Computing the 2-by-1 CS decomposition

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Three decompositions:

- 2-by-2 CS decomposition (CSD)

$$\underbrace{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}}_{\text{orthogonal}} = \underbrace{\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} C & -S \\ S & C \end{bmatrix}}_{\substack{\text{diagonal} \\ \text{blocks}}} \underbrace{\begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^T}_{\text{orthogonal}}$$

- 2-by-1 CSD

$$\underbrace{\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}}_{\substack{\text{orthonormal} \\ \text{columns}}} = \underbrace{\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} C \\ S \end{bmatrix}}_{\substack{\text{diagonal} \\ \text{blocks}}} \underbrace{V_1^T}_{\text{orthogonal}}$$

- Generalized singular value decomposition (GSVD)

$$\begin{bmatrix} A \\ B \end{bmatrix} = \underbrace{\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} C \\ S \end{bmatrix}}_{\substack{\text{diagonal} \\ \text{blocks}}} W^T$$

Earlier work [Sutton 2009, 2012] emphasized the 2-by-2 CSD. This talk moves to the 2-by-1 CSD.

A CSD code must make *consistent* choices even when those choices are *arbitrary*.

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^T$$

- The columns of U_1 are the left singular vectors of X_{11} and X_{12}.
- The columns of U_2 are the left singular vectors of X_{21} and X_{22}.
- The columns of V_1 are the right singular vectors of X_{11} and X_{21}.
- The columns of V_2 are the right singular vectors of X_{12} and X_{22}.

- Linear Algebra 101: $\lambda_i = \lambda_{i+1} \implies$ two-dimensional eigen*space*.
- Numerical LA: $\lambda_i \approx \lambda_{i+1} \implies$ *ill-conditioned* eigenvectors.

- CSD requires four simultaneous SVD's.
- The computed singular vectors must be identical.

Which is easier to compute?

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^T$$

or

$$\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V_1^T$$

2-by-1 CSD?

- Earlier algorithms (Stewart '82, Van Loan '85, Paige '86, Bai-Demmel '93, Drmač '98) compute the 2-by-1 CSD.
- The 2-by-1 CSD involves less sharing of singular vectors.

2-by-2 CSD?

- In "Computing the Complete CS Decomposition" (2009), I argue that the 2-by-2 CSD is easier....

Contrasted with the 2-by-1 CSD

$$\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V_1^T,$$

the input to the 2-by-2 CSD

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^T$$

- provides more information—X_{12}, X_{22},
- is more constrained—columns *and rows* are orthonormal.

Computing the 2-by-2 CSD . . .

$$\left[\begin{array}{cccc|cccc}
0.58 & 0.15 & 0.19 & 0.15 & 0.57 & -0.45 & -0.21 & 0.14 \\
0.21 & 0.66 & -0.14 & -0.10 & -0.34 & -0.18 & -0.05 & -0.58 \\
-0.53 & 0.68 & 0.23 & 0.01 & 0.19 & 0.05 & -0.08 & 0.40 \\
0.32 & 0.11 & 0.04 & -0.82 & 0.16 & 0.34 & 0.23 & 0.15 \\
\hline
-0.43 & -0.13 & -0.03 & -0.26 & 0.44 & -0.47 & 0.40 & -0.40 \\
-0.05 & 0.01 & -0.74 & -0.22 & -0.14 & -0.42 & -0.14 & 0.43 \\
-0.15 & -0.22 & 0.49 & -0.43 & -0.31 & -0.34 & -0.54 & -0.01 \\
-0.16 & -0.04 & -0.33 & -0.09 & 0.44 & 0.37 & -0.64 & -0.33
\end{array}\right]$$

$$= X$$

$$\left[\begin{array}{rrrr|rrrr}
\mathbf{0.58} & 0.15 & 0.19 & 0.15 & 0.57 & -0.45 & -0.21 & 0.14 \\
\mathbf{0.21} & 0.66 & -0.14 & -0.10 & -0.34 & -0.18 & -0.05 & -0.58 \\
\mathbf{-0.53} & 0.68 & 0.23 & 0.01 & 0.19 & 0.05 & -0.08 & 0.40 \\
\mathbf{0.32} & 0.11 & 0.04 & -0.82 & 0.16 & 0.34 & 0.23 & 0.15 \\
\hline
\mathbf{-0.43} & -0.13 & -0.03 & -0.26 & 0.44 & -0.47 & 0.40 & -0.40 \\
\mathbf{-0.05} & 0.01 & -0.74 & -0.22 & -0.14 & -0.42 & -0.14 & 0.43 \\
\mathbf{-0.15} & -0.22 & 0.49 & -0.43 & -0.31 & -0.34 & -0.54 & -0.01 \\
\mathbf{-0.16} & -0.04 & -0.33 & -0.09 & 0.44 & 0.37 & -0.64 & -0.33
\end{array}\right]$$

$$= X$$

$$
\left[\begin{array}{cccc|cccc}
\mathbf{0.87} & -0.11 & -0.03 & -0.23 & 0.24 & -0.25 & -0.02 & -0.24 \\
 & -0.66 & 0.16 & 0.08 & 0.46 & 0.07 & 0.02 & 0.57 \\
 & -0.70 & -0.29 & 0.02 & -0.48 & 0.20 & 0.16 & -0.37 \\
 & -0.10 & -10^{-3} & 0.80 & 0.01 & -0.49 & -0.28 & -0.17 \\
\hline
\mathbf{0.49} & 0.19 & 0.06 & 0.41 & -0.42 & 0.44 & 0.04 & 0.42 \\
 & 0.03 & -0.73 & -0.18 & -0.19 & -0.37 & -0.17 & 0.48 \\
 & -0.16 & 0.50 & -0.32 & -0.45 & -0.19 & -0.60 & 0.12 \\
 & 0.01 & -0.31 & 0.03 & 0.29 & 0.53 & -0.71 & -0.19
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} X
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & \text{-}\mathbf{0.11} & \text{-}\mathbf{0.03} & \text{-}\mathbf{0.23} & \mathbf{0.24} & \text{-}\mathbf{0.25} & \text{-}\mathbf{0.02} & \text{-}\mathbf{0.24} \\
 & \text{-}0.66 & 0.16 & 0.08 & 0.46 & 0.07 & 0.02 & 0.57 \\
 & \text{-}0.70 & \text{-}0.29 & 0.02 & \text{-}0.48 & 0.20 & 0.16 & \text{-}0.37 \\
 & \text{-}0.10 & \text{-}10^{\text{-}3} & 0.80 & 0.01 & \text{-}0.49 & \text{-}0.28 & \text{-}0.17 \\
\hline
0.49 & \mathbf{0.19} & \mathbf{0.06} & \mathbf{0.41} & \text{-}\mathbf{0.42} & \mathbf{0.44} & \mathbf{0.04} & \mathbf{0.42} \\
 & 0.03 & \text{-}0.73 & \text{-}0.18 & \text{-}0.19 & \text{-}0.37 & \text{-}0.17 & 0.48 \\
 & \text{-}0.16 & 0.50 & \text{-}0.32 & \text{-}0.45 & \text{-}0.19 & \text{-}0.60 & 0.12 \\
 & 0.01 & \text{-}0.31 & 0.03 & 0.29 & 0.53 & \text{-}0.71 & \text{-}0.19
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} X
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -\mathbf{0.26} & & & -\mathbf{0.42} & & & \\
 & -0.18 & -0.24 & -0.61 & 0.11 & 0.21 & 0.03 & 0.69 \\
 & -0.31 & 0.19 & -0.66 & 0.19 & -0.06 & 0.14 & -0.61 \\
 & 0.67 & 0.05 & -0.44 & -0.41 & -0.33 & -0.27 & -0.02 \\
\hline
0.49 & \mathbf{0.46} & & & \mathbf{0.74} & & & \\
 & -0.24 & 0.71 & 0.04 & 0.15 & -0.50 & -0.18 & 0.35 \\
 & -0.29 & -0.54 & 0.03 & 0.18 & -0.43 & -0.62 & -0.11 \\
 & -0.01 & 0.31 & -0.03 & 0.01 & 0.64 & -0.70 & -0.08
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & -\mathbf{0.18} & -0.24 & -0.61 & \mathbf{0.11} & 0.21 & 0.03 & 0.69 \\
 & -\mathbf{0.31} & 0.19 & -0.66 & \mathbf{0.19} & -0.06 & 0.14 & -0.61 \\
 & \mathbf{0.67} & 0.05 & -0.44 & -\mathbf{0.41} & -0.33 & -0.27 & -0.02 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & -\mathbf{0.24} & 0.71 & 0.04 & \mathbf{0.15} & -0.50 & -0.18 & 0.35 \\
 & -\mathbf{0.29} & -0.54 & 0.03 & \mathbf{0.18} & -0.43 & -0.62 & -0.11 \\
 & -\mathbf{0.01} & 0.31 & -0.03 & \mathbf{0.01} & 0.64 & -0.70 & -0.08
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & \mathbf{0.76} & 0.03 & 0.03 & -\mathbf{0.47} & -0.32 & -0.30 & 0.07 \\
 & & 0.28 & -0.45 & & -0.23 & 0.03 & -0.82 \\
 & & -0.14 & -0.89 & & 0.04 & -0.03 & 0.43 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & \mathbf{0.38} & -0.05 & -0.05 & -\mathbf{0.23} & 0.64 & 0.61 & -0.14 \\
 & & -0.90 & -0.01 & & 0.10 & -0.26 & -0.34 \\
 & & 0.30 & -0.03 & & 0.65 & -0.69 & -0.09
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & \mathbf{0.03} & \mathbf{0.03} & -0.47 & \mathbf{-0.32} & \mathbf{-0.30} & \mathbf{0.07} \\
 & & 0.28 & -0.45 & & -0.23 & 0.03 & -0.82 \\
 & & -0.14 & -0.89 & & 0.04 & -0.03 & 0.43 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & \mathbf{-0.05} & \mathbf{-0.05} & -0.23 & \mathbf{0.64} & \mathbf{0.61} & \mathbf{-0.14} \\
 & & -0.90 & -0.01 & & 0.10 & -0.26 & -0.34 \\
 & & 0.30 & -0.03 & & 0.65 & -0.69 & -0.09
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{rrrr|rrr}
0.87 & -0.26 & & & -0.42 & & \\
& 0.76 & -\mathbf{0.04} & & -0.47 & -\mathbf{0.44} & \\
& & 0.13 & -0.51 & & -0.01 & -0.13 & 0.84 \\
& & 0.74 & -0.51 & & -0.06 & 0.03 & -0.43 \\
\hline
0.49 & 0.46 & & & 0.74 & & \\
& 0.38 & \mathbf{0.07} & & -0.23 & \mathbf{0.89} & \\
& & 0.63 & 0.64 & & -0.05 & 0.28 & 0.34 \\
& & -0.19 & -0.24 & & 0.02 & 0.95 & 0.03
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
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$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & -0.04 & & -0.47 & -0.44 & & \\
 & & \mathbf{0.13} & -0.51 & & -\mathbf{0.01} & -0.13 & 0.84 \\
 & & \mathbf{0.74} & -0.51 & & -\mathbf{0.06} & 0.03 & -0.43 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & \mathbf{0.63} & 0.64 & & -\mathbf{0.05} & 0.28 & 0.34 \\
 & & -\mathbf{0.19} & -0.24 & & \mathbf{0.02} & 0.95 & 0.03
\end{array}\right]
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$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & -0.04 & & -0.47 & -0.44 & & \\
 & & \mathbf{0.75} & -0.60 & & -\mathbf{0.06} & 0.01 & -0.27 \\
 & & & -0.41 & & & -0.13 & 0.90 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & \mathbf{0.65} & 0.69 & & -\mathbf{0.05} & -0.01 & 0.31 \\
 & & & 0.05 & & & -0.99 & -0.12
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & -0.04 & & -0.47 & -0.44 & & \\
 & & 0.75 & -\mathbf{0.60} & & -0.06 & \mathbf{0.01} & -\mathbf{0.27} \\
 & & & -0.41 & & & -0.13 & 0.90 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & 0.65 & \mathbf{0.69} & & -0.05 & -\mathbf{0.01} & \mathbf{0.31} \\
 & & & 0.05 & & & -0.99 & -0.12
\end{array}\right]
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$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
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\left[\begin{array}{cccc|cccc}
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 & & & -0.41 & & & 0.90 & -0.11 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & 0.65 & \mathbf{0.69} & & -0.05 & \mathbf{0.31} & \\
 & & & 0.05 & & & -0.10 & -0.99
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & -0.04 & & -0.47 & -0.44 & & \\
 & & 0.75 & -0.60 & & -0.06 & -0.27 & \\
 & & & -\mathbf{0.41} & & & \mathbf{0.90} & -0.11 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & 0.65 & 0.69 & & -0.05 & 0.31 & \\
 & & & \mathbf{0.05} & & & -\mathbf{0.10} & -0.99
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
& 0.76 & -0.04 & & -0.47 & -0.44 & & \\
& & 0.75 & -0.60 & & -0.06 & -0.27 & \\
& & & \mathbf{0.41} & & & -\mathbf{0.90} & 0.11 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
& 0.38 & 0.07 & & -0.23 & 0.89 & & \\
& & 0.65 & 0.69 & & -0.05 & 0.31 & \\
& & & \mathbf{0.05} & & & -\mathbf{0.10} & -0.99
\end{array}\right]
$$

$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}
$$

$$\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & -0.04 & & -0.47 & -0.44 & & \\
 & & 0.75 & -0.60 & & -0.06 & -0.27 & \\
 & & & 0.41 & & & -0.90 & \mathbf{0.11} \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & 0.65 & 0.69 & & -0.05 & 0.31 & \\
 & & & 0.05 & & & -0.10 & -\mathbf{0.99}
\end{array}\right]$$

$$= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix}$$

$$
\left[\begin{array}{rrrr|rrrr}
0.87 & -0.26 & & & -0.42 & & & \\
& 0.76 & -0.04 & & -0.47 & -0.44 & & \\
& & 0.75 & -0.60 & & -0.06 & -0.27 & \\
& & & 0.41 & & & -0.90 & -\mathbf{0.11} \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
& 0.38 & 0.07 & & -0.23 & 0.89 & & \\
& & 0.65 & 0.69 & & -0.05 & 0.31 & \\
& & & 0.05 & & & -0.10 & \mathbf{0.99}
\end{array}\right]
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$$
= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} I & \\ & F \end{bmatrix}
$$

$$\left[\begin{array}{cccc|cccc}
0.87 & -0.26 & & & -0.42 & & & \\
 & 0.76 & -0.04 & & -0.47 & -0.44 & & \\
 & & 0.75 & -0.60 & & -0.06 & -0.27 & \\
 & & & 0.41 & & & -0.90 & -0.11 \\
\hline
0.49 & 0.46 & & & 0.74 & & & \\
 & 0.38 & 0.07 & & -0.23 & 0.89 & & \\
 & & 0.65 & 0.69 & & -0.05 & 0.31 & \\
 & & & 0.05 & & & -0.10 & 0.99
\end{array}\right]$$

$$= \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} X \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} I & \\ & F \end{bmatrix}$$

Now apply simultaneous QR iteration [Sutton 2009] or divide-and-conquer [Sutton (under review)].

In simultaneous bidiagonalization for the 2-by-2 CSD, a Householder reflector is

- constructed from two collinear columns or rows,
- applied to two collinear columns or rows.

The extra information in the 2-by-2 CSD is *helpful*—one of the two vectors is guaranteed to have norm $\geq 1/\sqrt{2}$, making the Householder reflector well determined.

For the 2-by-1 CSD $\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V_1^T$,

- half of the input is, in a sense, missing,
- rows are not orthonormal.

Can we simultaneously bidiagonalize X_{11} and X_{21} stably?

For the 2-by-1 CSD $\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V_1^T$,

- half of the input is, in a sense, missing,
- rows are not orthonormal.

Can we simultaneously bidiagonalize X_{11} and X_{21} stably?

Yes!

- Full QR decomposition.

$$\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

- 2-by-2 CSD.

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}^T$$

- Discard the right half.

$$\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} V_1^T$$

Can we simultaneously bidiagonalize X_{11} and X_{21} stably *and efficiently*?

$$\underset{\substack{\text{orthonormal}\\ \text{columns}}}{\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}} = \underset{\text{orthogonal}}{\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix}} \underset{\substack{\text{bidiagonal}\\ \text{blocks}}}{\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}} \underset{\text{orthogonal}}{V_1^T}$$

This is our problem!

The key trick in simultaneous bidiagonalization is to construct a Householder reflector from two collinear columns or rows.

With the 2-by-1 CSD, we never have the second column.

Can it simply be ignored?

The key trick in simultaneous bidiagonalization is to construct a Householder reflector from two collinear columns or rows.

With the 2-by-1 CSD, we never have the second column.

Can it simply be ignored?

No!

$$
\left[\begin{array}{rrrr}
0.58 & 0.13 & -0.16 & 0.78 \\
-0.05 & -0.76 & 0.17 & 0.28 \\
-0.12 & -0.18 & 0.70 & 0.17 \\
-0.05 & -0.55 & -0.41 & -0.02 \\
\hline
0.68 & -0.24 & 0.10 & -0.44 \\
-0.22 & -0.01 & -0.20 & 0.11 \\
0.36 & -0.02 & 0.12 & -0.25 \\
0.02 & 0.16 & 0.47 & 0.11
\end{array}\right]
$$

$$
\left\| I - X^T X \right\|_2 \approx 2.2 \times 10^{-15}
$$

$$\left[\begin{array}{rrrr} \mathbf{0.58} & 0.13 & -0.16 & 0.78 \\ -\mathbf{0.05} & -0.76 & 0.17 & 0.28 \\ -\mathbf{0.12} & -0.18 & 0.70 & 0.17 \\ -\mathbf{0.05} & -0.55 & -0.41 & -0.02 \\ \hline \mathbf{0.68} & -0.24 & 0.10 & -0.44 \\ -\mathbf{0.22} & -0.01 & -0.20 & 0.11 \\ \mathbf{0.36} & -0.02 & 0.12 & -0.25 \\ \mathbf{0.02} & 0.16 & 0.47 & 0.11 \end{array}\right]$$

$$\left[\begin{array}{rrrr}
\mathbf{0.59} & 0.27 & -0.28 & 0.71 \\
\mathbf{10^{-18}} & 0.74 & -0.15 & -0.34 \\
\mathbf{10^{-17}} & 0.14 & -0.66 & -0.31 \\
\mathbf{10^{-18}} & 0.53 & 0.43 & -0.04 \\
\hline
\mathbf{0.81} & -0.20 & 0.20 & -0.52 \\
\mathbf{10^{-17}} & 0.08 & 0.16 & 0.03 \\
\mathbf{-10^{-17}} & -0.09 & -0.05 & 0.02 \\
\mathbf{-10^{-18}} & -0.16 & -0.46 & -0.12
\end{array}\right]$$

$$\left[\begin{array}{rrrr} 0.59 & \mathbf{0.27} & -\mathbf{0.28} & \mathbf{0.71} \\ 10^{-18} & 0.74 & -0.15 & -0.34 \\ 10^{-17} & 0.14 & -0.66 & -0.31 \\ 10^{-18} & 0.53 & 0.43 & -0.04 \\ \hline 0.81 & -\mathbf{0.20} & \mathbf{0.20} & -\mathbf{0.52} \\ 10^{-17} & 0.08 & 0.16 & 0.03 \\ -10^{-17} & -0.09 & -0.05 & 0.02 \\ -10^{-18} & -0.16 & -0.46 & -0.12 \end{array}\right]$$

$$\left[\begin{array}{cccc}
0.59 & -\mathbf{0.81} & \mathbf{10^{-16}} & -\mathbf{10^{-16}} \\
10^{-18} & -10^{-11} & 0.04 & -0.83 \\
10^{-17} & 10^{-11} & -0.62 & -0.40 \\
10^{-18} & 10^{-11} & 0.56 & -0.38 \\
\hline
0.81 & \mathbf{0.59} & -\mathbf{10^{-17}} & \mathbf{0} \\
10^{-17} & -10^{-11} & 0.18 & -0.02 \\
-10^{-17} & 10^{-11} & -0.07 & 0.08 \\
-10^{-18} & -10^{-11} & -0.51 & -0.01
\end{array}\right]$$

$$\left[\begin{array}{rrrr}
0.59 & -0.81 & 10^{-16} & -10^{-16} \\
10^{-18} & -\mathbf{10^{-11}} & 0.04 & -0.83 \\
10^{-17} & \mathbf{10^{-11}} & -0.62 & -0.40 \\
10^{-18} & \mathbf{10^{-11}} & 0.56 & -0.38 \\
\hline
0.81 & 0.59 & -10^{-17} & \\
10^{-17} & -\mathbf{10^{-11}} & 0.18 & -0.02 \\
-10^{-17} & \mathbf{10^{-11}} & -0.07 & 0.08 \\
-10^{-18} & -\mathbf{10^{-11}} & -0.51 & -0.01
\end{array}\right]$$

$$\left[\begin{array}{rrrr}
0.59 & -0.81 & 10^{-16} & -10^{-16} \\
10^{-18} & \mathbf{10^{-11}} & 0.03 & -0.13 \\
10^{-17} & \mathbf{10^{-27}} & -0.62 & -0.67 \\
10^{-18} & \mathbf{10^{-27}} & 0.57 & -0.73 \\
\hline
0.81 & 0.59 & -10^{-17} & \\
-10^{-16} & \mathbf{10^{-11}} & -0.02 & 0.08 \\
10^{-17} & \mathbf{10^{-26}} & 0.07 & 0.01 \\
-10^{-17} & \mathbf{-10^{-27}} & -0.54 & 10^{-3}
\end{array}\right]$$

$$
\left[\begin{array}{cccc}
0.59 & \text{-}0.81 & 10^{\text{-}16} & \text{-}10^{\text{-}16} \\
10^{\text{-}18} & 10^{\text{-}11} & \mathbf{0.03} & \text{-}\mathbf{0.13} \\
10^{\text{-}17} & 10^{\text{-}27} & \text{-}0.62 & \text{-}0.67 \\
10^{\text{-}18} & 10^{\text{-}27} & 0.57 & \text{-}0.73 \\
\hline
0.81 & 0.59 & \text{-}10^{\text{-}17} & \\
\text{-}10^{\text{-}16} & 10^{\text{-}11} & \text{-}\mathbf{0.02} & \mathbf{0.08} \\
10^{\text{-}17} & 10^{\text{-}26} & 0.07 & 0.01 \\
\text{-}10^{\text{-}17} & \text{-}10^{\text{-}27} & \text{-}0.54 & 10^{\text{-}3}
\end{array}\right]
$$

$$\left[\begin{array}{cccc}
0.59 & -0.81 & -10^{-16} & 10^{-17} \\
10^{-18} & 10^{-11} & -\mathbf{0.13} & -\mathbf{10^{-6}} \\
10^{-17} & 10^{-27} & -0.50 & -0.76 \\
10^{-18} & 10^{-27} & -0.84 & 0.38 \\
\hline
0.81 & 0.59 & 10^{-18} & -10^{-17} \\
-10^{-16} & 10^{-11} & \mathbf{0.08} & -\mathbf{10^{-6}} \\
10^{-17} & 10^{-26} & -0.01 & 0.07 \\
-10^{-17} & -10^{-27} & 0.13 & -0.52
\end{array}\right]$$

$$\left[\begin{array}{cccc}
0.59 & -0.81 & -10^{-16} & 10^{-17} \\
10^{-18} & 10^{-11} & -0.13 & -10^{-6} \\
10^{-17} & 10^{-27} & -\mathbf{0.50} & -0.76 \\
10^{-18} & 10^{-27} & -\mathbf{0.84} & 0.38 \\
\hline
0.81 & 0.59 & 10^{-18} & -10^{-17} \\
-10^{-16} & 10^{-11} & 0.08 & -10^{-6} \\
10^{-17} & 10^{-26} & -\mathbf{0.01} & 0.07 \\
-10^{-17} & -10^{-27} & \mathbf{0.13} & -0.52
\end{array}\right]$$

$$\left[\begin{array}{rrrr}
0.59 & -0.81 & -10^{-16} & 10^{-17} \\
10^{-18} & 10^{-11} & -0.13 & -10^{-6} \\
-10^{-17} & -10^{-27} & \mathbf{0.98} & 0.07 \\
-10^{-18} & -10^{-27} & \mathbf{10^{-16}} & 0.85 \\
\hline
0.81 & 0.59 & 10^{-18} & -10^{-17} \\
-10^{-16} & 10^{-11} & 0.08 & -10^{-6} \\
-10^{-17} & -10^{-27} & \mathbf{0.13} & -0.52 \\
10^{-17} & 10^{-26} & \mathbf{10^{-18}} & 0.03
\end{array}\right]$$

$$\left[\begin{array}{rrrr}
0.59 & -0.81 & -10^{-16} & 10^{-17} \\
10^{-18} & 10^{-11} & -0.13 & -10^{-6} \\
-10^{-17} & -10^{-27} & 0.98 & \mathbf{0.07} \\
-10^{-18} & -10^{-27} & 10^{-16} & 0.85 \\
\hline
0.81 & 0.59 & 10^{-18} & -10^{-17} \\
-10^{-16} & 10^{-11} & 0.08 & -10^{-6} \\
-10^{-17} & -10^{-27} & 0.13 & -\mathbf{0.52} \\
10^{-17} & 10^{-26} & 10^{-18} & 0.03
\end{array}\right]$$

$$\left[\begin{array}{rrrr}
0.59 & -0.81 & -10^{-16} & -10^{-17} \\
10^{-18} & 10^{-11} & -0.13 & 10^{-6} \\
-10^{-17} & -10^{-27} & 0.98 & -\mathbf{0.07} \\
-10^{-18} & -10^{-27} & 10^{-16} & -0.85 \\
\hline
0.81 & 0.59 & 10^{-18} & 10^{-17} \\
-10^{-16} & 10^{-11} & 0.08 & 10^{-6} \\
-10^{-17} & -10^{-27} & 0.13 & \mathbf{0.52} \\
10^{-17} & 10^{-26} & 10^{-18} & -0.03
\end{array}\right]$$

$$
\left[\begin{array}{rrrr}
0.59 & -0.81 & -10^{-16} & -10^{-17} \\
10^{-18} & 10^{-11} & -0.13 & 10^{-6} \\
-10^{-17} & -10^{-27} & 0.98 & -0.07 \\
-10^{-18} & -10^{-27} & 10^{-16} & -\mathbf{0.85} \\
\hline
0.81 & 0.59 & 10^{-18} & 10^{-17} \\
-10^{-16} & 10^{-11} & 0.08 & 10^{-6} \\
-10^{-17} & -10^{-27} & 0.13 & 0.52 \\
10^{-17} & 10^{-26} & 10^{-18} & -\mathbf{0.03}
\end{array}\right]
$$

$$
\left[\begin{array}{rrrr}
0.59 & -0.81 & -10^{-16} & -10^{-17} \\
10^{-18} & 10^{-11} & -0.13 & 10^{-6} \\
-10^{-17} & -10^{-27} & 0.98 & -0.07 \\
10^{-18} & 10^{-27} & -10^{-16} & \mathbf{0.85} \\
\hline
0.81 & 0.59 & 10^{-18} & 10^{-17} \\
-10^{-16} & 10^{-11} & 0.08 & 10^{-6} \\
-10^{-17} & -10^{-27} & 0.13 & 0.52 \\
-10^{-17} & -10^{-26} & -10^{-18} & \mathbf{0.03}
\end{array}\right]
$$

$$
\left[\begin{array}{cccc}
0.59 & -0.81 & & \\
 & 10^{-11} & -0.13 & 10^{-6} \\
 & & 0.98 & -0.07 \\
 & & & 0.85 \\
\hline
0.81 & 0.59 & & \\
 & 10^{-11} & 0.08 & 10^{-6} \\
 & & 0.13 & 0.52 \\
 & & & 0.03
\end{array}\right]
$$

$$
\left[\begin{array}{cccc}
0.59 & -0.81 & & \\
 & 10^{-11} & -0.13 & \mathbf{10^{-6}} \\
 & & 0.98 & -0.07 \\
 & & & 0.85 \\
\hline
0.81 & 0.59 & & \\
 & 10^{-11} & 0.08 & \mathbf{10^{-6}} \\
 & & 0.13 & 0.52 \\
 & & & 0.03
\end{array}\right]
$$

Some entries are not numerically zero as they should be.

This does not happen for the 2-by-2 case. The proof relies on the orthogonality of the rows in addition to the columns.

Can we simultaneously bidiagonalize X_{11} and X_{21} stably and efficiently?

Yes, but we need information from X_{12} and X_{22}.

Instead of generating the right half of the matrix explicitly with a QR factorization—

$$\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

—generate it implicitly, one column at a time, without using any extra storage.

$$\left[\begin{array}{cccc|cccc}
\times & \times & \times & \times & ? & ? & ? & ? \\
\times & \times & \times & \times & ? & ? & ? & ? \\
\times & \times & \times & \times & ? & ? & ? & ? \\
\times & \times & \times & \times & ? & ? & ? & ? \\
\hline
\times & \times & \times & \times & ? & ? & ? & ? \\
\times & \times & \times & \times & ? & ? & ? & ? \\
\times & \times & \times & \times & ? & ? & ? & ? \\
\times & \times & \times & \times & ? & ? & ? & ?
\end{array}\right]$$

Problem: Simultaneously bidiagonalize the blocks on the left.

Method: Pretend that the right half of the matrix exists but has not been observed. Complete it one column at a time.

$$\left[\begin{array}{cccc|cccc}
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\hline
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ? \\
\boldsymbol{\times} & \times & \times & \times & ? & ? & ? & ?
\end{array}\right]$$

Householder reflectors from left...

$$
\left[\begin{array}{cccc|cccc}
\boldsymbol{c_1} & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
\hline
\boldsymbol{s_1} & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ?
\end{array}\right]
$$

$c_i = \cos\theta_i, \qquad s_i = \sin\theta_i$

$$
\left[\begin{array}{cccc|cccc}
c_1 & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \textbf{?} & \textbf{?} & \textbf{?} & \textbf{?} \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
\hline
s_1 & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \textbf{?} & \textbf{?} & \textbf{?} & \textbf{?} \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ?
\end{array}\right]
$$

Householder reflector from right; complete rows 1 and 5...

$$
\left[\begin{array}{cccc|cccc}
c_1 & \boldsymbol{-s_1 s_1'} & & & \boldsymbol{-s_1 c_1'} & & & \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
\hline
s_1 & \boldsymbol{c_1 s_1'} & & & \boldsymbol{c_1 c_1'} & & & \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ?
\end{array}\right]
$$

$$
c_i = \cos\theta_i, \qquad s_i = \sin\theta_i, \qquad c_i' = \cos\phi_i, \qquad s_i' = \sin\phi_i
$$

Rows 1 and 5 are orthonormal.

$$
\left[\begin{array}{cccc|cccc}
c_1 & -s_1 s_1' & & & -s_1 c_1' & & & \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
\hline
s_1 & c_1 s_1' & & & c_1 c_1' & & & \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ? \\
 & \times & \times & \times & ? & ? & ? & ?
\end{array}\right]
$$

Now for the tricky part. We would like to construct Householder reflectors from columns 2 and 5, but we don't have column 5.

The second column must have a specific form by orthogonality. . . .

$$
\left[\begin{array}{cccc|cccc}
c_1 & -s_1 s_1' & & & -s_1 c_1' & & & \\
 & \vdots & \times & \times & ? & ? & ? & ? \\
 & c_1' x & \times & \times & ? & ? & ? & ? \\
 & \vdots & \times & \times & ? & ? & ? & ? \\
\hline
s_1 & c_1 s_1' & & & c_1 c_1' & & & \\
 & \vdots & \times & \times & ? & ? & ? & ? \\
 & c_1' y & \times & \times & ? & ? & ? & ? \\
 & \vdots & \times & \times & ? & ? & ? & ?
\end{array}\right]
$$

Orthogonality forces $\|(x, y)\| = 1$.

Notice that we cannot measure (x, y) reliably; we only have $c_1'(x, y)$. Orthogonalize against columns 3, 4 using Kahan's "twice is enough." (If the projection is zero, then choose another vector arbitrarily.)

Then, complete column 5. . . .

$$
\left[\begin{array}{cccc|cccc}
c_1 & -s_1 s_1' & & & -s_1 c_1' & & & \\
 & \vdots & \times & \times & \vdots & ? & ? & ? \\
 & c_1' \bar{x} & \times & \times & -s_1' \bar{x} & ? & ? & ? \\
 & \vdots & \times & \times & \vdots & ? & ? & ? \\
\hline
s_1 & c_1 s_1' & & & c_1 c_1' & & & \\
 & \vdots & \times & \times & \vdots & ? & ? & ? \\
 & c_1' \bar{y} & \times & \times & -s_1' \bar{y} & ? & ? & ? \\
 & \vdots & \times & \times & \vdots & ? & ? & ?
\end{array}\right]
$$

Columns $1, \ldots, 5$ are orthonormal.

$$
\left[\begin{array}{cccc|cccc}
c_1 & -s_1 s_1' & & & -s_1 c_1' & & & \\
 & \vdots & \times & \times & \vdots & ? & ? & ? \\
 & \boldsymbol{c_1' \bar{x}} & \times & \times & \boldsymbol{-s_1' \bar{x}} & ? & ? & ? \\
 & \vdots & \times & \times & \vdots & ? & ? & ? \\
\hline
s_1 & c_1 s_1' & & & c_1 c_1' & & & \\
 & \vdots & \times & \times & \vdots & ? & ? & ? \\
 & \boldsymbol{c_1' \bar{y}} & \times & \times & \boldsymbol{-s_1' \bar{y}} & ? & ? & ? \\
 & \vdots & \times & \times & \vdots & ? & ? & ?
\end{array}\right]
$$

$$
\left[\begin{array}{cccc|cccc}
c_1 & -s_1 s_1' & & & -s_1 c_1' & & & \\
 & \boldsymbol{c_2 c_1'} & \times & \times & \boldsymbol{-c_2 s_1'} & ? & ? & ? \\
 & & \times & \times & & ? & ? & ? \\
 & & \times & \times & & ? & ? & ? \\
\hline
s_1 & c_1 s_1' & & & c_1 c_1' & & & \\
 & \boldsymbol{s_2 c_1'} & \times & \times & \boldsymbol{-s_2 s_1'} & ? & ? & ? \\
 & & \times & \times & & ? & ? & ? \\
 & & \times & \times & & ? & ? & ?
\end{array}\right]
$$

Continue....

$$\left[\begin{array}{cccc|cccc}
c_1 & -s_1 s_1' & & & -s_1 c_1' & & & \\
 & c_2 c_1' & -s_2 s_2' & & -c_2 s_1' & -s_2 c_2' & & \\
 & & c_3 c_2' & -s_3 s_3' & & -c_3 s_2' & -s_3 c_3' & \\
 & & & c_4 c_3' & & & -c_4 s_3' & -s_4 \\
\hline
s_1 & c_1 s_1' & & & c_1 c_1' & & & \\
 & s_2 c_1' & c_2 s_2' & & -s_2 s_1' & c_2 c_2' & & \\
 & & s_3 c_2' & c_3 s_3' & & -s_3 s_2' & c_3 c_3' & \\
 & & & s_4 c_3' & & & -s_4 s_3' & c_4
\end{array}\right]$$

We've actually computed

$$\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} V_1 & \\ & I \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with $X_{12} := U_1 B_{12}$ and $X_{22} := U_2 B_{22}$.

This has been done without storing X_{12} or X_{22} in computer memory.

To obtain the 2-by-1 CSD, discard the right half:

$$\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix}^T \begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} V_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}.$$

Summary:

- If $\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}$ has orthonormal columns, then its blocks can be simultaneously bidiagonalized.
- The reduction is numerically stable.
  - The naive approach does not work.
  - The solution uses an extra orthogonalization step.
- The reduction is efficient.
  - Typically, the orthogonalization step is dominated by a single matrix-vector multiply. (If the numerical projection is zero, then the search for a vector in the orthogonal complement requires one or more additional matrix-vector multiplies.)

References:

[Sutton 2009] "Computing the complete CS decomposition." *Numer. Algorithms.* 50 (2009), no. 1, 33–65.