MEMORANDUM RESPECTING A NEW SYSTEM OF ROOTS OF UNITY

By

William Rowan Hamilton

(Philosophical Magazine, 12 (1856), p. 446.)

Edited by David R. Wilkins

2000

Memorandum respecting a new System of Roots of Unity. By Sir WILLIAM ROWAN HAMILTON, *LL.D., M.R.I.A., F.R.A.S., &c., Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland*.*

[*The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 4th series, vol. xii (1856), p. 446.]

I have lately been led to the conception of a new system, or rather *family of systems*, of *non-commutative roots of unity*, which are entirely distinct from the ijk of the quaternions, though having some general analogy thereto; and which admit, even more easily than the quaternion symbols do, of *geometrical interpretation*. In the system which seems at present to be the most interesting one, among those included in this new family, I assume three symbols, ι, κ, λ, such that

$$\left.\begin{array}{c}\iota^2 = 1, \quad \kappa^3 = 1, \quad \lambda^5 = 1,\\ \lambda = \iota\kappa;\end{array}\right\} \tag{A}$$

where $\iota\kappa$ must be *distinguished* from $\kappa\iota$, since otherwise we should have $\lambda^6 = 1$, $\lambda = 1$. As a very simple *specimen* of the symbolical conclusions deduced from these fundamental assumptions, I may mention that if we make

$$\mu = \iota\kappa^2 = \lambda\iota\lambda,$$

we shall have also†

$$\mu^5 = 1, \quad \lambda = \mu\iota\mu;$$

so that μ is a new fifth root of unity, connected with the former fifth root λ by relations of perfect reciprocity. A long train of such symbolical deductions is found to follow: and every one of the results may be *interpreted*, as having reference to the passage from *face to face* (or from corner to corner) of the *icosahedron* (or of the dodecahedron): on which account, I am at present disposed to give the name of the "Icosian Calculus," to this new system of symbols, and of rules for their operation. Some additional remarks on this subject may soon

* Communicated by the Author.

† In fact, by (A),

$$\iota\kappa = (\iota\kappa)^{-4} = (\kappa^{-1}\iota^{-1})^4 = (\kappa^2\iota)^4,$$

$$1 = \iota \,.\, \iota\kappa \,.\, \kappa^2 = \iota(\kappa^2\iota)^4\kappa^2 = (\iota\kappa^2)^5;$$

also

$$\mu\iota\mu = \mu\kappa^2 = \iota\kappa^4 = \iota\kappa = \lambda.$$

be offered to the Philosophical Magazine, under the title, already sanctioned by the Editors, of "Extensions of the Quaternions."

Observatory of Trinity College, Dublin,
October 29, 1856.