining the auxiliary variable t, which variable he proposes to call the *argument of probability*, because it is the argument with which Encke's Table should be entered, in order to obtain from that Table the value of the probability p. He shows how to improve several of Laplace's approximate expressions for the argument t, and uses in many such questions a transformation of a certain double definite integral, of the form,

$$\frac{4s^{\frac{1}{2}}}{\pi} \int_0^r dr \int_0^\infty du\, e^{-su^2} \mathrm{U} \cos(2s^{\frac{1}{2}} ru \mathrm{V}) = \theta(r(1 + \nu_1 s^{-1} + \nu_2 s^{-2} + \ldots));$$

in which

$$\mathrm{U} = 1 + \alpha_1 u^2 + \alpha_2 u^4 + \ldots$$
$$\mathrm{V} = 1 + \beta_1 u^2 + \beta_2 u^4 + \ldots$$

while $\nu_1, \nu_2, \ldots$ depend on $\alpha_1, \ldots\ \beta_1, \ldots$ and on r; thus

$$\nu_1 = \tfrac{1}{2}\alpha_1 - \beta_1 r^2.$$

The function θ has the same form as before, so that if, for sufficiently large values of the quantity s (which represents, in many questions, the number of observations or events to be combined), a probability p can be expressed, exactly or nearly, by the foregoing double definite integral, then the *argument* t, of this probability p, will be expressed nearly by the formula,

$$t = r(1 + \nu_1 s^{-1} + \nu_2 s^{-2}).$$

Numerical examples were given, in which the approximations thus obtained appeared to be very close. For instance, if a common die (supposed to be perfectly fair) be thrown six times, the probability that the sum of the six numbers which turn up in these six throws shall not be less than 18, nor more than 24, is represented rigorously by the integral

$$p = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} dx \, \frac{\sin 7x}{\sin x} \left(\frac{\sin 6x}{6 \sin x} \right)^6, \quad \text{or by the fraction } \tfrac{27448}{46656};$$

while the approximate formula deduced by the foregoing method gives 27449 for the numerator of this fraction, or for the product $6^6 p$; the error of the resulting probability being therefore in this case only 6^{-6}. The advantage of the method is that the quantity which has here been called the argument of probability, depends in general more simply than does the probability itself on the conditions of a question; while the introduction of this new conception and nomenclature allows some of the most important known results respecting the mean results of many observations to be enunciated in a simple and elegant manner.