ON THE CALCULATION OF THE NUMERICAL VALUES OF A CERTAIN CLASS OF MULTIPLE AND DEFINITE INTEGRALS

By

William Rowan Hamilton

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On the Calculation of the Numerical Values of a certain class of Multiple and Definite Integrals. By Sir William Rowan Hamilton, *LL.D., M.R.I.A., F.R.A.S., &c., Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland**.

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Section I.

[1.] The results, in part numerical, of which a sketch is here to be given, may serve to illustrate some points in the theory of functions of large numbers, and in that of definite and multiple integrals. In stating them, it will be convenient to employ a notation which I have formerly published, and have often found to be useful; namely the following,

$$\mathrm{I}_t = \int_0^t dt; \tag{1}$$

or more fully,

$$\mathrm{I}_t ft = \int_0^t ft\, dt; \tag*{(1)$'$}$$

with which I am now disposed to combine this other symbol,

$$\mathrm{J}_t = \int_t^\infty dt; \tag{2}$$

in such a manner as to write,

$$\mathrm{J}_t ft = \int_t^\infty ft\, dt; \tag*{(2)$'$}$$

and therefore

$$\mathrm{I}_t + \mathrm{J}_t = \int_0^\infty dt. \tag{3}$$

I shall also retain, for the present, the known notation of Vandermonde for factorials, which has been described and used by Lacroix, and in which, for any positive whole value of n,

$$[x]^n = x(x-1)(x-2)\dots(x-n+1); \tag{4}$$

* Communicated by the Author.

so that there are the transformations,

$$[x]^n = [x]^m [x-m]^{n-m} = [x]^{n+m} : [x-n]^m, \ \text{\&c.}; \tag{4$'$}$$

which are extended by definition to the case of null and negative indices, and give, in particular,

$$[0]^{-n} = \frac{1}{[n]^n} = \frac{1}{1\,.\,2\,.\,3\,\ldots\,n}. \tag{4$''$}$$

For example,

$$(1+x)^n = \textstyle\sum_{m=0}^{m=\infty} [n]^m [0]^{-m} x^m. \tag{5}$$

It is easy, if it be desired, to translate these into other known notations of factorials, but they may suffice on the present occasion.

[2.] With the notations above described, it is evident that

$$\mathrm{I}_t^n 1 = [0]^{-n} t^n; \tag{6}$$

and more generally, that

$$\mathrm{I}_t^n t^m = \frac{t^{m+n}}{[m+n]^n} = [m]^{-n} t^{m+n}. \tag{6$'$}$$

Hence results the series,

$$(1 + \mathrm{I}_t + \mathrm{I}_t^2 + \cdots)1 = (1 - \mathrm{I}_t)^{-1} 1 = e^t; \tag{7}$$

and accordingly, we have the finite relation,

$$\mathrm{I}_t e^t = e^t - 1. \tag{7$'$}$$

The imaginary equation,

$$e^{t\sqrt{-1}} = (1 - \sqrt{-1}\,\mathrm{I}_t)^{-1} 1, \tag{8}$$

breaks up into the two real expressions,

$$\cos t = (1 + \mathrm{I}_t^2)^{-1} 1, \tag{8$'$}$$

$$\sin t = \mathrm{I}_t (1 + \mathrm{I}_2^2)^{-1} 1. \tag{8$''$}$$

The series of Taylor may be concisely denoted by the formula,

$$f(x+t) = (1 - \mathrm{I}_t D_x)^{-1} fx; \tag{9}$$

and accordingly,

$$\mathrm{I}_t D_x f(x+t) = \mathrm{I}_t f'(x+t) = f(x+t) - f(x). \tag{9$'$}$$

And other elementary applications of the symbol I_t may easily be assigned, whereof some have been elsewhere indicated.

[3.] The following investigations relate chiefly to the function,

$$F_{n,r}t = \mathrm{I}_t^n (1 + 4\mathrm{I}_t^2)^{-r-\frac{1}{2}} 1; \tag{10}$$

or

$$F_{n,r}t = \mathrm{I}_t^n (1 + 4\mathrm{I}_t^2)^{-r} ft, \tag{10$'$}$$

where

$$ft = F_{0,0}t = (1 + 4\mathrm{I}_t^2)^{-\frac{1}{2}} 1. \tag{11}$$

Developing by (5) and (6), and observing that

$$2^{2m}[-\tfrac{1}{2}]^m = (-1)^m [2m]^m, \tag{12}$$

and that therefore

$$2^{2m}[-\tfrac{1}{2}]^m [0]^{-m} [0]^{-2m} = (-1)^m ([0]^{-m})^2, \tag{12$'$}$$

we find the well-known series,

$$ft = 1 - \left(\frac{t}{1}\right)^2 + \left(\frac{t^2}{1\,.\,2}\right)^2 - \left(\frac{t^3}{1\,.\,2\,.\,3}\right)^2 + \&\text{c.}, \tag{13}$$

which admits of being summed as follows,

$$ft = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\omega \, \cos(2t \cos\omega); \tag{13$'$}$$

the function ft being thus equal to a celebrated definite integral, which is important in the mathematical theory of heat, and has been treated by Fourier and by Poisson.

[4.] It was pointed out* by the great analyst last named, that if there were written the equation,

$$y = \int_0^\pi \cos(k \cos\omega) \, d\omega, \tag{14}$$

so that, in our recent notation,

$$y = \pi f\left(\frac{k}{2}\right), \tag{14$'$}$$

then for large, real, and positive values of k, the function $y\sqrt{k}$ might be developed in a series of the form,

$$y\sqrt{k} = \left(A + \frac{A'}{k} + \frac{A''}{k^2} + \&\text{c.}\right) \cos k + \left(B + \frac{B'}{k} + \frac{B''}{k^2} + \&\text{c.}\right) \sin k; \tag{15}$$

* In his Second Memoir on the Distribution of Heat in Solid Bodies, *Journal de l'Ecole Polytechnique*, tome xii. cahier 19, Paris, 1823, pages 349, &c.

where a certain differential equation of the second order, which $y\sqrt{k}$ was obliged to satisfy, was proved to be sufficient for the successive deduction of as many of the other constant coefficients, $A', A'', \ldots$ and $B', B'', \ldots$ of the series, as might be desired, through an assigned system of equations of condition, after the two first constants, A and B, were determined; and certain processes of definite integration gave for them the following values,

$$A = B = \sqrt{\pi}; \tag{15)$'$}$$

so that when k is very large, we have nearly, as Poisson, showed,

$$y\sqrt{k} = (\cos k + \sin k)\sqrt{\pi}. \tag{15)$''$}$$

[5.] In my own paper on Fluctuating Functions*, I suggested a different process for arriving at this important formula of approximation, $(15)''$, which, with some slight variation, may be briefly stated as follows. Introducing the two definite integrals,

$$\left.\begin{aligned} A_t &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\omega\ \cos(2t \operatorname{vers}\omega), \\ B_t &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\omega\ \sin(2t \operatorname{vers}\omega), \end{aligned}\right\} \tag{16}$$

which give the following *rigorous transformation* of the expression $(13)'$, or of the function ft,

$$ft = A_t \cos 2t + B_t \sin 2t; \tag{16)$'$}$$

and employing the *limiting values*,

$$\left.\begin{aligned} \lim_{t=\infty} .t^{\frac{1}{2}} A_t &= \frac{2}{\pi}\int_0^{\infty} dx\ \cos(x^2) = (2\pi)^{-\frac{1}{2}}, \\ \lim_{t=\infty} .t^{\frac{1}{2}} B_t &= \frac{2}{\pi}\int_0^{\infty} dx\ \sin(x^2) = (2\pi)^{-\frac{1}{2}}; \end{aligned}\right\} \tag{16)$''$}$$

(which two last and well-known integrals have indeed been used by Poisson also,) I obtained (and, as I thought, more rapidly than by his method) the following *approximate expression*, equivalent to that lately marked as $(15)''$, for large, real, and positive values of t:

$$ft = (\pi t)^{-\frac{1}{2}} \sin\left(2t + \frac{\pi}{4}\right); \tag{17}$$

which is sufficient to show that the *large and positive roots* of the transcendental equation,

$$\int_0^{\frac{\pi}{2}} d\omega\ \cos(2t\cos\omega) = 0, \tag{17)$'$}$$

* In the Transactions of the Royal Irish Academy, vol. xix. part 2, p. 313; Dublin 1843. Several copies of the paper alluded to were distributed at Manchester in 1842, during the Meeting of the British Association for that year: one was accepted by the great Jacobi.

are (as is known)* very nearly of the form

$$t = \frac{n\pi}{2} - \frac{\pi}{8}, \tag{17}''$$

where n is a large whole number.

[6.] Poisson does not appear to have required, for the applications which he wished to make, any more than the *two* constants, which he called A and B, of his descending series (15); although (as has been said) he showed how all the *other* constants of that series *could be successively* computed, from them, if it had been thought necessary or desirable to do so. In other words, he seems to have been content with assigning the values (15)′, and the formula (15)″, as sufficient for the purpose which he had in view. In my own paper, already cited, I gave the *general term of the descending series* for ft, by assigning a formula, which (with one or two unimportant differences of notation) was the following:

$$(\pi t)^{\frac{1}{2}} ft = \textstyle\sum_{m=0}^{m=\infty} [0]^{-m} ([-\frac{1}{2}]^m)^2 (4t)^{-m} \cos\left(2t - \dfrac{\pi}{4} - \dfrac{m\pi}{2}\right). \tag{18}$$

As an example of the numerical approximation attainable hereby, when t was a moderately large number, (not necessarily whole,) I assumed $t = 20$; and found that *sixty terms* of the ultimately convergent, but initially divergent series (13), gave

$$\begin{aligned} f(20) &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\omega\, \cos(40 \cos\omega) \\ &= +7\,447\,387\,396\,709\,949 \cdot 965\,7957 \\ &\quad\; -7\,447\,387\,396\,709\,949 \cdot 958\,4289 \\ &= +0 \cdot 007\,3668; \end{aligned} \tag{19}$$

while only *three terms* of the ultimately divergent, but initially convergent series (18) sufficed to give almost exactly the same result, under the form,

$$\begin{aligned} f(20) &= \left(1 - \frac{9}{204800}\right) \frac{\cos 86^\circ\, 49'\, 52''}{\sqrt{20\pi}} + \frac{1}{320} \frac{\sin 86^\circ\, 49'\, 52''}{\sqrt{20\pi}} \\ &= 0 \cdot 0069736 + 0 \cdot 0003936 = +0 \cdot 0073672. \end{aligned} \tag{19}'$$

[7.] The function ft becomes infinitely small, when t becomes infinitely great, on account of the indefinite fluctuation which $\cos(2t \cos\omega)$ then undergoes, under the sign of integration in (13)′; so that we may write

$$F_{0,0}\infty = f\infty = 0. \tag{20}$$

* It must, I think, be a misprint, by which, in p. 353 of Poisson's memoir, the expression $k = i\pi + \dfrac{\pi}{4}$, is given, instead of $k = i\pi - \dfrac{\pi}{4}$, for the large roots of the transcendental equation $y = 0$. It is remarkable, however, that this error of sign, if it be such, does not appear to have had any influence on the correctness of the physical conclusions of the memoir: which, no doubt, arises from the circumstance that the number i is treated as infinite, in the applications.

But it is by no means true that the value of this *other* series,

$$F_{1,0}t = I_t ft = \frac{t}{1} - \frac{t}{3}\left(\frac{t}{1}\right)^2 + \frac{t}{5}\left(\frac{t}{1\,.\,2}\right)^2 + \&\text{c.}, \tag{21}$$

which may be expressed by the definite integral,

$$F_{1,0}t = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\omega\ \sec\omega\ \sin(2t\cos\omega), \tag{21$'$}$$

is infinitesimal when t is infinite. On the contrary, by making

$$2t\cos\omega = x, \quad d\omega\ \sec\omega = -\frac{dx}{x}\left(1 - \frac{x^2}{4t^2}\right)^{-\frac{1}{2}}, \tag{22}$$

the integral (21)$'$ becomes, at the limit in question,

$$F_{1,0}\infty = \frac{1}{\pi}\int_0^{\infty} \frac{dx}{x}\sin x = \tfrac{1}{2}. \tag{21$''$}$$

Accordingly I verified, many years ago, that the series (21) takes *nearly* this constant value, $\frac{1}{2}$, when t is a large and positive number. But I have lately been led to inquire what is the *correction* to be applied to this approximate value, in order to obtain a more accurate numerical estimate of the function $F_{1,0}t$, or of the integral $I_t ft$, when t is large. In other words, having here, by (3) and (21)$''$, the *rigorous* relation,

$$F_{1,0}t = I_t ft = \tfrac{1}{2} - J_t ft, \tag{23}$$

I wished to evaluate, at least *approximately*, this *other* definite integral, $-J_t ft$, for large and positive values of t. And the result to which I have arrived may be considered to be a very simple one; namely, that

$$-J_t ft = D_t^{-1} ft; \tag{24}$$

where $D_t^{-1} ft$ is a development analogous to the series (18), and reproduces that series, when the operation D_t is performed.

[8.] As an example, it may be sufficient here to observe that if we thus operate by D_t on the function,

$$f\grave{}t = \left(1 - \frac{129}{2^9 t^2}\right)\frac{\sin\left(2t - \frac{\pi}{4}\right)}{2\sqrt{\pi t}} - \frac{5\cos\left(2t - \frac{\pi}{4}\right)}{2^5 t\sqrt{\pi t}}, \tag{25}$$

and suppress $t^{-\frac{1}{2}}$ in the result, we are led to this other function of t,

$$D_t f\grave{}t = \left(1 - \frac{9}{2^9 t^2}\right)\frac{\cos\left(2t - \frac{\pi}{4}\right)}{\sqrt{\pi t}} + \frac{\sin\left(2t - \frac{\pi}{4}\right)}{2^4 t\sqrt{\pi t}}; \tag{25$'$}$$

which coincides, so far as it has been developed, with the expression (18) for ft: so that we may write, as at least approxmimately true, the equation

$$f^{\backprime}t = D_t^{-1} ft. \tag{25''}$$

Substituting the value 20 for t, in order to obtain an arithmetical comparison of results, we find

$$\begin{aligned} f^{\backprime}(20) &= \left(1 - \frac{129}{204800}\right) \frac{\sin 86^\circ\, 49'\, 52''}{\sqrt{20\pi}} - \frac{\cos 86^\circ\, 49'\, 52''}{128\sqrt{20\pi}} \\ &= 0 \cdot 062942 - 0 \cdot 000054 = +0 \cdot 062888; \end{aligned} \tag{26}$$

which ought, if the present theory be correct, to be nearly equal to the definite integral, $-J_t ft$, for the case where $t = 20$. In other words, I am thus led to expect, after adding the constant term $\frac{1}{2}$, that the value of the connected integral,

$$I_t ft = \pi^{-1} \int_0^{\frac{\pi}{2}} d\omega \, \sec\omega \, \sin(40 \cos\omega), \tag{26'}$$

must be nearly equal to the following number,

$$+0 \cdot 562888. \tag{26''}$$

And accordingly, when this last integral (26)′ is developed by means of the *ascending* series (21), I find that the sum of the first sixty terms (beyond which it would be useless for the present purpose to go) gives, as the difference of two large but nearly equal numbers, (which are *themselves* of interest, as representing certain *other* definite integrals,) the value:

$$\begin{aligned} \pi^{-1} \int_0^{\frac{\pi}{2}} d\omega \, \sec\omega \, \sin(40 \cos\omega), &= +3\,772\,428\,770\,679\,800 \cdot 537\,7058 \\ &\quad -3\,772\,428\,770\,679\,799 \cdot 974\,8177 \\ &= +0 \cdot 562\,888\,1; \end{aligned} \tag{26'''}$$

which can scarcely (as I estimate) be wrong in its last figure, the calculation having been pushed to more decimals than are here set down; and which exhibits as close an agreement as could be desired with the result (26)″ of an entirely different method.

[9.] It must however be stated, that in extending the method thus exemplified to higher orders of integrals, the development denoted by $D_t^{-n} ft$, or the definite and multiple integral $(-J_t)^n ft$, to which it is equivalent, comes to be *corrected*, in passing to the *other* integral $I_t^n ft$, not by a *constant term*, such as $\frac{1}{2}$, but by a *finite algebraical function*, which I shall here call $f_n t$, and of which I happened to perceive the existence and the law, while pursuing some unpublished researches respecting vibration, a considerable time ago. Lest anything should prevent me from soon submitting a continuation of the present little paper, (for I wish to write on one or two other subjects,) let me at least be permitted now to mention, that the spirit of the process alluded to, for determining this finite and algebraical *correction**,

$$I_t^n ft - (-J_t)^n ft = T_t^n ft - D_t^{-n} ft = f_n t, \tag{27}$$

(where $D_t^{-n} ft$ still denotes a descending and periodical series, analogous to and including those above marked (18) and (25),) consists in *developing the algebraical expression* (10), (for the case $r = 0$, but with a corresponding development for the more general case,) *according to descending powers of the symbol* I_t, and *retaining only those terms in which the exponent of that symbol is positive or zero*: which process gives the formula,

$$f_n t = \tfrac{1}{2} I_t^{n-1} (1 + 2^{-2} I_t^{-2})^{-\frac{1}{2}} 1 = (\tfrac{1}{2} I_t^{n-1} - \tfrac{1}{16} I_t^{n-3} + \tfrac{3}{256} I_t^{n-5} - \ldots) 1; \tag{28}$$

that is, by (5) and (6),

$$f_n t = \textstyle\sum_{m=0}^{m=\infty} 2^{-2m-1} [-\tfrac{1}{2}]^m [0]^{-m} [0]^{-(n-2m-1)} t^{n-2m-1}; \tag*{(28)$'$}$$

where the series may be written as if it were an infinite one, but the terms involving negative powers of t have each a null coefficient, and are in this question to be suppressed.

For instance, I have arithmetically verified, at least for the case $t = 10$, that the two finite algebraical functions,

$$f_6 t = \frac{t^5}{240} - \frac{t^3}{96} + \frac{3t}{256}, \tag*{(28)$''$}$$

$$f_7 t = \frac{t^6}{1440} - \frac{t^4}{384} + \frac{3t^2}{512} - \frac{5}{2048}, \tag*{(28)$'''$}$$

express the values of the two following sums or differences of integrals,

$$f_6 t = I_t^6 ft - J_t^6 ft, \tag*{(27)$'$}$$

$$f_7 t = I_t^7 ft + J_t^7 ft; \tag*{(27)$''$}$$

the calculations having been carried to several places of decimals, and the integrals $I_t^6 ft$, $I_t^7 ft$ having each been found as the difference of two large numbers.

Observatory of Trinity College, Dublin,
September 29, 1857.

[To be continued.]

* Although this *algebraical part*, $f_n t$, of the multiple integral $I_t^n ft$, is *here* spoken of as a *correction* of the *periodical part*, denoted above by $D_t^{-n} ft$, yet for *large* and *positive values* of t it is, *arithmetically* speaking, by much the *most important portion* of the whole: and accordingly I perceived (although I did not publish) it long ago, whereas it is only very lately that I have been led to *combine* with it the *trigonometrical series*, deduced by a sort of extension of Poisson's analysis.—When I thus venture to speak of any result on this subject as being my own, it is with every deference to the superior knowledge of other Correspondents of this Magazine, who may be able to point out many anticipations of which I am not yet informed. The formulæ (27) (28) are perhaps those which have the best chance of being new.