ELEMENTARY PROOF, THAT EIGHT PERIMETERS, OF THE REGULAR INSCRIBED POLYGON OF TWENTY SIDES, EXCEED TWENTY-FIVE DIAMETERS OF THE CIRCLE

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Elementary Proof, that Eight Perimeters, of the Regular inscribed Polygon of Twenty Sides, exceed Twenty-five Diameters of the Circle. By Professor Sir William Rowan Hamilton, *LL.D., &c.**

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It was proved by Archimedes that 71 perimeters, of a regular polygon of 96 sides inscribed in a circle, exceed 223 diameters; whence follows easily the well-known theorem, that eight circumferences of a circle exceed twenty-five diameters, or that $8\pi > 25$. Yet the following elementary proof, that eight perimeters of the regular inscribed polygon of *twenty sides* are greater than twenty-five diameters, has not perhaps hitherto appeared in any scientific† work or periodical; and if a page of the Philosophical Magazine can be spared for its insertion, some readers may find it interesting from its extreme simplicity. In fact, for completely understanding it, no preparation is required beyond the four first Books of Euclid, and the few first Rules of Arithmetic, together with some rudimentary knowledge of the connexion between arithmetic and geometry.

1. It follows from the Fourth Book of Euclid's 'Elements,' that the rectangle under the side of the regular decagon inscribed in a circle, and the same side increased by the radius, is equal to the square of the radius. But the product of the two numbers, 791 and 2071, whereof the latter is equal to the former increased by 1280, is less than the square of 1280 (because 1638161 is less than 1638400). If then the radius be divided into 1280 equal parts, the side of the inscribed decagon must be greater than a line which consists of 791 such parts; or briefly, if the radius be equal to 1280, the side of the decagon exceeds 791.

2. When a diameter of a circle bisects a chord, the square of the chord is equal, by the Third Book, to the rectangle under the doubled segments of that diameter. But the product of the two numbers, 125 and 4995, which together make up 5120, or the double of the double of 1280, is less than the square of 791 (because 624375 is less than 625681). If then the radius be still represented by 1280, and therefore the doubled diameter by 5120, and if the bisected chord be a side of the regular decagon, and therefore greater (by what has been just proved) than 791, the lesser segment of the diameter is greater than the line represented by 125.

* Communicated by the Author.

† A sketch of the proof was published, at the request of a friend, in an eminent literary journal last summer, but in a connexion not likely to attract the attention of mathematical readers in general. At all events, it pretends to no merit but that of brevity, and the simplicity of the principles on which it rests.

3. The rectangle under this doubled segment and the radius, is equal to the square of the side of the regular inscribed polygon of twenty sides. But the product of 125 and 1280 is equal to the square of 400; and if the radius be still 1280, it has been proved that the doubled segment exceeds 125; with this representation of the radius, the side of the inscribed polygon of twenty sides exceeds therefore the line represented by 400; and the perimeter of that polygon is consequently greater than 8000.

4. Dividing then the numbers 1280 and 8000 by their greatest common measure 320, we find that if the radius be now represented by the number 4, or the diameter by 8, the perimeter of the polygon will be greater than the line represented by 25; or in other words, that *eight perimeters of the regular inscribed polygon of twenty sides* (and by a still stronger reason, *eight circumferences of the circle* itself) *exceed twenty five diameters.*

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