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Semilinear Geometric Optics for Generalized Solutions

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Abstract. This paper is devoted to the study of nonlinear geometric optics in Colombeau algebras of generalized functions in the case of Cauchy problems for semilinear hyperbolic systems in one space variable. Extending classical results, we establish a generalized variant of nonlinear geometric optics. As an application, a nonlinear superposition principle is obtained when distributional initial data are perturbed by rapid oscillations.

Keywords: *Semilinear hyperbolic systems, Cauchy problems, nonlinear geometric optics, generalized solutions, delta waves*

AMS subject classification: 35 L 45, 35 B 05, 35 D 05, 46 F 30

1. Introduction

Consider the following model problem

$$\left.\begin{aligned} (\partial_t + A(t,x)\partial_x)u^\varepsilon &= f(t,x,u^\varepsilon) && \text{on } \mathrm{R}^2 \\ u^\varepsilon(0,x) &= a^\varepsilon(x) && \text{for } x \in \mathrm{R} \end{aligned}\right\} \tag{1.1}$$

where $A(t,x)$ is a smooth $m \times m$ matrix such that the operator $L = \partial_t + A(t,x)\partial_x$ is strictly hyperbolic with respect to t, $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)^T$, and $f(t,x,u)$ is smooth with $f(t,x,0) = 0$. Assume the initial data $a^\varepsilon(x)$ admit an asymptotic expansion

$$a^\varepsilon(x) = a\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big) + o(1)$$

with $a(x,\theta_0)$ almost periodic in the variable θ_0. The method of geometric optics establishes an asymptotic expansion for the solution

$$u^\varepsilon(t,x) = U\Big(t,x; \frac{\vec{\phi}(t,x)}{\varepsilon}\Big) + o(1)$$

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where $U(t, x; \theta)$ is almost periodic in θ, and the oscillation phases $\vec{\phi}(t, x)$ of U need to be determined. In the nonlinear case, there are resonance phenomena and thus the three basic problems:

- existence of u^ε
- determination of $U(t, x; \theta)$ as well as $\vec{\phi}(t, x)$
- proof of asymptotics

require an elaborate theory. For classical solutions, both the formal derivation of the nonlinear equation satisfied by the function $U(t, x; \theta)$ as well as the justification of the asymptotic analysis have been treated in the recent literature in detail (see Joly, Métivier and Rauch [7], and Majda and Rosales [9]).

On the other hand, generalized solutions to problem (1.1) can be constructed when the initial data are distributions. In some cases, these can be described as weak limits of approximate solutions (delta waves, see Rauch and Reed [16], and [14, 15]). An appropriate general framework for studying nonlinear equations with distributional data is provided by the algebras of generalized functions developed by Colombeau [3 - 5]. Existence and uniqueness results for solutions in these algebras are known ([12, 13]; see also Rosinger [17, 18] for a general theory).

The purpose of this paper is to develop a theory of nonlinear geometric optics for problem (1.1) in Colombeau algebras of generalized functions. In this way distributional data perturbed by rapid oscillations can be studied. The present investigation of the model problem (1.1) is a contribution towards a nonlinear asymptotic theory in the Colombeau setting.

The paper is arranged as follows: In Section 2 we recall basic notions on function spaces as well as the theory of Colombeau algebras and introduce convergence and asymptotic expansions in these algebras. In Section 3 we establish the general theory of semilinear geometric optics in Colombeau algebras in the case of the Cauchy problem for one space dimensional first order hyperbolic systems. We apply our theory to initial data having a delta function part and obtain a superposition principle in nonlinear geometric optics analogous to the non-oscillatory case considered in [14, 15] and in Rauch and Reed [16]. The notion of asymptotics in the Colombeau algebra requires estimates in the representing sequences for all derivatives. For completeness, these estimates, as far as not appearing in the literature, are presented in an appendix.

What concerns geometric optics for nonclassical solutions, we mention that bounded variation solutions in conservative hyperbolic systems have been studied by DiPerna and Majda [6] and Schochet [19], and oscillatory shock waves have been studied by [21, 22] and Williams [23].

2. Spaces of functions and generalized functions

In this section we first collect a number of classical notions which we need. Then we recall various basic constructions of differential algebras containing the space of distributions and introduce convergence structures and asymptotic expansions.

Let Ω be an open subset or the closure of an open subset of R^n. For the purpose of geometric optics, it will be convenient to introduce the ε-dependent norm

$$\|v\|_{k,\varepsilon,K} = \sum_{|\alpha|\leq k} \varepsilon^{|\alpha|} \sup_{x\in K} |\partial^\alpha v(x)| \tag{2.1}$$

where K is a relatively compact subset of $\Omega, k \in \mathrm{N}$ and $v \in C^\infty(\Omega)$.

Definition 2.1. We say that a net $(u^\varepsilon)_{\varepsilon>0} \subset C^\infty(\Omega)$ has the *classical asymptotic expansion*

$$u^\varepsilon(x) = u_0^\varepsilon(x) + o(1)$$

($u_0^\varepsilon \in C^\infty(\Omega)$) if, for all $K \subset\subset \Omega$ and all $k \in \mathrm{N}$, $\|u^\varepsilon - u_0^\varepsilon\|_{k,\varepsilon,K} \longrightarrow 0$ as $\varepsilon \to 0$.

Recall that the space of *almost periodic functions* on a finite-dimensional real vector space V is the Banach subspace of $L^\infty(V)$ generated by the exponentials $e^{i\langle\lambda,\theta\rangle}$ with $\theta \in V$ and $\lambda \in V^*$, the dual space of V (see Katznelson [8]). Denote by $C_p^0(V)$ the subspace of real-valued almost periodic functions on V.

Notation 2.2. With Ω and V as above, we denote by

$$\mathcal{C}_p^0(\Omega : V) = C^0(\Omega : C_p^0(V))$$

the space of continuous functions from Ω into $C_p^0(V)$. For $k \in \mathrm{N}$, the space

$$\mathcal{C}_p^k(\Omega : V)$$

is the subspace of $C_p^0(\Omega : V)$ of those functions whose derivatives (with respect to $x \in \Omega$ and $\theta \in V$) up to order k belong to $\mathcal{C}_p^0(\Omega : V)$. Let

$$\mathcal{C}_p^\infty(\Omega : V) = \cap_{k\geq 0}\mathcal{C}_p^k(\Omega : V).$$

What concerns Colombeau algebras of generalized functions [3 - 5], we employ the following definitions. Let $I = (0, 1]$ the semi-open unit interval. The set of all nets $(u_\eta)_{\eta\in I}$ of smooth functions $u_\eta \in C^\infty(\Omega)$ forms a differential algebra $\mathcal{E}[\Omega]$ under componentwise multiplication and partial differentiation. The subalgebra $\mathcal{E}_M[\Omega]$ is defined by those elements $(u_\eta)_{\eta\in I}$ of $\mathcal{E}[\Omega]$ which grow only moderately as $\eta \to 0$, i.e. satisfy the property: for all $K \subset\subset \Omega$ and $\alpha \in \mathrm{N}^n$ there exists $p \geq 0$ such that

$$\sup_{\eta\in I}\sup_{x\in K} \eta^p |\partial^\alpha u_\eta(x)| < \infty. \tag{2.2}$$

The differential ideal $\mathcal{N}(\Omega)$ of $\mathcal{E}_M[\Omega]$ is composed of those $(u_\eta)_{\eta\in I}$ with the property that, for all $K \subset\subset \Omega$, $\alpha \in \mathrm{N}^n$ and $q \geq 0$,

$$\sup_{\eta\in I}\sup_{x\in K} \eta^{-q} |\partial^\alpha u_\eta(x)| < \infty. \tag{2.3}$$

The *Colombeau algebra* $\mathcal{G}(\Omega)$ is defined to be the factor algebra $\mathcal{E}_M[\Omega]/\mathcal{N}(\Omega)$.

If Ω is an open subset of R^n, the space of distributions $\mathcal{D}'(\Omega)$ can be imbedded into $\mathcal{E}_M[\Omega]$ and $\mathcal{G}(\Omega)$ as follows. Take $\varphi \in \mathcal{S}(\mathrm{R}^n)$ with all moments vanishing, i.e.

$$\int \varphi(x)\, dx = 1$$
$$\int x^p \varphi(x)\, dx = 0 \quad \forall\, p \in \mathrm{N}, p \geq 1$$

and define the mollifier φ_η by $\varphi_\eta(x) = \frac{1}{\eta}\varphi(\frac{x}{\eta})$. The assignment

$$\iota : w \longrightarrow (w * \varphi_\eta)_{\eta \in I} \tag{2.4}$$

defines an imbedding of $\mathcal{E}'(\Omega)$ into $\mathcal{E}_M[\Omega]$. By using a locally finite smooth partition of unity, this can be extended to an imbedding of $\mathcal{D}'(\Omega)$. Taking the equivalence class of the expression on the right-hand side of (2.4) produces an imbedding of $\mathcal{E}'(\Omega)$ and then of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$.

For detailed explanations of this construction we refer to the literature, for example [1, 3 - 5, 12]. Here we just recall some further properties of $\mathcal{E}_M[\Omega]$ and $\mathcal{G}(\Omega)$ needed in our study of nonlinear hyperbolic problems. First, if $f : \mathrm{C}^N \to \mathrm{C}$ is a smooth map all whose derivatives are polynomially bounded, and $(u_\eta)_{\eta \in I} \in (\mathcal{E}_M[\Omega])^N$, then $(f(u_\eta))_{\eta \in I}$ belongs to $\mathcal{E}_M[\Omega]$ as well. Thus polynomially bounded nonlinear maps are defined on $\mathcal{E}_M[\Omega]$. Next, the elements of $\mathcal{E}_M[\Omega]$ have restrictions to open subsets of Ω as well as to coordinate hyperplanes. Thus if $(u_\eta)_{\eta \in I}$ is a member of $\mathcal{E}_M[[0,\infty) \times \mathrm{R}]$, then $(u_\eta|\{t=0\})_{\eta \in I}$ is an element of $\mathcal{E}_M[\mathrm{R}]$. All this is easily seen to be true of $\mathcal{G}(\Omega)$ as well. It follows that initial value problems like (1.1) can be formulated in the setting of $\mathcal{E}_M$ and of $\mathcal{G}$.

The notion of *association* identifies elements of $\mathcal{E}_M[\Omega]$, Ω an open subset of R^n, if they behave equivalently in the sense of distributions: we say that $u = (u_\eta)_{\eta \in I}$, $v = (v_\eta)_{\eta \in I} \in \mathcal{E}_M[\Omega]$ are *associated*, $u \approx v$, if

$$\lim_{\eta \to 0} (u_\eta - v_\eta) = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

In the case u is associated with $\iota(w)$ for a distribution $w \in \mathcal{D}'(\Omega)$, we say that u admits w as associated distribution.

We shall have need of *almost periodic generalized functions* as well. V denotes again a finite-dimensional vector space. Let $\mathcal{E}_{M,p}[\Omega \times V]$ be the subalgebra of $\mathcal{E}_M[\Omega \times V]$ whose elements $(u_\eta(x,\theta))_{\eta \in I}$ are almost periodic in $\theta \in V$. Let $\mathcal{N}_p(\Omega \times V)$ be the ideal in $\mathcal{E}_{M,p}[\Omega \times V]$ characterized by the property in (2.3). We define the factor algebra $\mathcal{G}_p(\Omega \times V) = \mathcal{E}_{M,p}[\Omega \times V]/\mathcal{N}_p(\Omega \times V)$. It is easy to see that $\mathcal{G}_p(\Omega \times V)$ is a subalgebra of $\mathcal{G}(\Omega \times V)$.

We now turn to the central question of this section, the notion of an asymptotic expansion in Colombeau algebras. Topologies on $\mathcal{G}(\Omega)$ have been studied in Biagioni [1], Biagioni and Colombeau [2], Nedeljkov, Pilipović and Scarpalézos [11]. However, topology on $\mathcal{G}(\Omega)$ is a delicate matter: the so-called sharp topology turns $\mathcal{G}(\Omega)$ into a Hausdorff topological ring, but induces the discrete topology on the space of distributions. On the other hand, weaker topologies for which the imbedding of $\mathcal{D}'(\Omega)$ is continuous may fail to have the Hausdorff property. Here we proceed by introducing a sequential convergence structure on $\mathcal{E}_M[\Omega]$ which is adapted to property (2.2). It reduces to weak convergence on the subspace $\mathcal{D}'(\Omega)$ and will be the basis for defining asymptotic expansions (see Mikusiński and Boehme [10] for a general discussion of sequential convergence).

Definition 2.3. A sequence $(u^n)_{n\in\mathrm{N}} \subset \mathcal{E}_M[\Omega]$ is said to converge to $u \in \mathcal{E}_M[\Omega]$, if the following property holds: for all $K \subset\subset \Omega$ and $\alpha \in \mathrm{N}^n$ there exists $p \geq 0$ such that

$$\sup_{\eta\in I}\sup_{x\in K} \eta^p \left|\partial^\alpha\left(u^n_\eta(x) - u_\eta(x)\right)\right| \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{2.5}$$

This determines a Hausdorff sequential convergence structure on $\mathcal{E}_M[\Omega]$ for which addition, multiplication, and differentiation are sequentially continuous. Further, the imbedding $\iota : \mathcal{D}'(\Omega) \to \mathcal{E}_M[\Omega]$ is sequentially continuous (the essential ingredient to see this is the characterization of distributional convergence by means of the representation theorem in Schwartz [20: Chapter III, §6, Theorem XXIII]). The definition of an asymptotic expansion is straightforward in this setting (with norms as defined at the beginning of this section) and extends Definition 2.1 from smooth to generalized functions:

Definition 2.4. We say that an ε-net $(u^\varepsilon)_{\varepsilon>0} \subset \mathcal{E}_M[\Omega]$ has the *asymptotic expansion*

$$u^\varepsilon = u^\varepsilon_0 + o(1)$$

if for all $K \subset\subset \Omega$ and $k \in \mathrm{N}$ there exists $p \geq 0$ such that

$$\sup_{\eta\in I} \eta^p \|u^\varepsilon_\eta - u^\varepsilon_{0,\eta}\|_{k,\varepsilon,K} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0. \tag{2.6}$$

We remark that the corresponding convergence structure on the factor algebra $\mathcal{G}(\Omega)$ is no longer Hausdorff. But the Hausdorff property is essential for asymptotics. In order to retain it we will consequently formulate and prove our results on asymptotic expansions in the algebra $\mathcal{E}_M[\Omega]$. Stronger uniqueness assertions in the factor algebra $\mathcal{G}(\Omega)$, when appropriate, will be stated separately.

3. Geometric optics for generalized data

Before stating our results on semilinear geometric optics in Colombeau algebras, we need to recall various facts from the classical, smooth case. For the subsequent definitions and notions we follow Joly, Métivier and Rauch [7].

Without loss of generality, we suppose in the Cauchy problem (1.1) that

$$A(t,x) = \Lambda(t,x) = \mathrm{diag}\,[\lambda_1(t,x),\ldots,\lambda_m(t,x)] \tag{3.1}$$

is diagonal with $\lambda_1(t,x) < \ldots < \lambda_m(t,x)$, $f(t,x,u)$ is independent of (t,x), and to avoid technicalities, we will always assume that

(HP) $f(u)$ and all its derivatives are polynomially bounded in u, $f(u)$ is globally Lipschitz in u and $f(0) = 0$.

Denote by $\omega = [x_-, x_+]$ an interval on the $x-$axis and by $\Omega \subset \mathrm{R}^2_+$ a determinacy domain of ω for the Cauchy problem (1.1),

$$X_k = \partial_t + \lambda_k(t,x)\partial_x$$

the k-th propagation field, $k \in \{1, \dots, m\}$, and

$$t \longrightarrow \Gamma_k(t;t',x) = (t, \gamma_k(t;t',x))$$

the integral curve of X_k passing through the point $(t', x) \in \Omega$ at $t = t'$.

Choose $T_0 > 0$ sufficiently small, such that the curves $\Gamma_k(t;0,x)$ are defined for $t \in [0, T_0], x \in [x_-, x_+]$, and $\gamma_m(T_0;0,x_-) \le \gamma_1(T_0;0,x_+)$. Then, we can choose Ω as

$$\Omega = \Big\{(t,x) \in \mathrm{R}^2_+ \Big| \, 0 \le t \le T_0 \ \text{ and } \ \gamma_m(t;0,x_-) \le x \le \gamma_1(t;0,x_+)\Big\}. \qquad (3.2)$$

Set $\Omega_T = \Omega \cap \{t \le T\}$ for any $0 < T \le T_0$. Let $T_k(x) > 0$ be such that for $x \in [x_-, x_+]$ the characteristic curve $t \to \Gamma_k(t;0,x)$ remains in Ω for t in the maximal interval $[0, T_k(x)] \subset [0, T_0]$.

We assume that the initial oscillation phase $\phi_0(x)$ in (1.2) is a scalar function with non-degeneracy, $\phi_0'(x) \ne 0$ for for all $x \in \omega$. For all $k \in \{1, \dots, m\}$, $\phi_k(t,x)$ is the unique solution to the Cauchy problem

$$\left.\begin{aligned} X_k\phi_k = \partial_t\phi_k + \lambda_k(t,x)\partial_x\phi_k = 0 \\ \phi_k(0,x) = \phi_0(x) \end{aligned}\right\}. \qquad (3.3)$$

We suppose that the space of phases

$$\Phi = \operatorname{span}\big\{\phi_1(t,x), \dots, \phi_m(t,x)\big\} \qquad \text{in } C^\infty(\Omega)$$

satisfies the following transversality condition:

(TC) for all $\phi \in \Phi$, if $X_k\phi \equiv 0$, then ϕ is transverse to X_k meaning that, for all $x \in \omega$, $X_k\phi(\cdot, \gamma_k(\cdot, 0, x)) \ne 0$ almost everywhere on $[0, T_k(x)]$.

In order to formulate the equations for the leading profile of u^ε, we need to define the averaging operators E_k $(k = 1, \dots, m)$ on the space of almost periodic functions.

Denote by $\mathcal{R} = \{\alpha \in \mathrm{R}^m | \sum_{i=1}^m \alpha_i\phi_i \equiv 0\}$ the resonances in Φ, $\Psi = \{\theta \in \mathrm{R}^m : \langle\alpha, \theta\rangle = 0$ for all $\alpha \in \mathcal{R}\}$ the orthogonal complement of $\mathcal{R}$ in R^m, and E_k the extension of the following action on the space of almost periodic functions on Ψ:

$$\mathrm{E}_k(e^{i\langle\alpha,\theta\rangle}) = \begin{cases} e^{i\langle\alpha,\theta\rangle} & \text{if } \alpha \in \mathcal{R} \oplus \tilde{\Phi}_k \\ 0 & \text{otherwise} \end{cases} \qquad (3.4)$$

where $\tilde{\Phi}_k = \{\alpha \in \mathrm{R}^m : \alpha_j = 0 \text{ for all } j \ne k\}$. It is not difficult to see that this averaging operator can be equivalently defined by an integral formula as

$$(\mathrm{E}_k u)(\theta) = \lim_{T\to\infty} T^{-\dim(\Psi_k)} \int_{TQ} u(\theta + \varphi)\, d\varphi \qquad (3.5)$$

for any almost periodic function $u(\theta)$ on Ψ, where $\Psi_k = \{\theta \in \Psi : \theta_k = 0\}$, Q is a cube in Ψ_k of measure one, and $d\varphi$ is the Lebesgue measure on Ψ_k.

The following result on nonlinear geometric optics for classical solutions with asymptotic estimates in terms of the L^∞-norm has been proven in [7: Theorem 2.8.1]. However, for the generalized asymptotic expansion in the Colombeau algebra, similar estimates on all derivatives are needed. We formulate these higher order estimates, together with the known classical assertions, in the proposition below; the proof of the additional estimates is deferred to the Appendix. We will denote by $P_*(\cdot)$ a polynomial, by $\|u^\varepsilon\|_T$ and $\|U\|_T$ the norm of u^ε and U in $L^\infty(\Omega_T)$ and $L^\infty(\Omega_T \times \Psi)$, respectively, and by $\|\cdot\|$ the norm in $L^\infty(\omega)$. Hypotheses (HP) and (TC) are assumed to hold throughout.

Proposition 3.1. *Assume the initial data $a^\varepsilon \in C^\infty(\omega)$ admit an asymptotic expansion*

$$a^\varepsilon(x) = a\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big) + o(1) \tag{3.6}$$

in the sense of Definition 2.1, where $a(x,\theta) \in \mathcal{C}_p^\infty(\omega : \mathrm{R})$. Then:

(1) *There is a unique solution $u^\varepsilon \in C^\infty(\Omega)$ to problem (1.1), and the estimate*

$$\|\partial^\alpha_{(t,x)} u^\varepsilon\|_T \le C e^{CT}\bigg(\|\partial_x^{|\alpha|} a^\varepsilon\| + P_\alpha\bigg(\sum_{|\beta|<|\alpha|} \|\partial^\beta_{(t,x)} u^\varepsilon\|_T\bigg)\bigg) \tag{3.7}$$

holds for any $\alpha \in \mathrm{N}^2$ with $C = C(\|\nabla f\|_{L^\infty}, \|\Lambda\|_{W^{1,\infty}(\Omega)})$.

(2) *There exist $U_k \in \mathcal{C}_p^\infty(\Omega : \mathrm{R})$ $(k = 1, \dots, m)$ such that*

$$u_k^\varepsilon(t,x) = U_k\Big(t, x; \frac{\phi_k(t,x)}{\varepsilon}\Big) + o(1) \tag{3.8}$$

in the sense of Definition 2.1.

(3) *The U_k's are the unique solutions to the problem*

$$\left.\begin{aligned} X_k U_k &= \mathrm{E}_k f_k(U) \\ U_k(0,x;\theta_k) &= a_k(x,\theta_k) \end{aligned}\right\} \qquad (k = 1, \dots, m). \tag{3.9}$$

Moreover, the estimate

$$\|\partial^\alpha_{(t,x)} \partial^\gamma_\theta U\|_T \le C e^{CT}\bigg(\|\partial_x^{|\alpha|}\partial_{\theta_0}^{|\gamma|} a\| + P_{\alpha,\gamma}\bigg(\sum_{|\beta|<|\alpha|} \|\partial^\beta_{(t,x)}\partial^\gamma_\theta U\|_T\bigg)\bigg) \tag{3.10}$$

holds for any multi-index (α, γ) with the same C as above.

Proof. See Appendix ∎

With these preparations, we are now in the position to formulate and prove our central result on geometric optics for Colombeau generalized solutions to the semilinear hyperbolic problem (1.1). As noted in Section 2, we consider problem (1.1) in the algebra $\mathcal{E}_M[\Omega]$ what concerns the asymptotics. Uniqueness of solutions to the systems of differential equations (1.1) and (3.9) actually holds even in the factor algebra $\mathcal{G}(\Omega)$. The initial data $\mathbf{a}^\varepsilon(x)$ will belong to the algebra $\mathcal{E}_M[\omega]$ and admit an asymptotic expansion in the sense of Definition 2.4 with profiles $\mathbf{a}(x,\theta_0) \in \mathcal{E}_{M,p}[\omega \times \mathrm{R}]$. For clarity, we adopt here and in the sequel the boldface notation for elements of the algebras $\mathcal{E}_M$ or $\mathcal{G}$.

Theorem 3.2. *Suppose given initial data* $\mathbf{a}^\varepsilon(x) = (a^\varepsilon_\eta(x))_{\eta\in I} \subset \mathcal{E}_M[\omega]$ *with profiles* $\mathbf{a}(x,\theta_0) = (a_\eta(x,\theta_0))_{\eta\in I} \in \mathcal{E}_{M,p}[\omega\times \mathrm{R}]$ *satisfying*

$$\mathbf{a}^\varepsilon(x) = \mathbf{a}\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big) + o(1)$$

in the sense of (2.6)*. Then:*

(1) *There are unique solutions* $\mathbf{u}^\varepsilon(t,x) \in \mathcal{E}_M[\Omega]$ *and* $\mathbf{U}(t,x;\theta) \in \mathcal{E}_{M,p}[\Omega\times\Psi]$ *to problems* (1.1) *and* (3.9)*, respectively, and they admit the asymptotic expansion*

$$\mathbf{u}^\varepsilon(t,x) = \mathbf{U}\Big(t,x;\frac{\phi_1(t,x)}{\varepsilon},\dots,\frac{\phi_m(t,x)}{\varepsilon}\Big) + o(1) \tag{3.11}$$

in the sense of (2.6)*.*

(2) *The solutions* $\mathbf{u}^\varepsilon$ *and* $\mathbf{U}$ *to problems* (1.1) *and* (3.9) *are unique in* $\mathcal{G}(\Omega)$ *and* $\mathcal{G}_p(\Omega\times\Psi)$*, respectively.*

Proof. For fixed $\eta > 0$, Proposition 3.1 provides smooth, classical solutions u^ε_η and U_η with initial data a^ε_η and a_η, respectively. To show that these η-nets of classical solutions determine a generalized solution, estimates (2.2) have to be established. These estimates follow from (3.7) and (3.10), thus $\mathbf{u}^\varepsilon = (u^\varepsilon_\eta)_{\eta\in I} \in \mathcal{E}_M[\Omega]$ and $\mathbf{U} = (U_\eta)_{\eta\in I} \in \mathcal{E}_{M,p}[\Omega\times\Psi]$ provide solutions to problems (1.1) and (3.9), respectively. Uniqueness of $\mathbf{u}^\varepsilon$ in $\mathcal{G}(\Omega)$ can be found in [12] under assumption (HP).

From the hypotheses on the initial data $\mathbf{a}^\varepsilon$ we have that there is $p \ge 0$ such that

$$\sup_{\eta\in I}\sup_{x\in\omega} \eta^p \Big| a^\varepsilon_\eta(x) - a_\eta\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big)\Big| \longrightarrow 0 \qquad \text{when } \varepsilon\to 0.$$

To prove the asymptotic expansion in (3.11) we follow first [7] for the zero order estimate. By checking each step of the simultaneous Picard iteration given in [7: Section 5] and using the global Lipschitz property of f, it is not difficult to see that for the solutions $\mathbf{u}^\varepsilon$ and $\mathbf{U}$ we have

$$\sup_{\eta\in I}\sup_{(t,x)\in\Omega} \eta^p \Big| u^\varepsilon_\eta(t,x) - U_\eta\Big(t,x,\frac{\phi_1(t,x)}{\varepsilon},\dots,\frac{\phi_m(t,x)}{\varepsilon}\Big)\Big| \longrightarrow 0 \tag{3.12}$$

when $\varepsilon\to 0$.

To obtain the corresponding estimates in (3.12) for the derivatives of $u^\varepsilon_\eta(t,x)$, we act with differential operators $\varepsilon^{i+j}\partial^i_t\partial^j_x$ on the equations and proceed exactly as in the proof of Proposition 3.1 in the Appendix. Thus, we conclude that the asymptotic property (3.11) holds.

It remains to establish the uniqueness of $\mathbf{U}$ in $\mathcal{G}_p(\Omega\times\Psi)$. Suppose that $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ are two solutions to problem (3.9) in $\mathcal{G}_p(\Omega\times\Psi)$, which means that there are $\mathbf{N}$ in $\mathcal{N}_p(\Omega\times\Psi)$ and $\mathbf{b}$ in $\mathcal{N}_p(\omega\times\Psi)$ such that

$$\left.\begin{aligned} X_k(\mathbf{U}^{(1)}_k - \mathbf{U}^{(2)}_k) &= \mathrm{E}_k\big(f_k(\mathbf{U}^{(1)}) - f_k(\mathbf{U}^{(1)})\big) + \mathbf{N}_k \\ (\mathbf{U}^{(1)} - \mathbf{U}^{(2)})(0,x;\theta) &= \mathbf{b}(x,\theta) \end{aligned}\right\}$$

which implies that $\mathbf{U} = \mathbf{U}^{(1)} - \mathbf{U}^{(2)}$ satisfies

$$\left.\begin{aligned} X_k \mathbf{U}_k &= \mathrm{E}_k(\nabla f_k(\mathbf{V})\mathbf{U}) + \mathbf{N}_k \\ \mathbf{U}(0, x; \theta) &= \mathbf{b}(x, \theta) \end{aligned}\right\} \tag{3.13}$$

where $\mathbf{V} = \sigma \mathbf{U}^{(1)} + (1-\sigma)\mathbf{U}^{(2)}$ for some $\sigma \in (0,1)$. By integrating (3.13) along characteristic lines and using the global Lipschitz property of f, we can easily obtain that the difference $\mathbf{U}$ belongs to the null ideal $\mathcal{N}_p(\Omega \times \Psi)$, thus $\mathbf{U}^{(1)} = \mathbf{U}^{(2)}$ in $\mathcal{G}_p(\Omega \times \Psi)$ ∎

We now turn to an application of this result to delta waves. Theorem 3.2 applies, in particular, for initial data distributions, viewed as members of the Colombeau algebra $\mathcal{E}_M[\omega]$ by means of the imbedding ι. The corresponding solutions $\mathbf{u}^\varepsilon$ with profiles $\mathbf{U}$ are Colombeau generalized functions. However, it has been observed in the non-oscillatory case that for distributions with discrete support and certain nonlinear functions f the generalized solution admits an associated distribution, which in turn is split into a sum of a regular and a singular part (see, e.g., [14, 15] and Rauch and Reed [16]). Our goal is to establish a similar result for the solution $\mathbf{u}^\varepsilon$ and its profile $\mathbf{U}$, and further to conclude that the regular part in the associated distribution of $\mathbf{u}^\varepsilon$ takes the corresponding term of $\mathbf{U}$ as its leading profile.

For simplicity, we study the particular case where the leading term in the initial data consists of a measure with discrete support (that is, a sum of delta functions at different points). In this case, the smooth function $f(t, x, u)$ in (1.1) is required to be *sublinear in* u, that is,

$$\lim_{|u| \to \infty} \frac{|f(t, x, u)|}{|u|} = 0$$

uniformly in $(t, x) \in K$ for any relatively compact subset K of Ω.

Let

$$\mu_k(x) = \sum_{j=1}^{n} a_{jk} \delta(x - \xi_{jk})$$

be a measure with discrete support $\{\xi_{1k}, \ldots, \xi_{nk}\} \subset \omega, \mu = (\mu_1, \ldots, \mu_m)$. The corresponding element of $\mathcal{E}_M[\omega]$ is given by

$$\iota\mu = (\mu * \varphi_\eta)_{\eta \in I}.$$

Further, let $a(x, \theta_0) \in \mathcal{C}_p^\infty(\omega : \mathrm{R}) \subset \mathcal{E}_{M,p}[\omega \times \mathrm{R}]$ be smooth and almost periodic with respect to $\theta_0 \in \mathrm{R}$. Consider the Cauchy problem

$$\left.\begin{aligned} L\mathbf{u}^\varepsilon &= f(\mathbf{u}^\varepsilon) \\ \mathbf{u}^\varepsilon(0, x) &= \iota\mu(x) + a\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big) \end{aligned}\right\}. \tag{3.14}$$

From Theorem 3.2, we know that this problem has a unique generalized solution $\mathbf{u}^\varepsilon = (u_\eta^\varepsilon)_{\eta \in I} \in \mathcal{E}_M[\Omega]$. In addition, the solution admits the asymptotic expansion

$$\mathbf{u}^\varepsilon(t, x) = \mathbf{U}\Big(t, x; \frac{\phi_1(t, x)}{\varepsilon}, \ldots, \frac{\phi_m(t, x)}{\varepsilon}\Big) + o(1) \tag{3.15}$$

in the sense of Definition 2.4, where the leading profile $\mathbf{U} = (U_\eta)_{\eta \in I} \in \mathcal{E}_{M,p}[\Omega \times \Psi]$ is the unique generalized solution to the problem

$$\left.\begin{aligned} X_k \mathbf{U}_k &= \mathrm{E}_k f_k(\mathbf{U}) \\ \mathbf{U}_k(0, x, \theta_k) &= \iota\mu_k(x) + a_k(x, \theta_k) \end{aligned}\right\}. \tag{3.16}$$

We will show that in this case both $\mathbf{u}^\varepsilon$ and $\mathbf{U}$ admit an associated distribution, both splitting in a singular and a regular part. To describe these, we introduce the distributional solution w to the linear system

$$\left.\begin{aligned} Lw &= 0 \\ w(0, x) &= \mu(x) \end{aligned}\right\} \tag{3.17}$$

and the smooth solution $v^\varepsilon \in \mathcal{C}^\infty(\Omega)$ of the nonlinear problem

$$\left.\begin{aligned} Lv^\varepsilon &= f(v^\varepsilon) \\ v^\varepsilon(0, x) &= a\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big) \end{aligned}\right\}. \tag{3.18}$$

According to the classical result in Proposition 3.1, v^ε has the classical asymptotic expansion

$$v^\varepsilon(t, x) = V\Big(t, x; \frac{\phi_1(t, x)}{\varepsilon}, \dots, \frac{\phi_m(t, x)}{\varepsilon}\Big) + o(1)$$

where the leading term V is the classical smooth solution of the problem

$$\left.\begin{aligned} X_k V_k &= \mathrm{E}_k f_k(V) \\ V_k(0, x, \theta_k) &= a_k(x, \theta_k) \end{aligned}\right\}. \tag{3.19}$$

The relation among these various classical and generalized parts is described by the following result:

Proposition 3.3. *Suppose f is sublinear and satisfies assumption* (HP). *Let $\mu, a, \mathbf{u}^\varepsilon, \mathbf{U}, v^\varepsilon, V$ and w be as described above. Then:*

(1) *For each $\varepsilon > 0$, $\mathbf{u}^\varepsilon$ admits $v^\varepsilon + w$ as associated distribution.*

(2) *$\mathbf{U}$ admits $V + w$ as associated distribution.*

Proof. Assertion (1) is a standard result on delta waves and can be found in [15] or in Rauch and Reed [16].

Let w_η be the smooth solution to problem (3.17), but with regularized initial data $w_\eta(0, x) = \mu * \varphi_\eta(x)$. Using equations (3.17) and (3.19) we know that, for any $\eta \in I$ and $k \in \{1, \dots, m\}$, $W_{k,\eta} = U_{k,\eta} - V_k - w_{k,\eta}$ satisfies

$$\left.\begin{aligned} \big(\partial_t + \lambda_k(t, x)\partial_x\big) W_{k,\eta} &= \mathrm{E}_k\big(f_k(U_\eta) - f_k(V + w_\eta) + Q\big) \\ W_{k,\eta}(0, x, \theta_k) &= 0 \end{aligned}\right\} \tag{3.20}$$

where $Q = f_k(V + w_\eta) - f_k(V)$. The $L^1_{loc}(\Omega)$-norm of the term Q is seen to go to zero when $\eta \to 0$ by using the sublinear property of f, the convergence $w_\eta \to 0$ for almost all $(t, x) \in \Omega$, and the Lebesgue dominated convergence theorem.

Integrating equations (3.20) along characteristic lines and using the global Lipschitz property of f, we obtain that the $L^1_{loc}(\Omega)$-norm of W_η goes to zero when $\eta \to 0$. Finally, it is clear that $w_\eta \to w$ in the sense of distributions as $\eta \to 0$. This establishes assertion (2) ∎

Remark 3.4. This result can be generalized in various directions. First, one may take f sublinear of order $s \in (0,1]$ as in [15]. Second, depending on this order of sublinearity s, the initial data μ_k may contain derivatives of delta functions as well, as in [15] and [16]. Third, the oscillatory term $a(\cdot)$ need not be smooth. It would suffice that $a(x,\theta_0) \in L^\infty_{loc}(\omega, C^0_p(\mathrm{R}))$; then an additional regularization is required, but the result remains true.

4. Appendix: Proof of Proposition 3.1

The local existence of solutions u^ε and U to problems (1.1) and (3.9) and the zero-order asymptotic expansion (3.8) in the L^∞-norm have been established in [7: Theorem 2.8.1]. It remains to prove the additional estimates (3.7) and (3.10) as well as the estimates on the derivatives needed for the asymptotic expansion (3.8) according to Definition 2.1.

(1) At first, we consider estimates (3.7). From (1.1), the solution u^ε can be expressed as

$$u_k^\varepsilon(t,x) = a_k^\varepsilon\big(\gamma_k(0;t,x)\big) + \int_0^t f_k\big(u^\varepsilon(s,\gamma_k(s;t,x))\big)ds$$

which gives rise to

$$\|u_k^\varepsilon\|_t \le \|a_k^\varepsilon\| + C\int_0^t \|u^\varepsilon\|_s\,ds \tag{A.1}$$

by using the global Lipschitz property of $f(u)$ and $f(0)=0$. It immediately follows from (A.1) that $\|u^\varepsilon\|_T \le e^{CT}\|a^\varepsilon\|$. Acting the operator ∂_x^p on problem (1.1), we get

$$\left.\begin{aligned} L(\partial_x^p u^\varepsilon) &= \nabla f(u^\varepsilon)\cdot\partial_x^p u^\varepsilon - \partial_x\Lambda(t,x)\partial_x^p u^\varepsilon + F(\{\partial_x^q u^\varepsilon\}_{0\le q<p}) \\ \partial_x^p u^\varepsilon(0,x) &= \partial_x^p a^\varepsilon(x)\end{aligned}\right\} \tag{A.2}$$

which implies

$$\|\partial_x^p u^\varepsilon\|_t \le \|\partial_x^p a^\varepsilon\| + C\int_0^t \|\partial_x^p u^\varepsilon\|_s\,ds + \int_0^t P\Big(\sum_{q=0}^{p-1}\|\partial_x^q u^\varepsilon\|_s\Big)ds \tag{A.3}$$

because in (A.2), $F(\cdot)$ is polynomially bounded. By using the Gronwall inequality in (A.3), estimate (3.7) for terms $\partial_x^p u^\varepsilon$ with any $p \ge 0$ follows. The estimates of $\partial_t^q\partial_x^p u^\varepsilon$ are easily obtained by using the equations for u^ε and induction on $q \ge 0$.

(2) Next we consider estimates (3.10). The solutions U_k of (3.9) can be expressed as

$$U_k(t,x;\theta_k) = a_k\big(\gamma_k(0;t,x),\theta_k\big) + \int_0^t \mathrm{E}_k f_k(U)\big(s,\gamma_k(s;t,x);\theta_k\big)ds$$

which implies estimate (3.10) for the term $\|U\|_T$ by using the boundedness of the operator E_k and condition (HP). The estimates for the general terms $\|\partial^\alpha_{(t,x)}\partial^\gamma_\theta U\|_T$ can be established in the same way as those for u^ε by acting $\partial_x^\alpha\partial_\theta^\gamma$ on problem (3.9).

(3) Finally, we study the asymptotic properties of $\varepsilon^{|\alpha|}\partial^\alpha_{(t,x)}u^\varepsilon(t,x)$ for any $|\alpha|>0$.

(a) Acting $\varepsilon\partial_x$ on problem (1.1), we obtain that $v^\varepsilon = \varepsilon\partial_x u^\varepsilon$ solves the problem

$$\left.\begin{aligned} X_k v_k^\varepsilon &= \nabla f_k(u^\varepsilon)v^\varepsilon - \partial_x\lambda_k(t,x)v_k^\varepsilon \\ v_k^\varepsilon(0,x) &= \varepsilon\partial_x a_k^\varepsilon(x) \end{aligned}\right\} \tag{A.4}$$

where the initial data satisfy $\varepsilon\partial_x a^\varepsilon(x) - \phi_0'(x)\partial_{\theta_0} a\big(x, \frac{\phi_0(x)}{\varepsilon}\big) = o(1)$ in $L^\infty(\omega)$. Applying the result of [7: Theorem 2.8.1] in problem (A.4), we obtain

$$v_k^\varepsilon(x) - V_k\Big(t, x; \frac{\phi_k(t,x)}{\varepsilon}\Big) = o(1) \tag{A.5}$$

in $L^\infty(\Omega)$ where the functions V_k satisfy

$$\left.\begin{aligned} X_k V_k &= \mathrm{E}_k(\nabla f_k(U)V) - \partial_x\lambda_k(t,x)V_k \\ V_k(0,x;\theta_k) &= \phi_0'(x)\partial_{\theta_k} a_k(x,\theta_k) \end{aligned}\right\}. \tag{A.6}$$

On the other hand, by acting the operators $\partial_x\phi_k(t,x)\partial_{\theta_k}$ on problem (3.9), we get that $V_k(t,x;\theta_k) = \partial_x\phi_k(t,x)\partial_{\theta_k}U_k(t,x;\theta_k)$ solve problem (A.6) by using [7: Lemma 4.3.1] and $X_k\phi_k = 0$. Thus, by invoking the uniqueness of solutions to (A.6), we know that (A.5) can be rewritten as

$$(\varepsilon\partial_x)u_k^\varepsilon(t,x) - \partial_x\phi_k(t,x)\partial_{\theta_k}U_k\Big(t, x; \frac{\phi_k(t,x)}{\varepsilon}\Big) = o(1) \tag{A.7}$$

in $L^\infty(\Omega)$.

(b) From the equations for u^ε we have $\varepsilon\partial_t u_k^\varepsilon = \varepsilon f_k(u^\varepsilon) - \lambda_k(\varepsilon\partial_x u_k^\varepsilon)$ which implies

$$(\varepsilon\partial_t)u_k^\varepsilon(t,x) - \partial_t\phi_k(t,x)\partial_{\theta_k}U_k\Big(t, x; \frac{\phi_k(t,x)}{\varepsilon}\Big) = o(1)$$

in $L^\infty(\Omega)$ by using (A.7) and $X_k\phi_k = 0$ again.

(c) Acting $\varepsilon\partial_x$ on problem (A.4), it follows that $z^\varepsilon(t,x) = (\varepsilon\partial_x)^2 u^\varepsilon(t,x)$ solves the problem

$$\left.\begin{aligned} X_k z_k^\varepsilon &= (\nabla^2 f_k(u^\varepsilon)v^\varepsilon, v^\varepsilon) + \nabla f_k(u^\varepsilon)\cdot z^\varepsilon - 2\partial_x\lambda_k z_k^\varepsilon - \varepsilon\partial_x^2\lambda_k v_k^\varepsilon \\ z_k^\varepsilon(0,x) &= (\varepsilon\partial_x)^2 a_k^\varepsilon(x) \end{aligned}\right\} \tag{A.8}$$

where $v^\varepsilon = \varepsilon\partial_x u^\varepsilon$, and the initial data satisfy $(\varepsilon\partial_x)^2 a^\varepsilon(x) - (\phi_0'(x))^2\partial_{\theta_0}^2 a\Big(x, \frac{\phi_0(x)}{\varepsilon}\Big) = o(1)$ in $L^\infty(\omega)$. Applying the result of [7: Theorem 2.8.1] again in problem (A.8), we obtain

$$z_k^\varepsilon(t,x) - Z_k\Big(t, x; \frac{\phi_k(t,x)}{\varepsilon}\Big) = o(1) \tag{A.9}$$

in $L^\infty(\Omega)$ where the functions Z_k satisfy

$$\left.\begin{aligned} X_k Z_k &= \mathrm{E}_k\big((\nabla^2 f_k(U)V, V) + \nabla f_k(U)Z\big) - 2\partial_x\lambda_k Z_k \\ Z_k(0,x;\theta_k) &= (\phi_0'(x))^2\partial_{\theta_k}^2 a_k(x,\theta_k) \end{aligned}\right\} \tag{A.10}$$

by using (A.5).

On the other hand, by acting the operators $\partial_x\phi_k\partial_{\theta_k}$ on problem (A.6) and using [7: Lemma 4.3.1] and $X_k\phi_k = 0$ again, we get that $Z_k(t,x,\theta_k) = (\partial_x\phi_k)^2\partial^2_{\theta_k}U_k(t,x,\theta_k)$ solves problem (A.10). Thus, (A.9) can be rewritten as

$$(\varepsilon\partial_x)^2 u^\varepsilon_k(t,x) - (\partial_x\phi_k(t,x))^2\partial^2_{\theta_k}U_k\Big(t,x;\frac{\phi_k(t,x)}{\varepsilon}\Big) = o(1) \qquad (A.11)$$

in $L^\infty(\Omega)$ for any $k \in \{1,\dots,m\}$.

(d) Acting $\varepsilon^2\partial_x$ on the equations for u^ε, it follows

$$\begin{aligned}\varepsilon^2\partial^2_{tx}u^\varepsilon_k(t,x) &= -\varepsilon^2\lambda_k(t,x)\partial^2_x u^\varepsilon_k(t,x) + o(1)\\ &= \partial_t\phi_k\partial_x\phi_k\partial^2_{\theta_k}U_k\Big(t,x;\frac{\phi_k(t,x)}{\varepsilon}\Big) + o(1)\end{aligned} \qquad (A.12)$$

in $L^\infty(\Omega)$ by using (A.11) and $X_k\phi_k = 0$.

(e) Acting $\varepsilon^2\partial_t$ on the equations for u^ε, it follows

$$\begin{aligned}\varepsilon^2\partial^2_t u^\varepsilon_k(t,x) &= -\varepsilon^2\lambda_k(t,x)\partial^2_{tx}u^\varepsilon_k(t,x) + o(1)\\ &= (\partial_t\phi_k)^2\partial^2_{\theta_k}U_k\Big(t,x;\frac{\phi_k(t,x)}{\varepsilon}\Big) + o(1)\end{aligned}$$

in $L^\infty(\Omega)$ by using (A.12) and $X_k\phi_k = 0$ again.

Summarizing the above results from (a) to (e), we obtain the asymptotic property (3.9) in the norm $\|\cdot\|_{2,\varepsilon,K}$ for any $K \subset\subset \Omega$. Successively, we can prove (3.9) in the norm $\|\cdot\|_{k,\varepsilon,K}$ for any $K \subset\subset \Omega$ and $k \geq 3$ ∎

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References

[1] Biagioni, H. A.: *A Nonlinear Theory of Generalized Functions.* Lect. Notes Math. 1401 (1990).

[2] Biagioni, H. A. and J. F. Colombeau: *New generalized functions and C^∞ functions with values in generalized complex numbers.* J. London Math. Soc. 33 (1986), 169 – 179.

[3] Colombeau, J. F.: *New Generalized Functions and Multiplication of Distributions* (North-Holland Math. Studies: Vol. 84). Amsterdam: North-Holland 1984.

[4] Colombeau, J. F.: *Elementary Introduction to New Generalized Functions* (North-Holland Math. Studies: Vol. 113). Amsterdam: North-Holland 1985.

[5] Colombeau, J. F.: *Multiplication of Distributions.* Lect. Notes Math. 1532 (1992).

[6] DiPerna, R. and A. Majda: *The validity of nonlinear geometric optics for weak solutions of conservation laws.* Commun. Math. Phys. 98 (1985), 313 – 347.

[7] Joly, J. L., Métivier, G. and J. Rauch: *Resonant one dimensional nonlinear geometric optics.* J. Funct. Anal. 114 (1993), 106 – 231.

[8] Katznelson, Y.: *Introduction to Harmonic Analysis.* New York: Dover 1976.

[9] Majda, A. and R. Rosales: *Resonantly interacting weakly nonlinear hyperbolic waves.* Part I: *A single space variable.* Stud. Appl. Math. 71 (1984), 149 – 179.

[10] Mikusiński, J. and T. K. Boehme: *Operational Calculus*, Vol II. Warszawa: Polish Sci. Pub. 1987.

[11] Nedeljkov, M., Pilipović, S. and D. Scarpalézos: *The Linear Theory of Colombeau Generalized Functions* (Pitman Res. Notes in Math.: Vol. 385). Harlow: Longman 1998.

[12] Oberguggenberger, M.: *Multiplication of Distributions and Applications to Partial Differential Equations* (Pitman Res. Notes in Math.: Vol. 259). Harlow: Longman 1992.

[13] Oberguggenberger, M.: *Generalized solutions to semilinear hyperbolic systems.* Monatsh. Math. 103 (1987), 133 – 144.

[14] Oberguggenberger, M.: *Weak limits of solutions to semilinear hyperbolic systems.* Math. Ann. 274 (1986), 599 – 607.

[15] Oberguggenberger, M. and Y. G. Wang: *Delta waves for semilinear hyperbolic Cauchy problems.* Math. Nachr. 166 (1994), 317 – 327.

[16] Rauch, J. and M. C. Reed: *Nonlinear superposition and absorption of delta waves in one space dimension.* J. Funct. Anal. 73 (1987), 152 – 178.

[17] Rosinger, E. E.: *Generalized Solutions of Nonlinear Partial Differential Equations* (North-Holland Math. Studies: Vol. 144). Amsterdam: North-Holland 1987.

[18] Rosinger, E. E.: *Nonlinear Partial Differential Equations. An Algebraic View of Generalized Solutions* (North-Holland Math. Studies: Vol. 164). Amsterdam: North-Holland 1990.

[19] Schochet, S.: *Resonant nonlinear geometric optics for weak solutions of conservation laws.* J. Diff. Eqs. 113 (1994), 473 – 504.

[20] Schwartz, L.: *Théorie des Distributions.* Paris: Hermann 1966.

[21] Wang, Y. G.: *Nonlinear geometric optics for shock waves.* Part I: *Scalar case* and Part II: *System case.* Z. Anal. Anw. 16 (1997), 607 – 619 and 857 – 918.

[22] Wang, Y. G.: *Nonlinear geometric optics for two shock waves.* Comm. Part. Diff. Eqs. 23 (1998), 1621 – 1692.

[23] Williams, M.: *Highly oscillatory multidimensional shocks.* Comm. Pure Appl. Math. 52 (1999), 129 – 192.

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