ON A GENERAL METHOD OF EXPRESSING THE PATHS OF LIGHT, AND OF THE PLANETS, BY THE COEFFICIENTS OF A CHARACTERISTIC FUNCTION

By

William Rowan Hamilton

(Dublin University Review and Quarterly Magazine, 1 (1833), pp. 795–826.)

Edited by David R. Wilkins

1999

On a general Method of expressing the Paths of Light, and of the Planets, by the Coefficients of a Characteristic Function.

By William R. Hamilton, *Royal Astronomer of Ireland.*

[*Dublin University Review and Quarterly Magazine*, Vol. I, 1833, pp. 795–826.]

The law of seeing in straight lines was known from the infancy of optics, being in a manner forced upon men's notice by the most familiar and constant experience. It could not fail to be observed that when a man looked at any object, he had it in his power to interrupt his vision of that object, and hide it at pleasure from his view, by interposing his hand between his eyes and it; and that then, by withdrawing his hand, he could see the object as before: and thus the notion of straight lines or rays of communication, between a visible object and a seeing eye, must very easily and early have arisen. This notion of straight lines of vision, was of of course confirmed by the obvious remark that objects can usually be seen on looking through a straight but not through a bent tube; and the most familiar facts of perspective supplied, we may suppose, new confirmations and new uses of the principle. A globe, for example, from whatever point it may be viewed, appears to have a circular outline; while a plate, or a round table, seems oval when viewed obliquely: and these facts may have been explained, and reduced to mathematical reasoning, by shewing that the straight rays or lines of vision, which touch any one given globe and pass through any one given point, are arranged in a hollow cone of a perfectly circular shape; but that the straight rays, which connect an eye with the round edge of a plate or table, compose, when they are oblique, an elliptical or oval cone. The same principle, of seeing in straight lines, must have been continually employed from the earliest times in the explanation of other familiar appearances, and in interpreting the testimony of sight respecting the places of visible bodies. It was, for example, an essential element in ancient as in modern astronomy.

The shapes and sizes of shadows, again, could not fail to suggest the notion of straight illuminating rays: although opinions, now rejected, respecting the nature of light and vision, led some of the ancients to distinguish the lines of luminous from those of visual communication, and to regard the latter as a kind of feelers by which the eye became aware of the presence of visible objects. It appears, however, that many persons held, even in the infancy of Optics, the modern view of the subject, and attributed vision, as well as illumination, to an influence proceeding from the visible or luminous body. But what finally established this view, and along with it the belief of a finite velocity of progress of the luminous influence, was the discovery made by Roemer, of the gradual propagation of light from objects to the eye, in the instance of the satellites of Jupiter; of which we have good reason to believe, from astronomical observation, that the eclipses are never seen by us, till more than half an hour after they have happened; the interval, besides, being found to be so much the greater, as Jupiter is more distant from the Earth. Galileo had indeed proposed terrestrial experiments to measure the velocity of light, which he believed to be finite; and Des Cartes, who held that the communication of light was instantaneous, had perceived that astronomical consequences ought to follow, if the propagation of light were gradual: but experiments such as Galileo proposed, were not, and could not, be made on a scale sufficient for the purpose; and the state of astronomical observation in the time of Des Cartes did not permit him to verify the consequences which he perceived, and seemed rather to justify the use that he made of their non-verification, as an argument against the opinion with which he had shown them to be logically connected. But when astronomers had actually observed appearances, which seemed and still seem explicable only by this opinion of the gradual propagation of light from objects to the eye, the opinion itself became required, and was adopted, in the legitimate process of induction.

By such steps, then, it has become an established theorem, fundamental in optical science, that the communication, whether between an illuminating body and a body illuminated, or between an object seen and a beholding eye, is effected by the gradual but very rapid passage of some thing, or influence, or state, called light, from the luminous or visible body, along mathematical or physical lines, usually called *rays*, and found to be, under the most common circumstances, exactly or nearly straight.

Again, it was very early perceived that in appearances connected with mirrors, flat or curved, the luminous or visual communication is effected in bent lines. When we look into a flat mirror, and seem to see an object, such as a candle, behind it, we should err if we were to extend to this new case the rules of our more familiar experience. We should not now come to touch the candle by continuing the straight line from the eye to a hand or other obstacle, so placed between the eye and the mirror as to hide the candle; this line continued would meet the mirror in a certain place from which it would be necessary to draw a new and different straight line, if we wished to reach the real or tangible candle: and the whole bent line, made up of these two straight parts, is found to be now the line of visual communication, and is to be regarded now as the linear path of the light. An opaque obstacle, placed any where on either part of this bent line, is found to hide the reflected candle from the eye; but an obstacle, placed any where else, produces no such interruption. And the law was very early discovered, that for every such bent line of luminous or visual communication, the angle between any two successive straight parts is bisected by the normal, or perpendicular, to the mirror at the point of bending.

Another early and important observation, was that of the broken or refracted lines of communication, between an object in water and an eye in air, and generally between a point in one ordinary medium and a point in another. A valuable series of experiments on such refraction was made and recorded by Ptolemy; but it was not till long afterwards that the law was discovered by Snellius. He found that if two lengths, in a certain ratio or proportion determined by the natures of the two media, be measured, from the point of breaking, or of bending, on the refracted ray and on the incident ray prolonged, these lengths have one common projection on the refracting surface, or on its tangent plane. This law of ordinary refraction has since been improved by Newton's discovery of the different refrangibility of the differently coloured rays; and has been applied to explain and to calculate the apparent elevation of the stars, produced by the atmosphere of the earth.

The phenomena presented by the passage of light through crystals were not observed until more lately. Bartolinus seems to have been the first to notice the double refraction of Iceland spar; and Huygens first discovered the laws of this refraction. The more complicated double refraction produced by biaxial crystals was not observed until the present century; and the discovery of conical refraction in such crystals is still more recent, the experiments of Professor Lloyd on arragonite (undertaken at my request) having been only made last year.

For the explanation of the laws of the linear propagation of light, two principal theories have been proposed, which still divide the suffrages of scientific men.

The theory of Newton is well known. He compared the propagation of light to the motion of projectiles; and as, according to that First Law of Motion, of which he had himself established the truth by so extensive and beautiful an induction, an ordinary projectile continues in rectilinear and uniform progress, except so far as its course is retarded or disturbed by the influence of some foreign body; so, he thought, do luminous and visible objects shoot off little luminous or light-making projectiles, which then, until they are accelerated or retarded, or deflected one way or another, by the attractions or repulsions of some refracting or reflecting medium, continue to move uniformly in straight lines, either because they are not acted on at all by foreign bodies, or because the foreign actions are nearly equal on all sides, and thus destroy or neutralise each other. This theory was very generally received by mathematicians during the last century, and still has numerous supporters.

Another theory however, proposed about the same time by another great philosopher, has appeared to derive some strong confirmations from modern inductive discoveries. This other is the theory of Huygens, who compared the gradual propagation of light, not to the motion of a projectile, but to the spreading of sound through air, or of waves through water. It was, according to him, no *thing*, in the ordinary sense, no *body*, which moved from the sun to the earth, or from a visible object to the eye; but a *state*, a *motion*, a *disturbance*, was first in one place, and afterwards in another. As, when we hear a cannon which has been fired at a distance, no bullet, no particle even of air, makes its way from the cannon to our ears; but only the aerial motion spreads, the air near the cannon is disturbed first, then that which is a little farther, and last of all the air that touches us. Or like the waves that spread and grow upon some peaceful lake, when a pebble has stirred its surface; the floating water-lilies rise and fall, but scarcely quit their place, while the enlarging wave passes on and moves them in succession. So that great ocean of ether which bathes the farthest stars, is ever newly stirred, by waves that spread and grow, from every source of light, till they move and agitate the whole with their minute vibrations: yet like sounds through air, or waves on water, these multitudinous disturbances make no confusion, but freely mix and cross, while each retains its identity, and keeps the impress of its proper origin. Such is the view of Light which Huygens adopted, and which justly bears his name; because, whatever kindred thoughts occurred to others before, he first shewed clearly how this view conducted to the laws of optics, by combining it with that essential principle of the undulatory theory which was first discovered by himself, the principle of accumulated disturbance.

According to this principle, the minute vibrations of the elastic luminous ether cannot perceptibly affect our eyes, cannot produce any sensible light, unless they combine and concur in a great and, as it were, infinite multitude; and on the other hand, such combination is possible, because particular or secondary waves are supposed in this theory to spread from every vibrating particle, as from a separate centre, with a rapidity of propagation determined by the nature of the medium. And hence it comes, thought Huygens, that light in any one uniform medium diffuses itself only in straight lines, so as only to reach those parts of space to which a straight path lies open from its origin; because an opaque obstacle, obstructing such straight progress, though it does not hinder the spreading of weak particular waves into the space behind it, yet prevents their accumulation within that space into one grand general wave, of strength enough to generate light. This want of accumulation of separate vibrations behind an obstacle, was elegantly proved by Huygens: the mutual destruction of such vibrations by interference, is an important addition to the theory, which has been made by Young and by Fresnel. Analogous explanations have been offered for the laws of reflexion and refraction.

Whether we adopt the Newtonian or the Huygenian, or any other physical theory, for the explanation of the laws that regulate the lines of luminous or visual communication, we may regard these laws themselves, and the properties and relations of these linear paths of light, as an important separate study, and as constituting a separate science, called often *mathematical optics*. This science of the laws and relations of luminous rays, is, however, itself a branch of another more general science, which may perhaps be called the *Theory of Systems of Rays*. I have published, in the XVth and XVIth volumes of the Transactions of the Royal Irish Academy, a series of investigations in that theory; and have attempted to introduce a new principle and method for the study of optical systems. Another supplementary memoir, which has been lately printed for the same Transactions, will appear in the XVIIth volume; but having been requested to resume the subject here, and to offer briefly some new illustrations of my view, I shall make some preliminary remarks on the state of deductive optics, and on the importance of a general method.

The science of optics, like every other physical science, has two different directions of progress, which have been called the ascending and the descending scale, the inductive and the deductive method, the way of analysis and of synthesis. In every physical science, we must ascend from facts to laws, by the way of induction and analysis; and must descend from laws to consequences, by the deductive and synthetic way. We must gather and groupe appearances, until the scientific imagination discerns their hidden law, and unity arises from variety: and then from unity must re-deduce variety, and force the discovered law to utter its revelations of the future.

It was with such convictions that Newton, when approaching to the close of his optical labours, and looking back on his own work, remarked, in the spirit of Bacon, that "As in Mathematics, so in Natural Philosophy, the investigation of difficult things by the method of Analysis ought ever to precede the method of Composition. This analysis consists in making experiments and observations, and in drawing general conclusions from them by induction, and admitting of no objections against the conclusions but such as are drawn from experiments or other certain truths." "And although the arguing from experiments and observations by induction be no demonstration of general conclusions; yet it is the best way of arguing which the nature of things admits of, and may be looked upon as so much the stronger, by how much the induction is more general. And if no exception occur from phenomena, the conclusion may be pronounced generally. But if at any time afterwards, any exception shall occur from experiments, it may then begin to be pronounced with such exceptions as occur. By this way of analysis, we may proceed from compounds to ingredients, and from motions to the forces producing them; and, in general, from effects to their causes, and from particular causes to more general ones, till the argument end in the most general. This is the method of analysis: and the synthesis consists in assuming the causes discovered, and established as principles, and by them explaining the phenomena proceeding from them, and proving the explanations." "And if Natural Philosophy in all its parts, by pursuing this method, shall at length be perfected, the bounds of Moral Philosophy will also be enlarged. For, so far as we can know by Natural Philosophy, what is the First Cause, what power He has over us, and what benefits we receive from Him, so far our duty towards Him, as well as that towards one another, will appear to us by the light of nature."

In the science of optics, which has engaged the attention of almost every mathematician for the last two thousand years, many great discoveries have been attained by both these ways. It is, however, remarkable that, while the laws of this science admit of being stated in at least as purely mathematical a form as any other physical results, their mathematical consequences have been far less fully traced than the consequences of many other laws; and that while modern experiments have added so much to the *inductive* progress of optics, the *deductive* has profited so little in proportion from the power of the modern algebra.

It was known to Euclid and to Ptolemy, that the communication between visible objects and a beholding eye is usually effected in straight lines; and that when the line of communication is bent, by reflexion, at any point of a plane or of a spheric mirror, the angle of bending at this point, between the two straight parts of the bent line, is bisected by the normal to the mirror. It was known also that this law extends to successive reflexions. Optical induction was therefore sufficiently advanced two thousand years ago, to have enabled a mathematician to understand, and, so far as depended on the knowledge of physical laws, to resolve the following problem: to determine the arrangement of the final straight rays, or lines of vision, along which a shifting eye should look, in order to see a given luminous point, reflected by a combination of two given spherical mirrors. Yet, of two capital deductions respecting this arrangement, without which its theory must be regarded as very far from perfect—namely, that the final rays are in general *tangents to a pair*, and that they are *perpendicular to a series* of surfaces—the one is a theorem new and little known, and the other is still under dispute. For Malus, who first discovered that the rays of an ordinary reflected or refracted system are in general tangents to a pair of caustic surfaces, was led, by the complexity of his calculations, to deny the general existence (discovered by Huygens) of surfaces perpendicular to such rays; and the objection of Malus has been lately revived by an eminent analyst of Italy, in a valuable memoir on caustics, which was published last year in the correspondence of the observatory of Brussels.

To multiply such instances of the existing imperfection of mathematical or deductive optics would be an unpleasant task, and might appear an attempt to depreciate the merit of living mathematicians. It is better to ascend to the source of the imperfection, the want of a general method, a presiding idea, to guide and assist the deduction. For although the deductive, as opposed to the inductive process, may be called itself a *method*, yet so wide and varied is its range, that it needs the guidance of some one central principle, to give it continuity and power.

Those who have meditated on the beauty and utility, in theoretical mechanics, of the general method of Lagrange—who have felt the power and dignity of that central dynamical theorem which he deduced, in the *Méchanique Analytique*, from a combination of the principle of virtual velocities with the principle of D'Alembert—and who have appreciated the simplicity and harmony which he introduced into the research of the planetary perturbations, by the idea of the variation of parameters, and the differentials of the disturbing function, must feel that mathematical optics can only *then* attain a coordinate rank with mathematical mechanics, or with dynamical astronomy, in beauty, power, and harmony, when it shall possess an appropriate method, and become the unfolding of a central idea.

This fundamental want forced itself long ago on my attention; and I have long been in possession of a method, by which it seems to me to be removed. But in thinking so, I am conscious of the danger of a bias. It may happen to me, as to others, that a meditation which has long been dwelt on shall assume an unreal importance; and that a method which has for a long time been practised shall acquire an only seeming facility. It must remain for others to judge how far my attempts have been successful, and how far they require to be completed, or set aside, in the future progress of the science.

Meanwhile it appears that if a general method in deductive optics can be attained at all, it must flow from some law or principle, itself of the highest generality, and among the highest results of induction. What, then, may we consider as the highest and most general axiom, (in the Baconian sense,) to which optical induction has attained, respecting the rules and conditions of the lines of visual and luminous communication? The answer, I think, must be, the principle or law, called usually the Law of Least Action; suggested by questionable views, but established on the widest induction, and embracing every known combination of media, and every straight, or bent, or curved line, ordinary or extraordinary, along which light (whatever light may be) extends its influence successively in space and time: namely, that this linear path of light, from one point to another, is always found to be such, that if it be compared with the other infinitely various lines by which in thought and in geometry the same two points might be connected, a certain integral or sum, called often *Action*, and depending by fixed rules on the length, and shape, and position of the path, and on the media which are traversed by it, is less than all the similar integrals for the other neighbouring lines, or, at least, possesses, with respect to them, a certain *stationary* property. From this Law, then, which may, perhaps, be named the LAW OF STATIONARY ACTION, it seems that we may most fitly and with best hope set out, in the synthetic or deductive process, and in search of a mathematical method.

Accordingly, from this known law of least or stationary action, I deduced (long since) another connected and coextensive principle, which may be called, by analogy, the LAW OF VARYING ACTION, and which seems to offer naturally a method such as we are seeking: the one law being as it were the last step in the ascending scale of induction, respecting linear paths of light, while the other law may usefully be made the first in the descending and deductive way. And my chief purpose, in the present paper, is to offer a few illustrations and consequences of these two coordinate laws.

The former of these two laws was discovered in the following manner. The elementary principle of straight rays shewed that light, under the most simple and usual circumstances, employs the direct, and, therefore, the shortest course to pass from one point to another. Again, it was a very early discovery, (attributed by Laplace to Ptolemy,) that in the case of a plane mirror, the bent line formed by the incident and reflected rays is shorter than any other bent line, having the same extremities, and having its point of bending on the mirror. These facts were thought by some to be instances and results of the simplicity and economy of nature; and Fermat, whose researches on maxima and minima are claimed by the continental mathematicians as the germ of the differential calculus, sought anxiously to trace some similar economy in the more complex case of refraction. He believed that by a metaphysical or cosmological necessity, arising from the simplicity of the universe, light always takes the course which it can traverse in the shortest time. To reconcile this metaphysical opinion with the law of refraction, discovered experimentally by Snellius, Fermat was led to suppose that the two lengths, or *indices*, which Snellius had measured on the incident ray prolonged and on the refracted ray, and had observed to have one common projection on a refracting plane, are inversely proportional to the two successive velocities of the light before and after refraction, and therefore that the velocity of light is diminished on entering those denser media in which it is observed to approach the perpendicular: for Fermat believed that the time of propagation of light along a line bent by refraction was represented by the sum of the two products, of the incident portion multiplied by the index of the first medium, and of the refracted portion multiplied by the index of the second medium; because he found, by his mathematical method, that this sum was less, in the case of a plane refractor, than if light went by any other than its actual path from one given point to another; and because he perceived that the supposition of a velocity inversely as the index, reconciled his mathematical discovery of the minimum of the foregoing sum with his cosmological principle of least time. Des Cartes attacked Fermat's opinions respecting light, but Leibnitz zealously defended them; and Huygens was led, by reasonings of a very different kind, to adopt Fermat's conclusions of a velocity inversely as the index, and of a *minimum time* of propagation of light, in passing from one given point to another through an ordinary refracting plane. Newton, however, by his theory of emission and attraction, was led to conclude that the velocity of light was *directly*, not *inversely*, as the index, and that it was *increased* instead of being *diminished* on entering a denser medium; a result incompatible with the theorem of shortest time in refraction. The theorem of shortest time was accordingly abandoned by many, and among the rest by Maupertuis, who, however, proposed in its stead, as a new cosmological principle, that celebrated *law of least action* which has since acquired so high a rank in mathematical physics, by the improvements of Euler and Lagrange. Maupertuis gave the name of *action* to the product of space and velocity, or rather to the sum of all such products for the various elements of any motion; conceiving that the more space has been traversed and the less time it has been traversed in, the more action may be considered to have been expended: and by combining this idea of action with Newton's estimate of the velocity of light, as increased by a denser medium, and as proportional to the refracting index, and with Fermat's mathematical theorem of the minimum sum of the products of paths and indices in ordinary refraction at a plane, he concluded that the course chosen by light corresponded always to the *least possible action*, though not always to the least possible time. He proposed this view as reconciling physical and metaphysical principles, which the results of Newton had seemed to put in opposition to each other; and he soon proceeded to extend his law of least action to the phenomena of the shock of bodies. Euler, attached to Maupertuis, and pleased with these novel results, employed his own great mathematical powers to prove that the law of least action extends to all the curves described by points under the influence of central forces; or, to speak more precisely, that if any such curve be compared with any other curve between the same extremities, which differs from it indefinitely little in shape and in position, and may be imagined to be described by a neighbouring point with the same law of velocity, and if we give the name of *action* to the integral of the product of the velocity and an element of a curve, the difference of the two neighbouring values of this action will be indefinitely less than the greatest linear distance (itself indefinitely small) between the two near curves; a theorem which I think may be advantageously expressed by saying that the action is *stationary.*

Lagrange extended this theorem of Euler to the motion of a system of points or bodies which act in any manner on each other; the action being in this case the sum of the masses by the foregoing integrals. Laplace has also extended the use of the principle in optics, by applying it to the refraction of crystals; and has pointed out an analogous principle in mechanics, for all imaginable connexions between force and velocity. But although the law of least action has thus attained a rank among the highest theorems of physics, yet its pretensions to a cosmological necessity, on the ground of economy in the universe, are now generally rejected. And the rejection appears just, for this, among other reasons, that the quantity pretended to be economised is in fact often lavishly expended. In optics, for example, though the sum of the incident and reflected portions of the path of light, in a single ordinary reflexion at a plane, is always the shortest of any, yet in reflexion at a curved mirror this economy is often violated. If an eye be placed in the interior but not at the centre of a reflecting hollow sphere, it may see itself reflected in two opposite points, of which one indeed is the nearest to it, but the other on the contrary is the furthest; so that of the two different paths of light, corresponding to these two opposite points, the one indeed is the shortest, but the other is the longest of any. In mathematical language, the integral called action, instead of being always a minimum, is often a maximum; and often it is neither the one nor the other: though it has always a certain *stationary* property, of a kind which has been already alluded to, and which will soon be more fully explained. We cannot, therefore, suppose the economy of this quantity to have been designed in the divine idea of the universe: though a simplicity of some high kind may be believed to be included in that idea. And though we may retain the name of *action* to denote the stationary integral to which it has become appropriated—which we may do without adopting either the metaphysical or (in optics) the physical opinions that first suggested the name—yet we ought not (I think) to retain the epithet *least*: but rather to adopt the alteration proposed above, and to speak, in mechanics and in optics, of the *Law of Stationary Action.*

To illustrate this great law, and that other general law, of *varying action*, which I have deduced from it, we may conveniently consider first the simple case of rectilinear paths of light. For the rectilinear course, which is evidently the *shortest* of any, is also distinguished from all others by a certain *stationary property*, and *law of variation*, which, being included in the general laws of stationary and varying action, may serve as preparatory examples.

The length V of any given line, straight or curved, may evidently be denoted by the following integral:

$$V = \int dV = \int \sqrt{dx^2 + dy^2 + dz^2}. \tag{1}$$

If now we pass from this to another neighbouring line, having the same extremities, and suppose that the several points of the latter line are connected with those of the former, by equations between their co-ordinates, of the form

$$x_\varepsilon = x + \varepsilon\xi, \quad y_\varepsilon = y + \varepsilon\eta, \quad z_\varepsilon = z + \varepsilon\zeta, \tag{2}$$

ε being any small constant, and ξ, η, ζ, being any arbitrary functions of x, y, z, which vanish for the extreme values of those variables, that is, for the extreme points of the given line, and do not become infinite for any of the intermediate points, nor for the value $\varepsilon = 0$, though they may in general involve the arbitrary constant ε; the length V_ε of the new line may be represented by the new integral,

$$\begin{aligned} V_\varepsilon &= \int \sqrt{dx_\varepsilon^2 + dy_\varepsilon^2 + dz_\varepsilon^2} \\ &= \int \sqrt{(dx + \varepsilon d\xi)^2 + (dy + \varepsilon d\eta)^2 + (dz + \varepsilon d\zeta)^2}, \end{aligned} \tag{3}$$

taken between the same extreme values of x, y, z, as the former; and this new length V_ε may be considered as a function of ε, which tends to the old length V, when ε tends to 0, the quotient

$$\frac{1}{\varepsilon}(V_\varepsilon - V)$$

tending in general at the same time to a finite limit, which may be thus expressed,

$$\begin{aligned} \lim.\ \frac{1}{\varepsilon}(V_\varepsilon - V) &= \int \frac{dx\,d\xi + dy\,d\eta + dz\,d\zeta}{\sqrt{dx^2 + dy^2 + dz^2}} \\ &= -\int \left(\xi\,d\frac{dx}{dV} + \eta\,d\frac{dy}{dV} + \zeta\,d\frac{dz}{dV}\right); \end{aligned} \tag{4}$$

the last of these forms being obtained from the preceding by integrating by parts, and by employing the condition already mentioned, that the functions ξ, η, ζ, vanish at the extremities of the integral. When the original line is such that the limit (4) vanishes, independently of the forms of the functions ξ, η, ζ, and therefore that the difference of the lengths $V_\varepsilon - V$ bears ultimately an evanescent ratio to the small quantity ε, (which quantity determines the difference between the second line and the first, and bears itself a finite ratio to the greatest distance between these two lines,) we may say that the original line has a *stationary length*, V, as compared with all the lines between the same extremities, which differ from it infinitely little in shape and in position. And since it easily follows, from the last form of the limit (4), that this limit cannot vanish independently of the forms of ξ, η, ζ, unless

$$d\frac{dx}{dV} = 0, \quad d\frac{dy}{dV} = 0, \quad d\frac{dz}{dV} = 0, \tag{5}$$

that is, unless the ratios

$$\frac{dx}{dV}, \quad \frac{dy}{dV}, \quad \frac{dz}{dV},$$

are constant throughout the original line, but that the limit vanishes when this condition is satisfied, we see that the property of *stationary length* belongs (in free space) to straight lines and to such only. The foregoing proof of this property of the straight line may, perhaps, be useful to those who are not familiar with the Calculus of Variations.

To illustrate, by examples, this stationary property of the length of a straight line, let us consider such a line as the common chord of a series of circular arcs, and compare its length with theirs, and theirs with one another. The length of the straight line being called V, let $\frac{1}{2}\varepsilon V$ be the height or sagitta of the circular arch upon this chord; so that $\frac{1}{2}(\varepsilon + \varepsilon^{-1})V$ shall be the diameter of the circle, and ε the trigonometric tangent of the quarter of an arc having the same number of degrees, to a radius equal to unity: we shall then have the following expression for the length V_ε of the circular arch upon the given chord V,

$$V_\varepsilon = V(\varepsilon + \varepsilon^{-1}) \tan.^{-1} \varepsilon. \tag{6}$$

This expression may be put under the form,

$$\frac{V_\varepsilon}{V} = 1 + 4 \left(\int_0^\varepsilon d\varepsilon \right)^2 \varepsilon^{-3} \int_0^\varepsilon \frac{\varepsilon^2 \, d\varepsilon}{(1+\varepsilon^2)^2}, \tag{7}$$

which shows not only that the ratio of the circular arch to its chord is always > 1, but also, that since

$$\frac{dV_\varepsilon}{d\varepsilon} = 4V \int_0^\varepsilon \left(\varepsilon^{-3} \int_0^\varepsilon \frac{\varepsilon^2 \, d\varepsilon}{(1+\varepsilon^2)^2} \right) d\varepsilon, \tag{8}$$

the arch V_ε increases continually with its height at an increasing rate; its differential coefficient being positive and increasing, when ε is positive and increases, but vanishing with ε, and showing, therefore, that in this series of circular arcs and chord the property of stationary length belongs to the straight line only.

Again, we may imagine a series of semi-ellipses upon a given common axis V, the other axis conjugate to this being a variable quantity εV. The length of such a semi-elliptic arch is

$$V_\varepsilon = V \int_0^{\frac{\pi}{2}} (\cos.\phi^2 + \varepsilon^2 \sin.\phi^2)^{\frac{1}{2}} \, d\phi, \tag{9}$$

an expression which may be thus transformed,

$$\frac{V_\varepsilon}{V} = 1 + \left(\int_0^\varepsilon d\varepsilon \right)^2 \int_0^{\frac{\pi}{2}} \frac{\sin.\phi^2 \cos.\phi^2 \, d\phi}{(\cos.\phi^2 + \varepsilon^2 \sin.\phi^2)^{\frac{3}{2}}}; \tag{10}$$

thus the ratio of the elliptic arch V_ε to its given base or axis V is not only greater than unity, and continually increases with the height, but increases at an increasing rate, which vanishes for an evanescent height; so that in this series of semi-elliptic arcs and axis, the latter alone has the property of stationary length.

In more familiar words, if we construct on a base of a given length, suppose one hundred feet, a series of circular or of semi-elliptic arches, having that base for chord or for axis, the lengths of those arches will not only increase with their heights, but every additional foot or inch of height will augment the length more than the foregoing foot or inch had done; and the lower or flatter any two such arches are made, the less will be the difference of their lengths as compared with the difference of their heights, till the one difference becomes less than any fraction that can be named of the other. For example, if we construct, on the supposed base of one hundred feet, two circular arches, the first fifty feet high, the second fifty-one feet high, of which the first will thus be a semicircle, and the second greater than a semicircle, the difference of lengths of these two arches will be a little more than double the difference of their heights, that is, it will be about two feet; but if on the same base we construct one circular arch with only one foot of height, and another with only two feet, the difference of lengths of these two low arches will not be quite an inch, though the difference of their heights remains a foot as before; and if we imagine the two circular arches, on the same base or common chord of one hundred feet, to have their heights reduced to one and two inches respectively, the difference of their lengths will thereby be reduced to less than the hundred-and-fiftieth part of an inch.

We see then that a straight ray, or rectilinear path of light, from one given point to another, has a *stationary length*, as compared with all the lines which differ little from it in shape and in position, and which are drawn between the same extremities. If, however, we suppose the extremities of the neighbouring line to differ from those of the ray, we shall then obtain in general a *varying* instead of stationary length. To investigate the law of this variation, which is the simplest case of the second general law above proposed to be illustrated, we may resume the foregoing comparison of the lengths V, V_ε, of any two neighbouring lines; supposing now that these two lines have different extremities, or in other words, that the functions ξ, η, ζ, do not vanish at the limits of the integral. The integration by parts gives now, along with the last expression (4) for the limit of

$$\frac{1}{\varepsilon}(V_\varepsilon - V),$$

the following additional terms,

$$\xi\frac{dx}{dV} + \eta\frac{dy}{dV} + \zeta\frac{dz}{dV} + \xi'\frac{d'x'}{d'V} + \eta'\frac{d'y'}{d'V} + \zeta'\frac{d'z'}{d'V},$$

which belong to the extremities of the given line, the accented being the initial quantities, and d' referring to the infinitesimal changes produced by a motion of the initial point along the initial element of the line, so that $d'V$ is this initial element taken negatively,

$$d'V = -\sqrt{d'x'^2 + d'y'^2 + d'z'^2}; \tag{11}$$

when, therefore, the last integral (4) vanishes, by the original line being straight, and when we compare this line with another infinitely near, the *law of varying length* is expressed by the following equation:

$$\begin{aligned} \lim. \frac{1}{\varepsilon}(V_\varepsilon - V) &= \xi\frac{dx}{dV} + \eta\frac{dy}{dV} + \zeta\frac{dz}{dV} + \xi'\frac{d'x'}{d'V} + \eta'\frac{d'y'}{d'V} + \zeta'\frac{d'z'}{d'V} \\ &= (\xi - \xi')\frac{dx}{dV} + (\eta - \eta')\frac{dy}{dV} + (\zeta - \zeta')\frac{dz}{dV}; \end{aligned} \tag{12}$$

it may also be thus expressed,

$$\delta V = \frac{dx}{dV}(\delta x - \delta x') + \frac{dy}{dV}(\delta y - \delta y') + \frac{dz}{dV}(\delta z - \delta z'), \tag{13}$$

and shows that the length $V + \delta V$ of any other line which differs infinitely little from the straight ray in shape and in position, may be considered as equal to its own projection on the ray.

It must be observed that in certain singular cases, the distance between two lines may be made less, throughout, than any quantity assigned, without causing thereby their lengths to tend to equality. For example, a given straight line may be subdivided into a great number of small parts, equal or unequal, and on each part a semicircle may be constructed; and then the waving line composed of the small but numerous semicircumferences will every where be little distant from the given straight line, and may be made as little distant as we please, to any degree short of perfect coincidence; while yet the length of the undulating line will not tend to become equal to the length of the straight line, but will bear to that length a constant ratio greater than unity, namely the ratio of π to 2. But it is evident that such cases as these are excluded from the foregoing reasoning, which supposes an approach of the one line to the other in shape, as well as a diminution of the linear distance between them.

From the *law of varying length* of a straight ray we may easily perceive (what is also otherwise evident) that the straight rays diverging from a given point $x'\ y'\ z'$, or converging to a given point $x\ y\ z$, are cut perpendicularly by a series of concentric spheres, having for their common equation,

$$V = \text{const.}; \tag{14}$$

and more generally, that if a set of straight rays be perpendicular to any one surface, they are also perpendicular to a series of surfaces, determined by the equation (14), that is, by the condition that the intercepted portion of a ray between any two given surfaces of the series shall have a constant length. Analogous consequences will be found to follow in general from the law of varying action.

It may be useful to dwell a little longer on the case of rectilinear paths, and on the consequences of the mathematical conception of luminous or visual communication as a motion from point to point along a mathematical straight line or ray, before we pass to the properties of other less simple paths.

It is an obvious consequence of this conception, that from any one point (A), considered as initial, we may imagine light, if unobstructed, as proceeding to any other point (B), considered as final, along one determined ray, or linear path; of which the shape, being straight, is the same whatever point its ends may be; but of which the length and the position depend on the places of those ends, and admit of infinite variety, corresponding to the infinite variety that can be imagined of pairs of points to be connected. So that if we express by one set of numbers the places of the initial and of the final points, and by another set the length and position of the ray, the latter set of numbers must, in mathematical language, be *functions* of the former; must admit of being deduced from them by some fixed mathematical rules. To make this deduction is an easy but a fundamental problem, which may be resolved in the following manner.

Let each of the two points A, B, be referred to one common set of three rectangular semiaxes OX, OY, OZ, diverging from any assumed origin O; let the positive or negative co-ordinates of the final point B, *to* which the light comes, be denoted by x, y, z, and let the corresponding co-ordinates of the initial point A, *from* which the light sets out, be denoted similarly by x', y' , z'; let V be the length of the straight ray, or line AB, and let α, β, γ, be the positive or negative cosines of the acute or obtuse angles which the direction of this ray makes with the positive semiaxes of co-ordinates: the problem is then to determine the laws of the functional dependence of the positive number V, and of the three positive or negative numbers α, β, γ, on the six positive or negative numbers x, y, z, x', y', z'; and this problem is resolved by the following evident formulæ;

$$V = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}; \tag{15}$$

$$\alpha = \frac{x - x'}{V}, \quad \beta = \frac{y - y'}{V}, \quad \gamma = \frac{z - z'}{V}. \tag{16}$$

It is a simple but important corollary to this solution, that the laws of the three cosines of direction α, β, γ, expressed by the equations (16), are connected with the law of the length V, expressed by the formula (15), in a manner which may be stated thus;

$$\alpha = \frac{\delta V}{\delta x}, \quad \beta = \frac{\delta V}{\delta y}, \quad \gamma = \frac{\delta V}{\delta z} : \tag{17}$$

δ being here a characteristic of partial differentiation. We find, in like manner,

$$\alpha = -\frac{\delta V}{\delta x'}, \quad \beta = -\frac{\delta V}{\delta y'}, \quad \gamma = -\frac{\delta V}{\delta z'}, \tag{18}$$

differentiating the function V with respect to the initial co-ordinates. And since the three cosines of direction α, β, γ, are evidently connected by the relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \tag{19}$$

we see that the function V satisfies simultaneously the two following partial differential equations of the first order and second degree,

$$\left.\begin{aligned} \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 = 1, \\ \left(\frac{\delta V}{\delta x'}\right)^2 + \left(\frac{\delta V}{\delta y'}\right)^2 + \left(\frac{\delta V}{\delta z'}\right)^2 = 1. \end{aligned}\right\} \tag{20}$$

The equations (17), (18), (20), will soon be greatly extended; but it seemed well to notice them here, because they contain the germ of my general method for the investigation of the paths of light and of the planets, by the partial differential coefficients of one *characteristic function*. For the equations (17) and (18), which involve the coefficients of the first order of the function V, that is, in the present case, of the length, may be considered as equations of the straight ray which passes with a given direction through a given initial or a given final point: and I have found analogous equations for all other paths of light, and even for the planetary orbits under the influence of their mutual attractions.

The equations (16) when put under the form

$$x - x' = \alpha V, \quad y - y' = \beta V, \quad z - z' = \gamma V, \tag{21}$$

give evidently by differentiation

$$dx = \alpha\, dV, \quad dy = \beta\, dV, \quad dz = \gamma\, dV, \tag{22}$$

and therefore

$$dV^2 = dx^2 + dy^2 + dz^2, \tag{23}$$

the symbol d referring here to an infinitesimal change of the final point B, by a motion along the ray prolonged at its extremity; in such a manner that the equations (22) may be regarded as differential equations of that ray. They give the expressions

$$\alpha = \frac{dx}{dV}, \quad \beta = \frac{dy}{dV}, \quad \gamma = \frac{dz}{dV}, \tag{24}$$

which may, by (23), be put under the form

$$\alpha = \frac{\delta dV}{\delta dx}, \quad \beta = \frac{\delta dV}{\delta dy}, \quad \gamma = \frac{\delta dV}{\delta dz}, \tag{25}$$

δ implying still a partial differentiation, and dV being treated here as a function of dx, dy, dz. And comparing the expressions (25) and (17), we obtain the following results, which we shall soon find to be very general, and to extend with analogous meanings to all linear paths of light,

$$\frac{\delta V}{\delta x} = \frac{\delta dV}{\delta dx}, \quad \frac{\delta V}{\delta y} = \frac{\delta dV}{\delta dy}, \quad \frac{\delta V}{\delta z} = \frac{\delta dV}{\delta dz}. \tag{26}$$

It must not be supposed that these equations are identical; for the quantities in the first members are the partial differential coefficients of one function, V, while those in the second members are the coefficients of another function dV.

In like manner, if we employ (as before) the characteristic d' to denote the infinitesimal changes arising from a change of the initial point A, by a motion along the initial element of the ray, we have the differential equations

$$d'x' = -\alpha\, d'V, \quad d'y' = -\beta\, d'V, \quad d'z' = -\gamma\, d'V, \tag{27}$$

$d'V$ being as before the initial element taken negatively, so that

$$d'V^2 = d'x'^2 + d'y'^2 + d'z'^2; \tag{28}$$

we have therefore

$$\alpha = -\frac{\delta d'V}{\delta d'x'}, \quad \beta = -\frac{\delta d'V}{\delta d'y'}, \quad \gamma = -\frac{\delta d'V}{\delta d'z'}, \tag{29}$$

and consequently, by (18),

$$\frac{\delta V}{\delta x'} = \frac{\delta d'V}{\delta d'x'}, \quad \frac{\delta V}{\delta y'} = \frac{\delta d'V}{\delta d'y'}, \quad \frac{\delta V}{\delta z'} = \frac{\delta d'V}{\delta d'z'}. \tag{30}$$

The same remarks apply to these last results, as to the equations (26).

The general law of stationary action, in optics, may now be thus stated.

The optical quantity called *action*, for any luminous path having i points of sudden bending by reflexion or refraction, and having therefore $i+1$ separate branches, is the sum of $i+1$ separate integrals,

$$\begin{aligned} Action = V &= \sum \int dV^{(r)} \\ &= V^{(1)} + V^{(2)} + V^{(3)} + \cdots + V^{(r)} + \cdots + V^{(i+1)}, \end{aligned} \tag{31}$$

of which each is determined by an equation of the form

$$V^{(r)} = \int dV^{(r)} = \int v^{(r)} \sqrt{dx^{(r)2} + dy^{(y)2} + dz^{(r)2}}, \tag{32}$$

the coefficient $v^{(r)}$ of the element of the path, in the rth medium, depending, in the most general case, on the optical properties of that medium, and on the position, direction, and colour of the element, according to rules discovered by experience, and such, for example, that if the rth medium be ordinary, $v^{(r)}$ is the index of that medium; so that $dV^{(r)}$ is always a homogeneous function of the first dimension of the differentials $dx^{(r)}$, $dy^{(r)}$, $dz^{(r)}$, which may also involve the undifferentiated co-ordinates $x^{(r)}$ $y^{(r)}$ $z^{(r)}$ themselves, and has in general a variation of the form

$$\begin{aligned} \delta dV^{(r)} &= \sigma^{(r)}\, \delta dx^{(r)} + \tau^{(r)}\, \delta dy^{(r)} + \upsilon^{(r)}\, \delta dz^{(r)} \\ &\quad + \left(\frac{\delta v^{(r)}}{\delta x^{(r)}} \delta x^{(r)} + \frac{\delta v^{(r)}}{\delta y^{(r)}} \delta y^{(r)} + \frac{\delta v^{(r)}}{\delta z^{(r)}} \delta z^{(r)} \right) ds^{(r)}, \end{aligned} \tag{33}$$

if we put for abridgment

$$\sigma^{(r)} = \frac{\delta dV^{(r)}}{\delta dx^{(r)}}, \quad \tau^{(r)} = \frac{\delta dV^{(r)}}{\delta dy^{(r)}}, \quad \upsilon^{(r)} = \frac{\delta dV^{(r)}}{\delta dz^{(r)}}, \tag{34}$$

and

$$ds^{(r)} = \sqrt{dx^{(r)2} + dy^{(r)2} + dz^{(r)2}} : \tag{35}$$

we have also, by the homogeneity of $dV^{(r)}$,

$$dV^{(r)} = \sigma^{(r)}\, dx^{(r)} + \tau^{(r)}\, dy^{(r)} + \upsilon^{(r)}\, dz^{(r)}. \tag{36}$$

If we now change the co-ordinates $x^{(r)}$ $y^{(r)}$ $z^{(r)}$ of the luminous path to any near connected co-ordinates

$$x_\varepsilon^{(r)} = x^{(r)} + \varepsilon \xi^{(r)}, \quad y_\varepsilon^{(r)} = y^{(r)} + \varepsilon \eta^{(r)}, \quad z_\varepsilon^{(r)} = z^{(r)} + \varepsilon \zeta^{(r)}, \tag{37}$$

ε being any small constant, and $\xi^{(r)}$ $\eta^{(r)}$ $\zeta^{(r)}$ any functions of ε and of the co-ordinates $x^{(r)}$ $y^{(r)}$ $z^{(r)}$, which do not become infinite for $\varepsilon = 0$, nor for any point on the rth portion of the path, and which satisfy at the meeting of two such portions the equation of the corresponding reflecting or refracting surface, and vanish at the ends of the whole path; we shall pass hereby to a near line having the same extremities as the luminous path, and having its points of bending on the same reflecting or refracting surfaces; and the *law of stationary action* is, that if we compare the integral or sum, $V = \sum \int dV^{(r)}$, for the luminous path, with the corresponding integral V_ε for this near line, the difference of these two integrals or actions bears an indefinitely small ratio to the quantity ε, (which makes the one line differ from the other,) when this quantity ε becomes itself indefinitely small: so that we have the limiting equation,

$$\lim. \frac{1}{\varepsilon}(V_\varepsilon - V) = 0, \tag{38}$$

that is

$$\lim. \sum \frac{1}{\varepsilon}(V_\varepsilon^{(r)} - V^{(r)}) = \sum \int \lim. \frac{1}{\varepsilon}(dV_\varepsilon^{(r)} - dV^{(r)}) = 0, \tag{39}$$

or finally

$$\sum \int \left(\frac{\delta dV_\varepsilon^{(r)}}{\delta \varepsilon} \right) = 0. \tag{40}$$

To develop this last equation, we have, by (33) and (37),

$$\begin{aligned} \left(\frac{\delta dV_\varepsilon^{(r)}}{\delta \varepsilon} \right) &= \sigma^{(r)}\, d\xi^{(r)} + \tau^{(r)}\, d\eta^{(r)} + \upsilon^{(r)}\, d\zeta^{(r)} \\ &\quad + \left(\xi^{(r)} \frac{\delta v^{(r)}}{\delta x^{(r)}} + \eta^{(r)} \frac{\delta v^{(r)}}{\delta y^{(r)}} + \zeta^{(r)} \frac{\delta v^{(r)}}{\delta z^{(r)}} \right) ds^{(r)}; \end{aligned} \tag{41}$$

and therefore, integrating by parts, and accenting the symbols which belong to the beginning of the rth portion of the path,

$$\begin{aligned} \int \left(\frac{\delta dV_\varepsilon^{(r)}}{\delta \varepsilon} \right) &= \sigma^{(r)}\xi^{(r)} - \sigma'^{(r)}\xi'^{(r)} + \tau^{(r)}\eta^{(r)} - \tau'^{(r)}\eta'^{(r)} + \upsilon^{(r)}\zeta^{(r)} - \upsilon'^{(r)}\zeta'^{(r)} \\ &\quad + \int \xi^{(r)} \left(\frac{\delta v^{(r)}}{\delta x^{(r)}} ds^{(r)} - d\sigma^{(r)} \right) \\ &\quad + \int \eta^{(r)} \left(\frac{\delta v^{(r)}}{\delta y^{(r)}} ds^{(r)} - d\tau^{(r)} \right) \\ &\quad + \int \zeta^{(r)} \left(\frac{\delta v^{(r)}}{\delta z^{(r)}} ds^{(r)} - d\upsilon^{(r)} \right). \end{aligned} \tag{42}$$

And since the extreme values, and values for the points of juncture, of the otherwise arbitrary functions ξ η ζ, are subject to the following conditions:

$$\xi'^{(1)} = 0, \quad \eta'^{(1)} = 0, \quad \zeta'^{(1)} = 0, \quad \xi^{(i+1)} = 0, \quad \eta^{(i+1)} = 0, \quad \zeta^{(i+1)} = 0, \tag{43}$$

and

$$\xi^{(r)} = \xi'^{(r+1)}, \quad \eta^{(r)} = \eta'^{(r+1)}, \quad \zeta^{(r)} = \zeta'^{(r+1)}, \tag{44}$$

r varying from 1 to i; and finally, for every value of r within the same range, to the condition

$$n_x^{(r)}\xi^{(r)} + n_y^{(r)}\eta^{(r)} + n_z^{(r)}\zeta^{(r)} = 0, \tag{45}$$

$n^{(r)}$ being either seminormal to the rth reflecting or refracting surface at the rth point of incidence, and $n_x^{(r)}$ $n_y^{(r)}$ $n_z^{(r)}$ being the cosines of the angles which $n^{(r)}$ makes with the three rectangular positive semiaxes of co-ordinates x y z; the law of stationary action (40) resolves itself into the following equations:

$$d\sigma^{(r)} = \frac{\delta\upsilon^{(r)}}{\delta x^{(r)}} ds^{(r)}; \quad d\tau^{(r)} = \frac{\delta\upsilon^{(r)}}{\delta y^{(r)}} ds^{(r)}; \quad d\upsilon^{(r)} = \frac{\delta\upsilon^{(r)}}{\delta z^{(r)}} ds^{(r)}; \tag{46}$$

and

$$\sigma'^{(r+1)} - \sigma^{(r)} = \lambda^{(r)} n_x^{(r)}; \quad \tau'^{(r+1)} - \tau^{(r)} = \lambda^{(r)} n_y^{(r)}; \quad \upsilon'^{(r+1)} - \upsilon^{(r)} = \lambda^{(r)} n_z^{(r)}; \tag{47}$$

in which $\lambda^{(r)}$ is an indeterminate multiplier. The three equations (46), which may by the condition (36) be shown to be consistent with each other, express the gradual changes, if any, of a ray, between its points of sudden bending; and the equations (47) contain the rules of ordinary and extraordinary reflexion and refraction. All these results of that known law, which I have called the law of stationary action, are fully confirmed by experience, when suitable forms are assigned to the functions denoted by $\upsilon^{(r)}$.

For example, in the case of an uniform medium, ordinary or extraordinary, the function $\upsilon^{(r)}$ is to be considered as independent of the undifferentiated co-ordinates $x^{(r)}$ $y^{(r)}$ $z^{(r)}$, and the differential equations (46) of the rth portion of the luminous path become simply

$$d\sigma^{(r)} = 0, \quad d\tau^{(r)} = 0, \quad d\upsilon^{(r)} = 0, \tag{48}$$

and give by integration

$$\sigma^{(r)} = \text{const.}, \quad \tau^{(r)} = \text{const.}, \quad \upsilon^{(r)} = \text{const.}; \tag{49}$$

they express, therefore, the known fact of the rectilinear propagation of light in a uniform medium, because in such a medium $\sigma^{(r)}$ $\tau^{(r)}$ $\upsilon^{(r)}$ depend only on the colour and direction, but not on the co-ordinates of the path, and are functions of $\alpha^{(r)}$ $\beta^{(r)}$ $\gamma^{(r)}$ not including $x^{(r)}$ $y^{(r)}$ $z^{(r)}$, if we put for abridgment

$$\alpha^{(r)} = \frac{dx^{(r)}}{ds^{(r)}}, \quad \beta^{(r)} = \frac{dy^{(r)}}{ds^{(r)}}, \quad \gamma^{(r)} = \frac{dz^{(r)}}{ds^{(r)}}, \tag{50}$$

so that $\alpha^{(r)}$, $\beta^{(r)}$, $\gamma^{(r)}$, represent the cosines of the inclination (in this case constant) of any element of the rth portion of the path to the positive semiaxes of co-ordinates. The formulæ (46) give also the known differential equations for a ray in the earth's atmosphere.

With respect to the rules of reflexion or refraction of light, expressed by the equations (47), they may in general be thus summed up;

$$t_x^{(r)}\Delta\sigma^{(r)} + t_y^{(r)}\Delta\tau^{(r)} + t_z^{(r)}\Delta\upsilon^{(r)} = 0 : \tag{51}$$

in which Δ refers to the sudden changes produced by reflexion or refraction, and $t_x^{(r)}$ $t_y^{(r)}$ $t_z^{(r)}$ are the cosines of the inclinations to the positive semiaxes of co-ordinates, of any arbitrary line $t^{(r)}$, which touches the rth reflecting or refracting surface, at the rth point of incidence, so that

$$t_x^{(r)} n_x^{(r)} + t_y^{(r)} n_y^{(r)} + t_z^{(r)} n_z^{(r)} = 0. \tag{52}$$

In the case of ordinary media, for example, we have

$$\sigma^{(r)} = v^{(r)} \alpha^{(r)}, \quad \tau^{(r)} = v^{(r)} \beta^{(r)}, \quad \upsilon^{(r)} = v^{(r)} \gamma^{(r)}; \tag{53}$$

and the equation (51) may be put under the form

$$\Delta . v^{(r)} v_t^{(r)} = 0, \tag{54}$$

in which

$$v_t^{(r)} = \alpha^{(r)} t_x^{(r)} + \beta^{(r)} t_y^{(r)} + \gamma^{(r)} t_z^{(r)}, \tag{55}$$

so that the unchanged quantity $v^{(r)} v_t^{(r)}$ is the projection of the index $v^{(r)}$ on the arbitrary tangent $t^{(r)}$, each index being measured from the point of incidence in the direction of the corresponding ray: which agrees with the law of Snellius. In general, if we put

$$\nu^{(r)} = \sqrt{\sigma^{(r)2} + \tau^{(r)2} + \upsilon^{(r)2}}, \tag{56}$$

and

$$\sigma^{(r)} = \nu^{(r)} \nu_x^{(r)}, \quad \tau^{(r)} = \nu^{(r)} \nu_y^{(r)}, \quad \upsilon^{(r)} = \nu^{(r)} \nu_z^{(r)}, \tag{57}$$

we may consider $\sigma^{(r)}$ $\tau^{(r)}$ $\upsilon^{(r)}$ as the projections, on the axes of co-ordinates, of a certain straight line $\nu^{(r)}$, of which the length and direction depend (according to rules expressed by the foregoing equations) on the form of the function $v^{(r)}$ or $dV^{(r)}$, and on the direction and colour of the element of the luminous path, before or after incidence; and if we put

$$\nu_t^{(r)} = \nu_x^{(r)} t_x^{(r)} + \nu_y^{(r)} t_y^{(r)} + \nu_z^{(r)} t_z^{(r)}, \tag{58}$$

the equation (51) will take the form

$$\Delta . \nu^{(r)} \nu_t^{(r)} = 0, \tag{59}$$

which expresses that the projection of this straight line $\nu^{(r)}$ on any arbitrary tangent $t^{(r)}$ to the reflecting or refracting surface, at the point of incidence, is not changed by reflection or refraction, ordinary or extraordinary: which is a convenient general form for all the known rules of sudden change of direction of a path of light. In the undulatory theory, I have found that the line $\nu^{(r)}$ is the reciprocal of the normal velocity of propagation of the wave; and its projections may therefore be called *components of normal slowness*: so that the foregoing property of unchanged projection of the line $\nu^{(r)}$, may be expressed, in the language of this theory, by saying that the component of normal slowness in the direction of any line which touches any ordinary or extraordinary reflecting or refracting surface at any point of incidence is not changed by reflection or refraction. It was, however, by a different method that I originally deduced this general enunciation of the rules of optical reflexion and refraction, namely, by employing my principle of the characteristic function, and that other general law, of which it is now time to speak.

This other general law, the *law of varying action*, results from the known law above explained, by considering the extreme points of a luminous path as variable: that is, by not supposing the six extreme functions (43) to vanish. Denoting, for abridgment, the three final functions of this set by $\xi\ \eta\ \zeta$, and the three initial functions by $\xi'\ \eta'\ \zeta'$, and writing similarly v, dV, &c., instead of the final quantities $v^{(i+1)}$, $dV^{(i+1)}$, &c. and v', dV', &c., instead of the initial quantities $v^{(1)}$, $dV^{(1)}$, &c., we find this new equation,

$$\begin{aligned} \lim.\frac{1}{\varepsilon}(V_\varepsilon - V) &= \sum \int \left(\frac{\delta dV_\varepsilon^{(r)}}{\delta\varepsilon}\right) \\ &= \sigma\xi - \sigma'\xi' + \tau\eta - \tau'\eta' + \upsilon\zeta - \upsilon'\zeta', \end{aligned} \tag{60}$$

which is a form of my general result. It may also be put conveniently under this other form,

$$\delta V = \sigma\,\delta x - \sigma'\,\delta x' + \tau\,\delta y - \tau'\,\delta y' + \upsilon\,\delta z - \upsilon'\,\delta z'; \tag{61}$$

in which

$$\left.\begin{aligned} \sigma &= \frac{\delta dV}{\delta dx} = \frac{\delta\,.\,v\,ds}{\delta dx}, \\ \tau &= \frac{\delta dV}{\delta dy} = \frac{\delta\,.\,v\,ds}{\delta dy}, \\ \upsilon &= \frac{\delta dV}{\delta dz} = \frac{\delta\,.\,v\,ds}{\delta dz}, \end{aligned}\right\} \tag{62}$$

and

$$\left.\begin{aligned} -\sigma' &= -\left(\frac{\delta\,.\,v\,ds}{\delta dx}\right)' = \frac{\delta d'V}{\delta d'x'}, \\ -\tau' &= -\left(\frac{\delta\,.\,v\,ds}{\delta dy}\right)' = \frac{\delta d'V}{\delta d'y'}, \\ -\upsilon' &= -\left(\frac{\delta\,.\,v\,ds}{\delta dz}\right)' = \frac{\delta d'V}{\delta d'z'}, \end{aligned}\right\} \tag{63}$$

the symbols

$$\left(\frac{\delta\,.\,v\,ds}{\delta dx}\right)',\ \text{\&c.},$$

representing the initial quantities which correspond to

$$\frac{\delta\,.\,v\,ds}{\delta dx},\ \text{\&c.};$$

and $d'V$ being, according to the same analogy of notation, the infinitesimal change of the whole integral V, arising from the infinitesimal changes $d'x'$, $d'y'$, $d'z'$, of the initial co-ordinates, that is, from a motion of the initial point $x'\ y'\ z'$ along the initial element of the luminous path; so that $d'V$ is the initial element of the integral taken negatively,

$$d'V = -v'\sqrt{d'x'^2 + d'y'^2 + d'z'^2}. \tag{64}$$

If then we consider the integral or *action* V as a function (which I have called the *characteristic function*) of the six extreme co-ordinates, and if we differentiate this function with respect to these co-ordinates, we see that its six partial differential coefficients of the first order may be represented generally by the equations (26) and (30), which were already proved to be true for the simple case of rectilinear paths of light. And as, in that simple case, those equations, being then equivalent to the formulae (17) and (18), were seen to determine the course of the straight ray, which passed with a given direction through a given initial or a given final point; so, generally, when we know the initial co-ordinates, direction, and colour of a luminous path, and the optical properties of the initial medium, we can determine, or at least restrict (in general) to a finite variety, the values of the initial coefficients

$$\frac{\delta d'V}{\delta d'x'}, \quad \frac{\delta d'V}{\delta d'y'}, \quad \frac{\delta d'V}{\delta d'z'},$$

which form the second members of the equations (30); and therefore we may regard as known the first members of the same equations, namely the partial differential coefficients

$$\frac{\delta V}{\delta x'}, \quad \frac{\delta V}{\delta y'}, \quad \frac{\delta V}{\delta z'},$$

of the characteristic function V, taken with respect to the known initial co-ordinates: so that if the form of the function V be known, we have between the final co-ordinates x, y, z, considered as variable, the three following equations of the path, or at least of its final branch,

$$\frac{\delta V}{\delta x'} = \text{const.}, \quad \frac{\delta V}{\delta y'} = \text{const.}, \quad \frac{\delta V}{\delta z'} = \text{const.} \tag{65}$$

These three equations are compatible with each other, and are equivalent only to two distinct relations between the variable co-ordinates x y z, because in general V must satisfy a partial differential equation of the form

$$0 = \Omega'(\sigma', \tau', \upsilon', x', y', z'), \tag{66}$$

in which, by what has been shown,

$$\sigma' = -\frac{\delta V}{\delta x'}, \quad \tau' = -\frac{\delta V}{\delta y'}, \quad \upsilon' = -\frac{\delta V}{\delta z'}, \tag{67}$$

and which is therefore analogous to the second formula (20): this equation (66) being obtained by eliminating the ratios of $d'x'$, $d'y'$, $d'z'$, between the general formulæ (30). In like manner the formulæ (26) give generally a partial differential equation of the form

$$0 = \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, x, y, z\right), \tag{68}$$

analogous to the first of those marked (20), and the three following compatible equations between the variable initial co-ordinates x', y', z', of a path of light which is obliged to pass with a given direction through a given final point,

$$\frac{\delta V}{\delta x} = \text{const.}, \quad \frac{\delta V}{\delta y} = \text{const.}, \quad \frac{\delta V}{\delta z} = \text{const.} \tag{69}$$

But for the integration and use of these partial differential equations, the limits of the present communication oblige me to refer to the volumes, already mentioned, of the Transactions of the Royal Irish Academy.

I may, however, mention here, that my employment of the characteristic function V, in all questions of reflexion and refraction, is founded on an equation in finite differences, which, by the integral nature of this function V, is evidently satisfied, namely,

$$\Delta V = 0 = \lambda u; \tag{70}$$

Δ referring, as before, to the sudden changes produced at any reflecting or refracting surface having for its equation

$$u = 0; \tag{71}$$

and λ being an indeterminate multiplier, employed for the purpose of being able to treat the co-ordinates of incidence as three independent variables. For example, the formulae (47), for a sudden change of direction, result immediately from (70), under the form

$$\Delta\frac{\delta V}{\delta x} = \lambda\frac{\delta u}{\delta x}, \qquad \Delta\frac{\delta V}{\delta y} = \lambda\frac{\delta u}{\delta y}, \qquad \Delta\frac{\delta V}{\delta z} = \lambda\frac{\delta u}{\delta z}, \tag{72}$$

by differentiating with respect to the co-ordinates of incidence, as three independent variables, and then reducing by the equation (71) of the ordinary or extraordinary reflecting or refracting surface. These results respecting the change of direction of a luminous path may be put under the form

$$\Delta\frac{\frac{\delta V}{\delta x}}{\frac{\delta u}{\delta x}} = \Delta\frac{\frac{\delta V}{\delta y}}{\frac{\delta u}{\delta y}} = \Delta\frac{\frac{\delta V}{\delta z}}{\frac{\delta u}{\delta z}}; \tag{73}$$

or under the following,

$$\left.\begin{aligned}
\Delta\left(\frac{\delta u}{\delta x}\frac{\delta V}{\delta y} - \frac{\delta u}{\delta y}\frac{\delta V}{\delta x}\right) &= 0,\\
\Delta\left(\frac{\delta u}{\delta y}\frac{\delta V}{\delta z} - \frac{\delta u}{\delta z}\frac{\delta V}{\delta y}\right) &= 0,\\
\Delta\left(\frac{\delta u}{\delta z}\frac{\delta V}{\delta x} - \frac{\delta u}{\delta x}\frac{\delta V}{\delta z}\right) &= 0:
\end{aligned}\right\} \tag{74}$$

and in general, all theorems respecting the changes produced by reflexion or refraction in the properties of an optical system, may be expressed, by the help of the formula (70), as *permanences* of certain other properties. The remarkable permanence, already stated, of the components of normal slowness of propagation of a luminous wave, was suggested to me by observing that my function V is (in the undulatory theory) the time of propagation of light from the initial to the final point, and therefore that the waves (in the same theory) are represented by the general equation

$$V = \text{const.}, \tag{75}$$

and the components of normal slowness by the partial differential coefficients of V of the first order. The properties of the function V, on which my whole optical method depends, supplied me also, long since, with a simple proof of the contested theorem of Huygens already mentioned, namely, that the rays of any ordinary homogeneous system, which after issuing originally from any luminous point, or being (in an initial and ordinary state) perpendicular to any common surface, have undergone any number of reflexions or refractions ordinary or extraordinary, before arriving at their final state, are in that state perpendicular to a series of surfaces, namely, to the series (75), which are waves in the theory of Huygens: because, by the properties of my function, the differential equation of that series is

$$\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z = 0, \tag{76}$$

α, β and γ being the cosines which determine the final direction of a ray. It was also by combining the properties of the same characteristic function V with the physical principles of Fresnel, that I was first led, (from perceiving an indeterminateness in two particular cases in the relations between the coefficients

$$\frac{\delta dV}{\delta dx}, \quad \frac{\delta dV}{\delta dy}, \quad \frac{\delta dV}{\delta dz},$$

and the ratios of dx, dy, dz,) to form that theoretical expectation of two kinds of *conical refraction* which I communicated in last October (1832) to the Royal Irish Academy and to Professor Lloyd, and which the latter has since verified experimentally. Mr. MacCullagh has lately informed me that the same two indeterminate cases in Fresnel's theory had occurred to him from geometrical considerations, some years ago, and that he had intended to try to what geometrical and physical consequences they would lead.

The method of the characteristic function has conducted me to many other consequences, besides those which I have already published in the Transactions of the Royal Irish Academy: and I think that it will hereafter acquire, in the hands of other mathematicians, a rank in deductive optics, of the same kind as that which the method of co-ordinates has attained in algebraical geometry. For as, by the last-mentioned method, Des Cartes reduced the study of a plane curve, or of a curved surface, to the study of that one function which expresses the law of the ordinate, and made it possible thereby to discover general formulæ for the tangents, curvatures, and all other geometrical properties of the curve or surface, and to regard them as included all in that one law, that central algebraical relation: so I believe that mathematicians will find it possible to deduce all properties of optical systems from the study of that one central relation which connects, for each particular system, the optical function V with the extreme co-ordinates and the colour, and which has its partial differential coefficients connected with the extreme directions of a ray, by the law of varying action, or by the formulæ (26) and (30).

It only now remains, in order to conclude the present remarks, that I should briefly explain the allusions already made to my view of an analogous function and method in the research of the planetary and cometary orbits under the influence of their mutual perturbations. The view itself occurred to me many years ago, and I gave a short notice or announcement of it in the XVth volume (page 80) of the Transactions of the Royal Irish Academy; but I have only lately resumed the idea, and have not hitherto published any definite statement on the subject.

To begin with a simple instance, let us attend first to the case of a comet, considered as sensibly devoid of mass, and as moving in an undisturbed parabola about the sun, which latter body we shall regard as fixed at the origin of co-ordinates, and as having an attracting mass equal to unity. Let r be the comet's radius vector at any moment t considered as final, and r' the radius vector of the same comet at any other moment t' considered as initial; let also r'' be the chord joining the ends of r and r', and let us put for abridgment

$$V = 2\sqrt{r + r' + r''} \mp 2\sqrt{r + r' - r''}; \tag{77}$$

then I find, that the final and initial components of velocity of the comet, parallel to any three rectangular semiaxes of co-ordinates, may be expressed as follows by the coefficients of the function V,

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{\delta V}{\delta x}, & \frac{dy}{dt} &= \frac{\delta V}{\delta y}, & \frac{dz}{dt} &= \frac{\delta V}{\delta z}, \\ \frac{dx'}{dt'} &= -\frac{\delta V}{\delta x'}, & \frac{dy'}{dt'} &= -\frac{\delta V}{\delta y'}, & \frac{dz'}{dt'} &= -\frac{\delta V}{\delta z'}, \end{aligned}\right\} \tag{78}$$

and that this function V satisfies the two following partial differential equations,

$$\left.\begin{aligned} \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 &= \frac{2}{r}, \\ \left(\frac{\delta V}{\delta x'}\right)^2 + \left(\frac{\delta V}{\delta y'}\right)^2 + \left(\frac{\delta V}{\delta z'}\right)^2 &= \frac{2}{r'}, \end{aligned}\right\} \tag{79}$$

which reconcile the expressions (78) with the known law of a comet's velocity. I find also that all the other properties of a comet's parabolic motion agree with and are included in the formulæ (78), when the form (77) is assigned to the function V. They give, for example, by an easy combination, the theorem discovered by Euler for the dependence of the time $(t - t')$ on the parabolic chord (r'') and on the sum $(r + r')$ of the radii drawn to its extremities.

More generally, in any system of points or bodies which attract or repel one another according to any function of the distance, for example, in the solar system, I have found that the final and initial components of momentum may be expressed in a similar manner, by the partial differential coefficients of the first order of some one central or *characteristic function* V of the final and initial co-ordinates; so that we have generally, by a suitable choice of V,

$$\left.\begin{aligned} &m_1 \frac{dx_1}{dt} = \frac{\delta V}{\delta x_1}; \quad m_1 \frac{dy_1}{dt} = \frac{\delta V}{\delta y_1}; \quad m_1 \frac{dz_1}{dt} = \frac{\delta V}{\delta z_1}; \\ &m_2 \frac{dx_2}{dt} = \frac{\delta V}{\delta x_2}; \quad \&\text{c.}; \end{aligned}\right\} \tag{80}$$

and

$$\left.\begin{aligned} &m_1 \frac{dx_1'}{dt'} = -\frac{\delta V}{\delta x_1'}; \quad m_1 \frac{dy_1'}{dt'} = -\frac{\delta V}{\delta y_1'}; \quad m_1 \frac{dz_1'}{dt'} = -\frac{\delta V}{\delta z_1'}; \\ &m_2 \frac{dx_2'}{dt'} = -\frac{\delta V}{\delta x_2'}; \quad \&\text{c.} : \end{aligned}\right\} \tag{81}$$

m_1, m_2, &c., being the masses of the system, and the function V being obliged to satisfy two partial differential equations of the first order and second degree, which are analogous to (79), and may be thus denoted

$$\left.\begin{aligned} \sum \cdot \frac{1}{m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\} = 2F, \\ \sum \cdot \frac{1}{m} \left\{ \left(\frac{\delta V}{\delta x'}\right)^2 + \left(\frac{\delta V}{\delta y'}\right)^2 + \left(\frac{\delta V}{\delta z'}\right)^2 \right\} = 2F'; \end{aligned}\right\} \tag{82}$$

the function F involving the final co-ordinates, and the function F' involving similarly the initial co-ordinates, and the common form of these two functions depending on the law of attraction or repulsion. In the solar system

$$\left.\begin{aligned} F = \sum \frac{m_i m_k}{\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2 + (z_i - z_k)^2}} + H, \\ F' = \sum \frac{m_i m_k}{\sqrt{(x'_i - x'_k)^2 + (y'_i - y'_k)^2 + (z'_i - z'_k)^2}} + H, \end{aligned}\right\} \tag{83}$$

H being a certain constant; and in general the partial differential equations (82) contain the law of living forces, which the other known general laws or integrals of the equations of motion are expressed by other general and simple properties of the same characteristic function V: the coefficients of which function, when combined with the relations (80) and (81), are sufficient to determine all circumstances of the motion of a system. By this view the research of the most complicated orbits, in lunar, planetary, and sidereal astronomy, is reduced to the study of the properties of a single function V; which is analogous to my optical function, and represents the *action* of the system from one position to another. If we knew, for example, the form of this one function V for a system of three bodies attracting according to Newton's law, (suppose the system of Sun, Earth, and Moon, or of the Sun, Jupiter and Saturn,) we should need no further integration in order to determine the separate paths and the successive configurations of these three bodies; the eight relations, independent of the time, between their nine variable co-ordinates, would be given at once by differentiating the one function V, and employing the nine initial equations of the form (81), which in consequence of the second equation (82) are only equivalent to eight distinct relations, the positions and velocities being given for some one initial epoch; and the variable time t of arriving at any one of the subsequent states of the system would be given by a single integration of any combination of these relations with the equations (80). The development of this view, including its extension to other analogous questions, appears to me to open in mechanics and astronomy an entirely new field of research. I shall only add, that the view was suggested by a general law of varying action in dynamics, which I had deduced from the known dynamical law of least or stationary action, by a process analogous to that general reasoning in optics which I have already endeavoured to illustrate.

Observatory of Trinity College, Dublin,
September, 1833.