ON THE GEOMETRICAL DEMONSTRATION OF SOME THEOREMS OBTAINED BY MEANS OF THE QUATERNION ANALYSIS

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(Proceedings of the Royal Irish Academy, 5 (1853), pp. 407–415.)

Edited by David R. Wilkins

2000

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Communicated April 11, 1853.

[*Proceedings of the Royal Irish Academy*, vol. 5 (1853), pp. 407–415.]

Professor Sir William Rowan Hamilton, LL.D., communicated a few remarks on the geometrical demonstration of some theorems lately obtained by means of the quaternion analysis.

1. (*Rule of Derivation*).—Let CD be a given right line, bisected in L, and LI a given perpendicular thereto. Assume at pleasure any point P on the same plane, and *derive* from it another point Q, by the conditions that

$$CQ \perp DP, \quad CQ \,.\, DP = CD \,.\, LI,$$

and that the rotation from the direction of CQ to the direction of DP shall be towards the same hand as that from LD to LI. From Q derive R, and from R derive S, &c., as Q has been derived from P. It is required to investigate geometrically the chief properties of the resulting arrangement.

2. (*First Case*: $LI < LD$).—If the given line LI be less than LD, then, parallel to the latter line, there can be drawn, through the extremity I of the former, a chord AB of the circle (L), described on CD as diameter: and we may suppose that the point B is nearer than A to C. Then,

$$CQ \,.\, DP = CA \,.\, DA = CB \,.\, DB,$$

and

$$ACQ = ADP, \quad BCQ = BDP,$$

even if *signs of angles* (or directions of rotation) be attended to. Thus the two triangles ACQ, PDA, and in like manner the two triangles BCQ, PDB, are equiangular, but oppositely turned, like a figure and its reflexion in a plane mirror, or like the two triangles ABC, BAD: which relations we may perhaps not inconveniently express, by saying that in each of these three pairs the two triangles are *inversely similar*, or by writing,

$$ABC \propto' BAD, \quad ACQ \propto' PDA, \quad BCQ \propto' PDB; \tag{1}$$

and then either of these two latter formulæ of inverse similarity of triangles is sufficient to express the *rule of derivation* of the point Q from P.

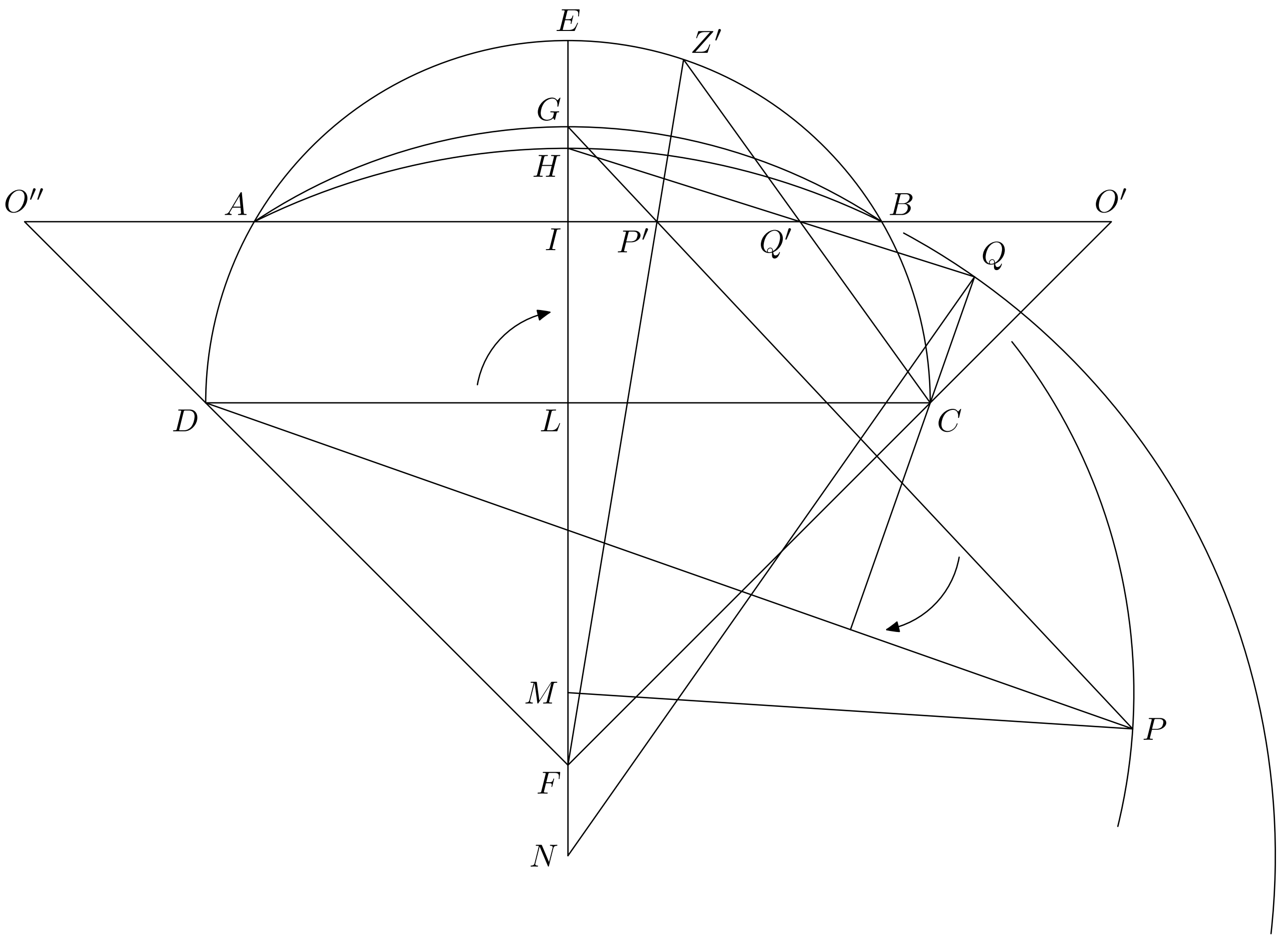

Fig. 1

3. Hence, attending still to signs of angles, we may see (even without referring to the figure) that

$$CQA = PAD, \quad CQB = PBD;$$

and that therefore

$$\begin{aligned} AQB &= CQB - CQA = PBD - PAD \\ &= (ADP + DPA) - (BDP + DPB) = ADB - APB = ACB - APB; \end{aligned}$$

or that

$$APB + AQB = ACB. \tag{2}$$

The *sum of the two angles subtended by the fixed chord* AB, at the assumed point P and at the derived point Q, is therefore *constant*, and equal to the angle which the same chord subtends at the point C (or D); these angles being supposed to change their signs when their vertices cross that fixed chord. (This result was given in the Philosophical Magazine for February 1853, as one of several which had been obtained by applying quaternions to the question.)

4. In like manner if we continue to derive successively other points, $R, S, T, U, \ldots$ we shall have

$$AQB + ARB = ACB, \quad \text{\&c.},$$

and therefore

$$\left.\begin{aligned} APB = ARB = ATB = \ldots, \\ AQB = ASB = AUB = \ldots; \end{aligned}\right\} \tag{3}$$

the *alternate points*, $P, R, T, \ldots$ are therefore situated on *one circular locus* (M), and the *other* alternate and derived points $Q, S, U, \ldots$ are on *another circular locus* (N); or rather these two sets of alternate points are contained on two circular *segments*, both resting on the fixed chord AB as the *common base* (as stated in the Phil. Mag. just cited).

5. It is evident also that if E, F be (as in fig. 1) the summits of the two semicircles on CD, of which the former contains the chord, and if MN be the centres of the two loci, then

$$APB = AMI, \quad AQB = ANI, \quad ACB = ALI = 2AFI;$$

so that, by (2),

$$AFI - ANI = AMI - AFI, \quad \text{or} \quad FAN = MAF: \tag{4}$$

wherefore the centres M, N are *harmonic conjugates* with respect to the given circle (L), or its diameter EF, and we may write

$$LM \,.\, LN = LF^2. \tag{5}$$

6. The similar triangles (1) give

$$\frac{QA}{PA} = \frac{QC}{DA}, \quad \frac{QB}{PB} = \frac{QC}{DB},$$

and therefore

$$\frac{QB \,.\, PA}{QA \,.\, PB} = \frac{DA}{DB} = \frac{CB}{CA} = \text{const.}, \tag{6}$$

(as stated in the same number of the Magazine). Hence (as there stated) the successively and *directly derived points* $Q, R, S, \ldots$ must *tend indefinitely to coincide with the fixed point* B, and in like manner the *inversely derived points* $R, Q, P, \ldots$ must tend indefinitely to coincide with the *other fixed point* A, as the limits of their positions, on account of the geometrical progressions of the quotients of distances from those two fixed points, *wherever the first point* P or S of the direct or inverse derivation may be: unless it happen to be exactly *at* either of those two fixed points A and B, in which case the derivation will produce no change of place. (It might therefore be not too fanciful to say that A and B are respectively positions of *unstable* and *stable* equilibrium for the *direct* mode of derivation, but of stable and unstable for the inverse mode.)

7. Let G and H be summits of the loci (M) and (N), so chosen that the lines PG and QH, crossing the fixed chord AB in the points P' and Q', are both internal or both external bisectors of the angles APB, AQB; and prolong FC, FD, or the external bisectors of ACB, ADB, to meet the same fixed chord prolonged in O' and O''. Then the formula (6) will still hold good, even with attention to the *signs of the segments*, after changing P, Q, C, D, to P', Q', O', O''; we have therefore the two following equations between *anharmonic ratios*,

$$(ABP'\infty) = (ABQ'O'), \quad (ABP'O'') = (ABQ'\infty), \tag{7}$$

which give

$$\frac{Q'O'}{AO'} = \frac{Q'B}{P'B} = \frac{O''A}{O''P'}, \tag{8}$$

and consequently,

$$Q'O' \,.\, O''P' = AO' \,.\, O''A = IO'^2 - IA^2 = \text{const.}, \tag{9}$$

where the constant may be variously transformed: for instance we may write,

$$Q'O' \,.\, Q''P' = 2FI \,.\, LI. \tag{10}$$

8. The equations (7) show that we have the *two involutions*,

$$(AB, P'O', Q'\infty) \quad \text{and} \quad (AB, P'\infty, Q'O''); \tag{11}$$

if then from *any point* Z, assumed at pleasure *on any one of the three circles*, we draw three successive chords of that circle, ZZ' through P', $Z'Z''$ through Q', and $Z''Z'''$ through O', or else ZZ' through Q', $Z'Z''$ through P', and $Z''Z'''$ through O'', the fourth or *closing chord* $Z'''Z$ will in each case pass through infinity; or in other words, this closing chord will be *parallel to the fixed chord* AB. In particular, by placing Z, and therefore also Z''' at F, which will oblige Z'' to be at C or D, we see that the lines FP', CQ' (or FQ', CR', &c.) must intersect each other (at an angle of 45°) in some point Z' of the given circle (L); and that the same thing holds for the lines FQ', DP' (or FR', DQ', &c.), as stated to the Academy at the Meeting of February 28th, 1853 (see the Proceedings of that date). And thus we might prove in a new way the indefinite *tendency* if the points $Q', R', \ldots$ on the fixed chord, and therefore also the corresponding tendency of the points $Q, R, \ldots$ in the plane, to *coincidence with the fixed point* B (that point being still supposed to be *real*).

9. By placing Z alternately at G and at H, it may be shewn, in like manner, that the alternate lines $PQ', RS', TU', \ldots$ all pass through one fixed point, namely, the point where GO' intersects (M) again, after meeting it in the summit G; the other alternate lines $QR', ST', \ldots$ all pass through another fixed point, namely, the second intersection of HO' with (N); again, $RQ', TS', \ldots$ pass through the analogous intersection of GO'' with (M); and $QP', SR', \ldots$ through that of HO'' with (N). The opposite summits E, G', H' might be employed in the same way to furnish other theorems, which would not, however, be essentially different from these.

10. (*Second Case*: $LI > LD$).—When the given line LI is greater than LD, or than the radius of the circle (L), that circle is no longer met by the line $O''IO'$ in any *real points*, A, B; but it is obvious, from the known principles of modern geometry, that this latter line is *still* the *common radical axis* of three circles (L) (M) (N), whereof the two latter have still their centres M and N *harmonic conjugates* with respect to the given circle (L), and are still the loci of the two systems of alternate points, $P, R, T, \ldots$ and $Q, S, U, \ldots$ namely, the assumed point and those derived from it by the rule stated in Art. 1, taken alternately; because that rule did not involve any reference to the points of intersection A and B. These circular loci will still have real summits G, H, which will serve to determine real points, $P', Q', R', \ldots$ upon the radical axis, by the alternate lines $GP, HQ, GR, \ldots$; and the same relations of *homography* and *involution* will still hold good, conducting to the same theorems of *real intersections of lines* as before, although the points A and B on the *circle* (L) have now become *imaginary*. For example, the lines FP', CQ', or the lines FQ', DP', still cross at an angle of 45° in some point Z' on that given circle (L): but because the radical axis is now *beyond* that circle, there is now *no tendency to any convergence* of the points $Q'R'S' \ldots$, nor of $R'Q'P' \ldots$, nor consequently of the points $Q\,R\,S \ldots$, nor of $R\,Q\,P \ldots$, *to any fixed position.*

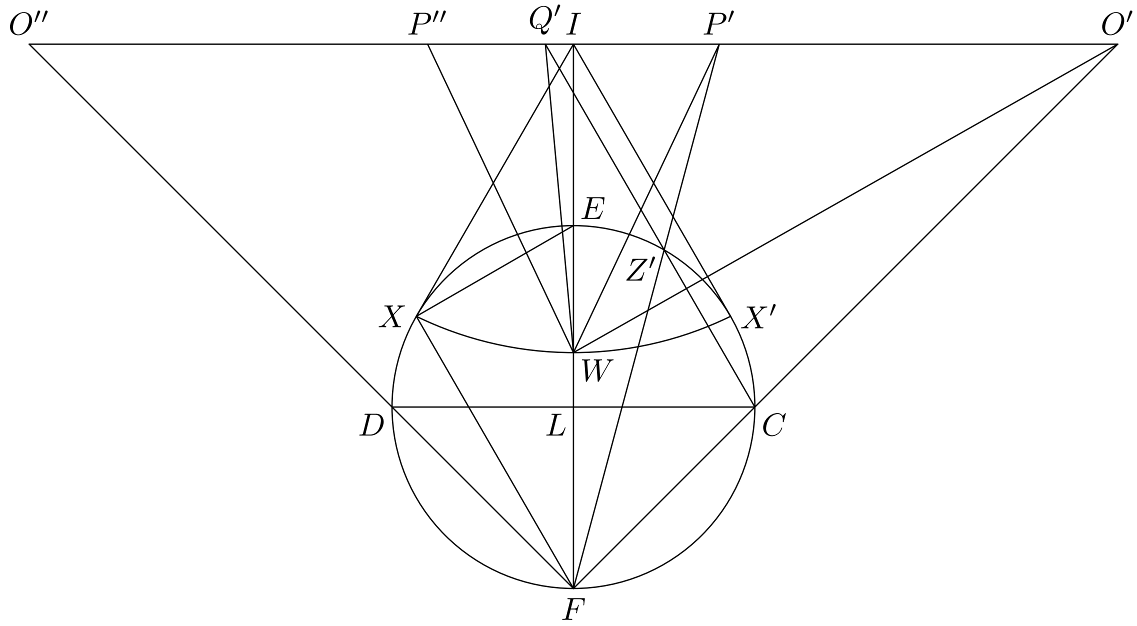

Fig. 2

11. There *may* however be, in this second case, when A and B are imaginary, a constant *circulation in a period*, among the derived points in the plane, or on the axis. For we have now, in the formula (9) (compare fig. 2),

$$-IA^2 = IL^2 - LA^2 = IL^2 - LX^2 = IX^2 = IW^2, \tag{12}$$

if IX be a tangent (now real) from I to (L), and if W be one of the two fixed points in which the common orthogonals to the three circles (now really) intersect each other: thus (9) becomes, in the present case,

$$O'Q' \,.\, O'P'' = Q'O' \,.\, O''P' = IO'^2 + IW^2 = O'W^2, \tag{13}$$

if P'' be so taken on the radical axis that I shall bisect $P'P''$ hence

$$O'WQ' = (\,WP''Q' =)\, IP'W;$$

subtracting therefore $O'WP'$ from each, and observing that the triangles $WO'I$, EFX are equiangular, we obtain the formula,

$$P'WQ' = IO'W = EFX = \text{const.} \tag{14}$$

If then the constant angle thus subtended by $P'Q'$ at W be commensurable with a right angle, or in other words if EX be a *side* or a *diagonal* of a *regular polygon* with n sides inscribed in (L), n derivations of Q' from P' will answer to one or more complete revolutions of the line WP', and will conduct from P' to P' again, and therefore also from P to P, if the number n be *even*: in this case, then, there will be a *period of n points* $P\,Q\,R\,\ldots$, arranged half on one locus, and half upon the other. For example, if $LI = FE = 2LD$, the chord EX will be the side of an inscribed *hexagon*; and wherever P may be assumed, we shall have a period of *six* points, $P\,Q\,R\,S\,T\,U$, three $(P\,R\,T)$ on one locus, and three $(Q\,S\,U)$ on the other. But if n be *odd* (for instance, if EX be the side of a regular *pentagon*), then the result of n derivations gives indeed the initial position P' *on the axis*, but this position now answers *in the plane* not to the first assumed point P on (M), but to a certain other point on (N): and the period therefore now consists of $2n$ points in the plane, whereof n are on the circle (M) and the n others on the alternate circle (N). An outline of these results respecting *periods of points* was lately submitted to the Academy, in the communication of last February.

12. (*Third Case*: $LI = LD$).—In the intermediate case, where the given line LI is equal to LD, the radical axis becomes a common tangent at E to the three circles, the centres M, N being harmonic conjugates as before; and because all former theorems respecting intersections of lines hold good, the lines FP', CQ' still cross on (L); and therefore the points $Q', R', S', \ldots$ and in like manner $R', Q', P', \ldots$ and consequently also the points $Q, R, S, \ldots$ and $R, Q, P, \ldots$ (the lines GPP', HQQ', &c. being now obtained by lines drawn from the summits G and H remote from the common summit E), must all indefinitely tend to that fixed position E as a limit. As regards the law of this tendency, it may be expressed by either of the formulæ

$$Q'O' \,.\, O''P' = EO'^2; \quad EP' \,.\, EQ' = EO' \,.\, Q'P'; \tag{15}$$

or more clearly by the following,

$$\frac{1}{EQ'} - \frac{1}{EP'} = \frac{1}{EO'} = \text{const.} \tag{16}$$

And instead of treating (as has here been done) this third case, or the case of *contact* at E, of the line $O'O''$ with the circle (L), as a *limit* of the first case, or of the case of *intersection* of that line with that circle in two real and distinct points A, B, we might have treated it *directly*, by a shorter but less general method.*

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* Some remarks on this case have appeared in the number of the Philosophical Magazine for the present month (April, 1853).

13. The readers of the excellent *Traité de Géometrie de Position*, by M. Chasles (Paris, 1852), with which the author of the present paper does not pretend to be more than partially acquainted, will not fail to recognise the *double homographic division* of the radical axis (whence such divisions on the circular loci can easily be obtained), with the *double points* A, B, and with O', O'' as *homologues of infinity.* That theory of homographic division may also be employed in the treatment of the case where A and B become imaginary, without any previous reference to the case where those two points are real. It was, however, almost entirely through the quaternion method, including, indeed (as lately stated to the Academy), some use of *biquaternions*, or combining the employment of the old imaginary of algebra with that of his peculiar symbols $i\,j\,k$, that Sir W. R. H. was led, not merely to the *results*, but even to the chief *constructions* of the present paper. In particular, he was led to perceive the theorem of *circulation* in Art. 11, and to make out the geometrical construction given in that article for exhibiting it, by endeavouring to interpret formulæ which presented themselves to him, in investigating the integral of an equation in finite differences of quaternions, which integral was found to contain a *periodical term.*